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Verification of Performance Bounds for A-Posteriori Quantized Explicit MPC Feedback Laws

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Abstract: We investigate how the performance of explicit MPC feedback laws is affected by rounding-based quantization of the control commands. Specifically, we address the problem of providing a rigorous certificate that a given quantized piecewise affine explicit MPC feedback is bounded from below and from above by specific functions. These functions are constructed as to reflect typical control requirements, such as recursive feasibility and closed-loop stability. We show how to obtain an analytical form of the quantized MPC feedback and how to provide the certificate by solving a set of linear programs.

Keywords: Predictive control for linear systems, Quantization, Verification.

1. INTRODUCTION

Control under quantized feedbacks is an important research field since a majority of control policies is nowadays implemented on digital platforms, which inherently induce quantization effects due to finite-precision arithmetics and employment of analog-to-digital and digital-to-analog conversions. Numerous techniques for designing quantized feedback strategies which provide desired properties (e.g., closed-loop stability and/or performance) were proposed recently, see e.g. Goodwin and Quevedo (2003); Nair et al. (2007) and references therein. Typically, the approaches consider the control of linear time-invariant systems by a linear feedback, quantized by memory-less static quantizers. When such a setup is considered, Huijun and Tongwen (2008) have shown how to derive maximum sectors bounds for which the quantized linear feedback attains closed-loop stability. In Fu and Xie (2009) the authors have studied which number of quantization levels is required to attain stability. If the feedback law is piecewise affine (PWA), and the system to be controlled contains just one state and one input, Fagnani and Zampieri (2003) have developed conditions under which the quantized feedback attains practical stability in the sense of providing guarantees that the closed-loop trajectories converge to certain intervals. A common deficiency of the aforementioned approaches, though, is that they do not consider constraints on system's states and inputs.

On the other hand, model predictive control (MPC) excels at providing optimal performance while rigorously enforcing constraint satisfaction. Design of MPC feedback for systems with quantized inputs is, however, non-trivial. The problem is typically tackled by devising a hybrid model (Bemporad and Morari, 1999) of the system where the system's inputs are enforced to belong to a finite alphabet (Picasso et al., 2003; Grancharova and Johansen, 2012). Then the MPC optimization problem can be solved as a mixed-integer optimization problem (Bemporad et al., 2002a). However, complexity of such strategies is often prohibitive for application on simple control platforms which offer only limited computational capabilities. To address this issue, the concept of *explicit* MPC was developed (Bemporad et al., 2002b). Here, the optimization problem is solved off-line for all possible initial conditions and the solution is recorded as a PWA function of state measurements. Then, once the optimal control action needs to be obtained at each sampling instant, it suffices to evaluate the function. This can be done efficiently even on simple hardware platforms. The limitation of traditional explicit MPC techniques for quantized systems is in the inherent complexity of the off-line optimization. In particular, the MPC piecewise affine feedback needs to be obtained by solving a parametric mixed-integer optimization problem, where the individual quantization levels are modeled by binary (or integer) variables. The total number of such discrete components is then proportional to the prediction horizon. As the horizon or the number of states increase, the problems become very challenging to solve. One notable exception is the work of Quevedo et al. (2002), in which the explicit representation of the quantized feedback is developed directly by employing Voronoi diagrams. The limitation, however, is that the resulting quantized control law does not posses a-priori guarantees of recursive feasibility and closed-loop stability.

Instead of attempting to devise a quantized explicit MPC feedback directly, in this paper we propose to employ the real-valued MPC feedback, synthesized for a linear system subject to real-valued control inputs. The optimal real-valued feedback is then quantized, a-posteriori, by a static memory-less quantizer with a finite number of quantization levels. The main objective of the paper is to certify that such an a-posteriori quantized feedback achieves desired properties. If a positive certificate is obtained, one can safely apply the real-valued feedback, whose construction is much simpler compared to designing a quantized feedback directly.

2. PROBLEM STATEMENT

We aim at controlling linear time-invariant systems in the discrete-time domain described by the state-update equation

$$x^+ = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$ are the states, $u \in \mathbb{R}^m$ are the inputs, and x^+ is the successor state, with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and the pair (A, B) controllable. The system in (1) is subject to constraints

$$x \in \mathcal{X},\ u \in \mathcal{U}, \tag{2}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are non-empty polyhedra.

For the system in (1), the finite-horizon MPC problem can be formulated as

$$\min_{u_0,\dots,u_{N-1}} \ x_N^T Q_\mathrm{N} x_N + \sum_{k=0}^{N-1} x_k^T Q_\mathrm{x} x_k + u_k^T Q_\mathrm{u} u_k \tag{3a}$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k, \tag{3b}$$

$$x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ x_N \in \mathcal{T}, \tag{3c}$$

where x_k and u_k denote, respectively, the state and input predictions at the k-th step. Q_N is the terminal penalty, Q_x and Q_u are stage penalties, and $\mathcal{T} \subset \mathbb{R}^n$ is the polytopic terminal set.

It is well known (see e.g. Bemporad et al. (2002b)) that by applying *parametric optimization*, the MPC feedback law [1]

$$u = f(x), \tag{4}$$

with $f : \mathbb{R}^n \to \mathbb{R}^m$ can be obtained by solving (3) as a parametric quadratic program. Then $f(\cdot)$ is a piecewise affine function of the state, given as

$$f(x) := F_i x + g_i \text{ if } x \in \mathcal{R}_i, \tag{5}$$

where $F_i \in \mathbb{R}^{m\times n}$, $g_i \in \mathbb{R}^m$, $\mathcal{R}_i \subseteq \mathbb{R}^n$, $i = 1,\dots,M$, and the polyhedra $\mathcal{R}_i$ do not overlap.

Assume now that we are given a total of d quantization levels $q_1,\dots,q_d$, $q_i \neq q_j\ \forall i \neq j$. Then the rounding-based quantized version of (4) is

$$u = f_\mathrm{q}(x) := q_i \text{ if } x \in \mathcal{P}_i, \tag{6}$$

where

$$\mathcal{P}_i = \{x \mid \|f(x) - q_i\|_2 \le \|f(x) - q_j\|_2,\ \forall j \neq i\}, \tag{7}$$

denotes the region of the state space where the control command generated by the real-valued feedback (4) is closer to the i-th quantization level than to any other level. In other words, (6) rounds the value of (4) to the nearest quantization level. Note that each region $\mathcal{P}_i$ in (7) can, in general, be a non-convex and disconnected set. However, the regions can be decomposed into a finite number D of connected sets. If $f(\cdot)$ in (4) is a linear function, then $D = d$. Otherwise, $D \ge d$.

The objective of this paper can be formally stated as follows:

Problem 1. Given are: the real-valued explicit MPC feedback $f(\cdot)$ in (4) and (5), performance bounds $\underline{V} : \mathbb{R}^n \to \mathbb{R}^m$, $\overline{V} : \mathbb{R}^n \to \mathbb{R}^m$ with $\mathrm{dom}(\underline{V}) = \mathrm{dom}(\overline{V}) = \mathrm{dom}(f_\mathrm{q}) = \Omega$, and the quantization levels $q_1,\dots,q_d$. Determine whether the quantized feedback (6) satisfies performance bounds

$$\underline{V}(x) \le f_\mathrm{q}(x) \le \overline{V}(x),\ \forall x \in \Omega. \tag{8}$$

□

[1] To simplify the notation, we will henceforth assume $u \equiv u_0$ and $x \equiv x_0$.

In Section 4 we will show how to design the bounds $\underline{V}(\cdot)$, $\overline{V}(\cdot)$ as to capture various performance and safety criteria.

The difficulty of solving Problem 1 stems from the fact that (8) has to hold for *all* points from the domain Ω. A further complication is that the quantized feedback $f_\mathrm{q}(\cdot)$ is a nonlinear function (cf. (6) and notice that the function is nonlinear due to presence of IF-THEN logic rules), and $\underline{V}(\cdot)$ with $\overline{V}(\cdot)$ can be nonlinear functions as well.

The condition under which we will provide an answer to Problem 1 in a non-conservative manner is summarized next.

Assumption 2. We assume that the performance bounds $\underline{V}(\cdot)$ and $\overline{V}(\cdot)$ are piecewise affine functions

$$\underline{V}(x) := \underline{\alpha}_{i,1} x + \underline{\alpha}_{i,0} \text{ if } x \in \mathcal{R}_i, \tag{9a}$$

$$\overline{V}(x) := \overline{\alpha}_{i,1} x + \overline{\alpha}_{i,0} \text{ if } x \in \mathcal{R}_i, \tag{9b}$$

with $\alpha_{i,1} \in \mathbb{R}^{m\times n}$, $\alpha_{i,0} \in \mathbb{R}^m$, and the polyhedra $\mathcal{R}_i$, $i = 1,\dots,M$ are the same as in (5). □

Remark 3. Although the procedures of this paper are applicable to multivariable systems with multiple inputs ($m > 1$), to simplify the presentation we will henceforth assume that all functions in (8) are scalar-valued. Note that with $m > 1$, (8) can be split into a set of m relations where each of them has to be satisfied to certify that a vector-valued quantized feedback $f_\mathrm{q}(\cdot)$ meets the prescribed performance bounds. □

3. CERTIFICATION OF QUANTIZED FEEDBACKS

In this section we show how to compute a *true/false* answer to Problem 1, provided that the bounding functions $\underline{V}(\cdot)$ and $\overline{V}(\cdot)$ satisfy the condition in Assumption 2.

3.1 Analytical Form of the Quantized Feedback

We start by developing an analytical form of the quantized feedback $f_\mathrm{q}(\cdot)$ in (6), provided we know the analytic expression for the real-valued explicit MPC control law $f(\cdot)$ in (5). In particular, we devise regions $\mathcal{P}_1,\dots,\mathcal{P}_D$, along with the quantization level active in each region, such that $f_\mathrm{q}(\cdot)$ is given as a piecewise affine constant function of the form

$$f_\mathrm{q}(x) := c_i \text{ if } x \in \mathcal{P}_i, \tag{10}$$

where $c_i \in \{q_1,\dots,q_d\}$ for $i = 1,\dots,D$, and $\mathcal{P}_i$ are polyhedra in $\mathbb{R}^n$.

Consider the k-th polyhedron $\mathcal{R}_k$ of the real-valued feedback $f(\cdot)$ where $f(x) = F_k x + g_k$ is the local feedback law. Then the subset of $\mathcal{R}_k$ where the control action $u = f(x)$ is rounded towards the i-th quantization level q_i is given by

$$\mathcal{P}_{k,i} = \{x \in \mathcal{R}_k \mid \|f(x) - q_i\|_2 \le \|f(x) - q_j\|_2,\ \forall j \neq i\}. \tag{11}$$

Recall that $\mathcal{R}_k$ is assumed to be a polyhedron and $f(x)$ is an affine function $\forall x \in \mathcal{R}_k$ by (5). Then $\mathcal{P}_{k,i}$ is a polyhedron. To see this, note that the inequalities that constitute $\mathcal{P}_{k,i}$ involve non-negative functions. Hence squaring both sides does not change the sign and we obtain

$$(f(x) - q_i)^T (f(x) - q_i) \le (f(x) - q_j)^T (f(x) - q_j), \tag{12}$$

which simplifies to

$$2f(x)^T(q_j - q_i) \leq q_j^T q_j - q_i^T q_i. \tag{13}$$

Note that (13) has to hold $\forall j \neq i$. Finally, since $f(x) = F_k x + g_k$ for all $x \in \mathcal{R}_k$ per (5), and because $q_1, \ldots, q_d$ are known and fixed, expressions in (13) become

$$2(F_k x + g_k)^T(q_j - q_i) \leq q_j^T q_j - q_i^T q_i, \ \forall j \neq i, \tag{14}$$

which are $d-1$ linear inequalities in x. Then $\mathcal{P}_{k,i}$ is given by

$$\mathcal{P}_{k,i} = \{x \mid 2(F_k x + g_k)^T(q_j - q_i) \leq q_j^T q_j - q_i^T q_i, \ \forall j \neq i\}, \tag{15}$$

which furthermore needs to be intersected with $\mathcal{R}_k$. More generally, $\mathcal{P}_{k,i}$ is the i-th cell of the Voronoi diagram (Aurenhammer, 1991) of the points $q_1, \ldots, q_d$, where each cell is intersected with $\mathcal{R}_k$. It follows directly from properties of Voronoi diagrams that, for a fixed k, $\mathrm{int}(\mathcal{P}_{k,i}) \cap \mathrm{int}(\mathcal{P}_{k,j}) = \emptyset$ and $\cup_i \mathcal{P}_{k,i} = \mathcal{R}_k$. Naturally, the sets $\mathcal{P}_{k,i}$ can be empty for some $i \in \{1, \ldots, d\}$. However, there always exists at least one index i for which $\mathcal{P}_{k,i}$ is not empty.

The analytic representation of the quantized feedback in (6) can be obtained per Algorithm 1. The algorithm iterates over polyhedra $\mathcal{R}_k$ which define the real-valued feedback $f(\cdot)$ in (5). Next, for each $\mathcal{R}_k$ the i-th cell of the Voronoi diagram is computed in Step 4. If the intersection of the cell with $\mathcal{R}_k$ is non-empty, it is added to the set of polyhedra $\mathcal{P}$, the counter is updated, and the "active" quantization level associated to $\mathcal{P}_{k,i}$ is recorder. Upon exit, the algorithm generates polyhedra $\mathcal{P}_1, \ldots, \mathcal{P}_D$ of $f_\mathrm{q}(\cdot)$, along with information of which quantization level is active in each region. We remark that the upper bound on the total number of regions generated by Algorithm 1 is $D = dM$.

Algorithm 1 Construction of the quantized feedback in (6)

1: $D \leftarrow 0$, $\mathcal{P} \leftarrow \{\}$, $c \leftarrow \{\}$
2: **for each** $k = 1, \ldots, M$ **do**
3:   **for each** $i = 1, \ldots, d$ **do**
4:     Compute $\mathcal{P}_{k,i}$ per (15)
5:     **if** $\mathcal{P}_{k,i} \cap \mathcal{R}_k \neq \emptyset$ **then**
6:       $D \leftarrow D + 1$
7:       $\mathcal{P} \leftarrow \mathcal{P} \cup \{\mathcal{P}_{k,i} \cap \mathcal{R}_k\}$
8:       $c \leftarrow c \cup \{q_i\}$
9:     **end if**
10:   **end for**
11: **end for**

3.2 *Certification of Bounded Performance*

With the analytic form of the quantized feedback (10) in hand, we will next show how to formulate the certification problem (8). Our first main result is that the certification of performance bounds in (8) can be approached by investigating the minima/maxima of the performance bound functions $\underline{V}(\cdot)$ and $\overline{V}(\cdot)$ when x is restricted to all non-empty intersections $\mathcal{P}_i \cap \mathcal{R}_j$, where $\mathcal{P}_i$ is a particular polyhedron of the piecewise constant quantized feedback $f_\mathrm{q}(\cdot)$ in (10), and $\mathcal{R}_j$ is a region of the piecewise affine performance bounds in (9).

Theorem 4. Let the quantized feedback $f_\mathrm{q}(\cdot)$ in (10) be given, along with piecewise affine performance bounds $\underline{V}(\cdot)$, $\overline{V}(\cdot)$ of (9). Furthermore, denote $v(x, \underline{\alpha}_j) = \underline{\alpha}_{j,1} x + \underline{\alpha}_{j,0}$ and $v(x, \overline{\alpha}_j) = \overline{\alpha}_{j,1} x + \overline{\alpha}_{j,0}$ for $j = 1, \ldots, M$. Then $\underline{V}(x) \leq f_\mathrm{q}(x) \leq \overline{V}(x)$ for all $x \in \Omega$, i.e., (8) holds, if and only if

$$\max_{x \in \mathcal{P}_i \cap \mathcal{R}_j} (c_i - v(x, \overline{\alpha}_j)) \leq 0 \leq \min_{x \in \mathcal{P}_i \cap \mathcal{R}_j} (c_i - v(x, \underline{\alpha}_j)) \tag{16}$$

is satisfied for all $i = 1, \ldots, D$, $j = 1, \ldots, M$ for which $\mathcal{P}_i \cap \mathcal{R}_j \neq \emptyset$. ■

Remark 5. Note that the optimization problems in (16) are always feasible since we assume that x is constrained to belong to a non-empty set $\mathcal{P}_i \cap \mathcal{R}_j$. □

To exploit Theorem 4 to certify satisfaction of (8) for *all* $x \in \Omega$ we thus need to solve up to D times M problems (16), each of which involves solving two optimization problems. In practice, the number will be less, since only non-empty intersections $\mathcal{P}_i \cap \mathcal{R}_j$ need to be considered. The complete certification procedure is reported as Algorithm 2. Checking whether $\mathcal{P}_i \cap \mathcal{R}_j = \emptyset$ in Step 3 can be performed at the cost of solving one linear program, see e.g. Borrelli (2003).

Algorithm 2 Certification of performance bounds in (8).

1: **for each** $i = 1, \ldots, D$ **do**
2:   **for each** $j = 1, \ldots, M$ **do**
3:     **if** $\mathcal{P}_i \cap \mathcal{R}_j \neq \emptyset$ **then**
4:       Solve the maximization/minimization problems in (16)
5:       **if** (16) is not satisfied **then**
6:         **return** (8) not satisfied
7:       **end if**
8:     **end if**
9:   **end for**
10: **end for**
11: **return** (8) satisfied $\forall x \in \mathrm{dom}(f_\mathrm{q})$

In a more general sense, Theorem 4 provides necessary and sufficient conditions for satisfaction of (8) for arbitrary bounding functions $\underline{V}(\cdot)$, $\overline{V}(\cdot)$, e.g. for piecewise quadratic or piecewise polynomial functions. Then, however, the main difficulty there is that the optimization problems in (16) can be non-convex. Determining globally optimal solutions to non-convex problems is computationally challenging. With $\underline{V}(\cdot)$, $\overline{V}(\cdot)$ restricted to piecewise affine functions, on the other hand, the problems in (16) are linear programs that can be solved in polynomial time. To see this, note that the objective functions are linear in x, and the constraints are linear as well, since $\mathcal{P}_i \cap \mathcal{R}_j$ are polyhedra. Therefore the problems in (16) can be solved even in large dimensions with off-the-shelf optimization packages, such as CPLEX or GUROBI, or even with open-source alternatives such as GLPK or CDD.

4. CONSTRUCTION OF PERFORMANCE BOUNDS

In this section we present construction of performance bounds $\underline{V}(\cdot)$, $\overline{V}(\cdot)$ in (8) that reflect typical control requirements, such as recursive satisfaction of state/input constraints and closed-loop stability. We focus on construction of piecewise affine bounds $\underline{V}(\cdot)$, $\overline{V}(\cdot)$ for which validity of (8) can be certified in a non-conservative manner by solving a series of linear programs in (16). The development will be based on the assumption that the constraints in (2) are polytopic.

4.1 Recursive Satisfaction of State and Input Constraints

For the system in (1) with constraints as in (2), the positive control invariant set is given by

$$\mathbb{C} = \{x_0 \mid \exists \pi : Ax_k + B\pi(x_k) \in \mathcal{X},\ \pi(x_k) \in \mathcal{U},\ \forall k > 0\}, \tag{17}$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is a feedback strategy, and x_k and u_k are the states and inputs at the discrete time step k. Under mild conditions, the set $\mathbb{C}$ is a polytope and can be computed by a set recursion (Blanchini, 1999; Dórea and Hennet, 1999). The important property of the control invariant set is that for any $x \in \mathbb{C}$ there always exist a control action $u \in \mathcal{U}$ that keeps the state update x^+ inside of the state constraints, i.e., $x^+ \in \mathcal{X}$ for all time. Let us denote by

$$\mathcal{C}_{\text{xu}} = \{(x, u) \mid x \in \mathbb{C},\ u \in \mathcal{U},\ Ax + Bu \in \mathbb{C}\} \tag{18}$$

the set of all state-input pairs for which the state update $x^+ \in \mathbb{C}$. Therefore for any x and u satisfying $(x, u) \in \mathcal{C}_{\text{xu}}$ we have that: 1) $x \in \mathcal{X}$, 2) $u \in \mathcal{U}$, and 3) $Ax + Bu \in \mathcal{X}$. Therefore if u is selected such that $(x, u) \in \mathcal{C}_{\text{xu}}$, recursive satisfaction of state and input constraints is enforced. Since $\mathcal{C}_{\text{xu}}$ is a polytope, it can be represented by

$$\mathcal{C}_{\text{xu}} = \{(x, u) \mid Gx + Hu \le h\}. \tag{19}$$

Then the bounds $\underline{V}(\cdot)$, $\overline{V}(\cdot)$ that reflect recursive satisfaction of state and input constraints can be computed as follows:

$$\underline{V}(x) = \{\min u \mid Gx + Hu \le h\}, \tag{20a}$$
$$\overline{V}(x) = \{\max u \mid Gx + Hu \le h\}. \tag{20b}$$

The problems in (20) are *parametric* linear programs with optimization variables $u \in \mathbb{R}^m$ and parameters $x \in \mathbb{R}^n$:

Lemma 6. (Borrelli (2003)). The solution to (20) are piecewise affine function

$$\underline{V}(x) := \underline{\alpha}_{i,1}x + \underline{\alpha}_{i,0} \text{ if } x \in \mathcal{R}_i, \tag{21a}$$
$$\overline{V}(x) := \overline{\alpha}_{i,1}x + \overline{\alpha}_{i,0} \text{ if } x \in \mathcal{R}_i, \tag{21b}$$

where $\mathcal{R}_i$ are non-overlapping polytopes [2] in $\mathbb{R}^n$. ■

To construct $\underline{V}(\cdot)$, $\overline{V}(\cdot)$ in (21), one needs to solve the parametric linear programs (20). This can be achieved e.g. by the Multi-Parametric Toolbox (Herceg et al., 2013).

Remark 7. The worst-case complexity of parametric linear programs in (20), i.e., the upper bound on the number of polytopes $\mathcal{R}_i$ in (21), is exponential in the number of constraints, which in turn depends on dimensions of u and x. However, recent development (Gupta et al., 2011) in the theory of parametric optimization allows to solve such problems even in large dimensions, say over 50. □

4.2 Closed-Loop Stability

Let a piecewise linear convex Lyapunov function $L : \mathbb{R}^n \to \mathbb{R}$ for the system in (1), i.e.,

$$L(x) := \max_i\ \ell_i^T x \tag{22}$$

with $\ell_i \in \mathbb{R}^n$, $i = 1, \ldots, V$ be given. Then any feedback policy $\pi(x)$ that guarantees

$$L(Ax + B\pi(x)) \le \gamma L(x),\ \forall x \in \text{dom}(\pi), \tag{23}$$

for some $0 \le \gamma < 1$ trivially provides closed-loop stability as it forces the Lyapunov function to decrease along the state trajectories, see e.g. Lazar et al. (2008). The minimal and maximal control actions, as a function of the states x, which render the closed-loop stable in the sense of (23) can thus be obtained by

$$\underline{V}(x) = \{\min u \mid L(Ax + Bu) \le \gamma L(x),\ u \in \mathcal{U}\}, \tag{24a}$$
$$\overline{V}(x) = \{\max u \mid L(Ax + Bu) \le \gamma L(x),\ u \in \mathcal{U}\}. \tag{24b}$$

With $L(\cdot)$ as in (22), problems (24) can be formulated and solved as *parametric mixed-integer linear programs* in decision variables u and parameters x by introducing additional binary variables to model the maxima in (22) as follows:

Proposition 8. (Kvasnica et al. (2012)). The maximum among linear functions of x, i.e., $z = \max_i \ell_i^T x$, is modeled by

$$-M(1 - \delta_i) \le z - \ell_i^T x \le M(1 - \delta_i), \tag{25a}$$
$$\ell_j^T x \le \ell_i^T x + M(1 - \delta_i),\ \forall j \ne i, \tag{25b}$$
$$\textstyle\sum_{i=1}^V \delta_i = 1, \tag{25c}$$

where $\delta_i \in \{0, 1\}$, $i = 1, \ldots, V$ are binary variables, M is a sufficiently large positive constant, and (25a)−(25b) are enforced for all $i = 1, \ldots, V$. ■

With the help of Proposition 8, problem (24a) can be rewritten as

$$\underline{V}(x) = \{\min u \mid y \le \gamma z,\ u \in \mathcal{U}\}, \tag{26}$$

where z is related to $\max_i \ell_i^T x$ via (25), and y models $\max_i \ell_i^T (Ax + Bu)$ via the same relations. Note that when $L(\cdot)$ in (22) is a piecewise linear function of x, $L(Ax+Bu)$ is a piecewise linear function of x and u.

As shown in Borrelli (2003), the solutions $\underline{V}(x)$ and $\overline{V}(x)$ to (24) are piecewise affine functions of x as in (21). The local affine expressions, as well as the associated regions of validity, can be obtained by the Multi-Parametric Toolbox.

5. EXAMPLES

5.1 Illustrative 1D Example

To illustrate the procedure, consider the system $x^+ = Ax + Bu$ with $A = 0.9$, $B = 1$, $\mathcal{X} = \{x \mid -5 \le x \le 5\}$, $\mathcal{U} = \{u \mid -1 \le u \le 1\}$. The system is controlled by the state-feedback regulator $u = Kx$ with $K = -0.2$. It is easy to verify that such a feedback achieves recursive satisfaction of input and state constraints and attains closed-loop stability. Next, we investigate whether constraint satisfaction and closed-loop stability are achieved by a-posteriori quantizing the real-valued feedback $f(x) := Kx$ using three quantization levels: $q_1 = -0.9$, $q_2 = 0$, $q_3 = 0.8$. To do so, we have first derived the analytical form of $f_{\text{q}}(\cdot)$ in (10). This was achieved by applying Algorithm 1, which produced

$$f_{\text{q}}(x) := \begin{cases} 0.8 & \text{if } -5 \le x \le -2, \\ 0 & \text{if } -2 \le x \le 2.25, \\ -0.9 & \text{if } \ \ 2.25 \le x \le 5 \end{cases} \tag{27}$$

[2] The polytopes in (21a) and (21b) are the same since both problems in (20) have identical constraints.

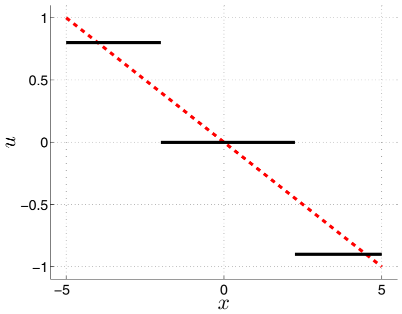


Fig. 1. Real-valued feedback $u = Kx$ (dashed red line) and its quantized version $u = f_{\mathrm{q}}(x)$ in (27) (solid black function) for the example in Section 5.1.

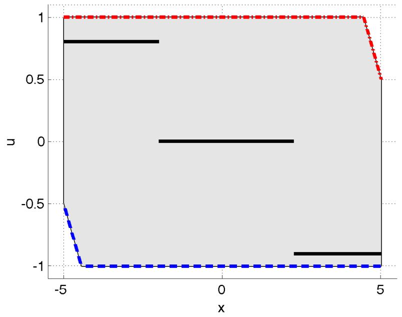


Fig. 2. Results for feasibility verification in Section 5.1: the set $\mathcal{C}_{\mathrm{xu}}$ in gray, $\underline{V}(\cdot)$ as the blue dashed function, $\overline{V}(\cdot)$ as the red dash-dotted function, and the quantized feedback in (27) as the black function.

after 0.02 seconds. The real-valued feedback $f(\cdot)$, as well as its quantized version (27), are shown in Fig. 1.

To verify whether $f_{\mathrm{q}}(\cdot)$ of (27) maintains recursive satisfaction of input and state constraints, we have proceed as in Section 4.1. First, the control invariant set $\mathbb{C} = \{x \mid -5 \leq x \leq 5\}$ was calculated by (17). Subsequently, the set $\mathcal{C}_{\mathrm{xu}}$ in (19) was constructed and represented as a polyhedron per (20). Finally, the bounding functions $\underline{V}(\cdot)$, $\overline{V}(\cdot)$ were computed by solving (20) as parametric linear programs. After 0.5 seconds we have obtained

$$\underline{V}(x) := \begin{cases} -0.9x - 5 & \text{if } -5 \leq x \leq -4.444, \\ -1 & \text{if } -4.444x \leq x \leq 5, \end{cases} \tag{28a}$$

$$\overline{V}(x) := \begin{cases} 1 & \text{if } -5 \leq x \leq 4.444, \\ -0.9x + 5 & \text{if } \ 4.444 \leq x \leq 5. \end{cases} \tag{28b}$$

The set $\mathcal{C}_{\mathrm{xu}}$, along with functions $\underline{V}(\cdot)$, $\overline{V}(\cdot)$, and $f_{\mathrm{q}}(\cdot)$, are shown in Fig. 2. As can be observed, the value of $f_{\mathrm{q}}(x)$ is always between $\underline{V}(x)$ and $\overline{V}(x)$ for all $x \in \mathbb{C}$. Hence $f_{\mathrm{q}}(\cdot)$ of (27) provides recursive satisfaction of state and input constraints. To give a rigorous certificate that such an observation is indeed true, we have applied Algorithm 2 to $f_{\mathrm{q}}(\cdot)$, $\underline{V}(\cdot)$ and $\overline{V}(\cdot)$. By solving a total of 4 linear programs in (16), which took 0.03 seconds, a positive certificate for Problem 1 was indeed obtained.

Then we have investigated whether $f_{\mathrm{q}}(\cdot)$ in (27) provides closed-loop stability. To do so, we have first constructed functions $\underline{V}(\cdot)$ and $\overline{V}(\cdot)$ by solving parametric mixed-integer problems in (24) with the Lyapunov function given as the Minkowski function of the control invariant set

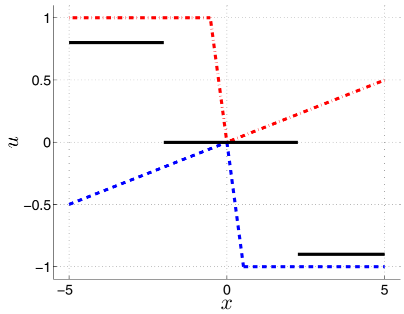


Fig. 3. Results for stability verification in Section 5.1: functions $\underline{V}(\cdot)$ (blue dashed), $\overline{V}(\cdot)$ (red dash-dotted) from (24), and the quantized feedback $f_{\mathrm{q}}(\cdot)$ in (27).

$\mathbb{C}$. The piecewise affine functions were constructed in 1 second and each of them was defined over 4 polyhedra. The resulting bounding functions, along with $f_{\mathrm{q}}(\cdot)$, are shown in Fig. 3. As can be observed from the figure, all values of $f_{\mathrm{q}}(x)$ is always bounded by $\underline{V}(x)$ and $\overline{V}(x)$, hence the quantized feedback (27) provides closed-loop stability. A rigorous certificate of such a property was obtained by applying Algorithm 2, which required solving 4 linear programs for 4 non-empty intersections $\mathcal{P}_i \cap \mathcal{R}_j$.

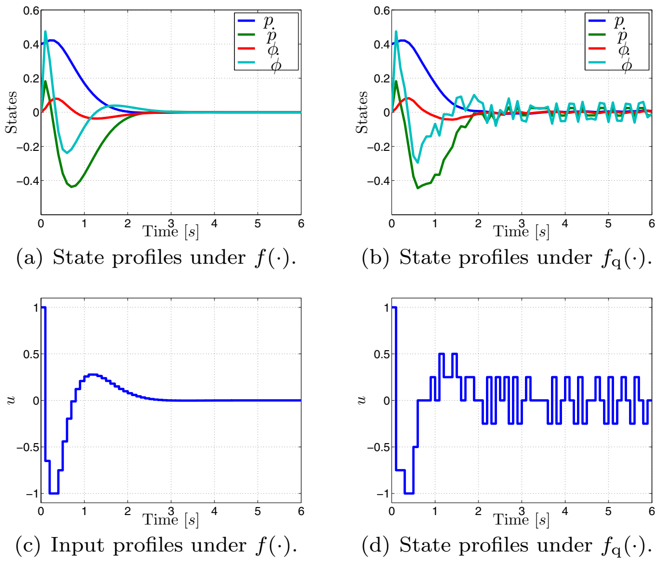
5.2 Inverted Pendulum on a Cart

Next we consider an inverted pendulum mounted on a moving cart. Linearizing the nonlinear dynamics around the upright, marginally stable position leads to the following linear model:

$$\begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.182 & 2.673 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.455 & 31.182 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.818 \\ 0 \\ 4.546 \end{bmatrix} u, \tag{29}$$

where p is the position of the cart (constrained by $|p| \leq 1.5$), $\dot{p}$ is the cart's velocity (with $|\dot{p}| \leq 1.5$), ϕ is the pendulum's angle from the upright position (with $|\phi| \leq 0.35$), and $\dot{\phi}$ denotes the angular velocity (restricted to $|\dot{\phi}| \leq 1.5$). The control input u, constrained to $|u| \leq 1$, is proportional to the force applied to the cart. System (29) was converted to (1) by assuming sampling time 0.1 seconds. For the discrete-time system we have first constructed the real-valued feedback in (5) by solving the MPC problem in (3) with the prediction horizon $N = 4$, and penalties $Q_{\mathrm{x}} = \mathrm{diag}(10, 1, 10, 1)$, $Q_{\mathrm{u}} = 0.1$. Moreover, the terminal penalty Q_{N} was selected as the solution to the algebraic Riccati equation, while $\mathcal{T}$ is the LQR terminal set. By solving (3) parametrically, the feedback $u_0 = f(x_0)$ was obtained as a piecewise affine function defined over 187 polytopes in $\mathbb{R}^4$.

Then we have investigated the properties of the quantized version of $f(\cdot)$ by assuming quantization levels $\{-1, -0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75, 1\}$. Here, the quantized controller $f_{\mathrm{q}}(\cdot)$ in (10) was first obtained by applying Algorithm 1 to $f(\cdot)$. After 5.7 seconds we have obtained the quantized feedback in (10) which was defined over 557 polytopes. A comparison of closed-loop performance of the real-valued MPC feedback $f(\cdot)$ versus its a-posteriori quantized version $f_{\mathrm{q}}(\cdot)$ is shown in Fig. 4.



(a) State profiles under $f(\cdot)$. (b) State profiles under $f_q(\cdot)$.

(c) Input profiles under $f(\cdot)$. (d) State profiles under $f_q(\cdot)$.

Fig. 4. Simulation results for the example in Section 5.2.

Subsequently, we have applied the procedure of Section 4.1 to check whether $f_q(\cdot)$ achieves recursive satisfaction of state and input constraints for an arbitrary controllable initial condition. The computation of the invariant set $\mathbb{C}$ in (17), along with construction of the performance bounds per (21), took 16.1 seconds in total. Here, the functions $\underline{V}(\cdot)$, $\overline{V}(\cdot)$ were both defined over 36 regions. Finally, Algorithm 2 was executed to check validity of (8). A positive certificate was obtained in 3.7 seconds. Therefore the a-posterior quantized feedback $f_q(\cdot)$ will never violate state/input constraints.

As can be observed from Fig. 4(b), for at least one initial condition the quantized feedback does not push all system's states asymptotically to the origin. Therefore $f_q(\cdot)$ does not provide guarantees of asymptotic closed-loop stability. This conclusion was verified per the procedure of Section 4.2. Here, we have first constructed the bounding functions by solving the parametric mixed-integer linear programs in (24). After 640 seconds we have obtained piecewise affine functions $\underline{V}(\cdot)$ and $\overline{V}(\cdot)$ defined over 714 polytopes in $\mathbb{R}^4$. The subsequent execution of Algorithm 2 took 1.8 seconds to find a violation of (16), hence certifying a negative answer to verification of closed-loop stability properties.

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