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Asymptotic Average Consensus of Continuous-time Multi-agent Systems with Dynamically Quantized Communication

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Abstract: This paper focuses on the consensus problem of continuous-time single-integrator and double-integrator multi-agent systems (MASs) with dynamically quantized information transmission. The connected undirected graphs are utilized to characterize the interaction topology between the agents. Dynamic quantizers are firstly introduced for a linear asymptotically stable system with a less conservative update interval. Through certainty equivalent quantized feedback controller and state transformation, the consensus problems of single-integrator and double-integrator MASs are then converted to the linear asymptotic stabilization problem, meanwhile the proposed dynamic quantization strategy is naturally applied to MAS to achieve asymptotic quantized average consensus. Finally, numerical examples are provided to illustrate the effectiveness of the theoretical results.

Keywords: Multi-Agent systems, Average Consensus, Dynamic Quantizer, Reduced Laplacian Matrix

1. INTRODUCTION

Distributed coordination control problem of MAS has attracted significant research interests in the past decade (Ren *et al*., 2011). A typical and widely investigated problem in distributed control is known as the consensus problem, which aims at coordinating the whole group of agents to achieve a global behavior of common decision through interacting each agent with its neighboring agents under a distributed protocol (Cortés 2008, Cao *et al*., 2008).

Under the assumption of accurate inter-agent information exchange, the accurate consensus to the average of the initial states can be arrived asymptotically or in finite time depending on the closed-loop dynamics (Wang *et al*., 2010, Li *et al*., 2011). However, due to the bandwidth constraints in communication, the information data should be quantized before transmission, i.e., the original precise data need to be truncated. Quantization can reduce the quantity of data transmission, whereas the imperfect information exchange may have a considerable impact on the performance of a MAS. Many researchers in control community have begun to investigate the quantization effects on distributed control of MASs, which leads to the concept of 'quantized consensus' (Kashay *et al*., 2007, Ceragioli *et al*., 2011, Liu *et al*., 2012, Guo *et al*., 2013).

There are three types of commonly used quantizers, namely, uniform quantizer, logarithmic quantizer and dynamic quantizer. The distributed control algorithms with uniform quantizers drive the single-integrator multi-agent systems to reach near consensus - a set around the accurate average consensus (Hui 2011). An encoding/decoding strategy with logarithmic quantizer to transmit information among agents reaches exact average consensus of first-order discrete-time MAS (Carli *et al*., 2010). The first-order continuous-time MAS with logarithmic quantizer can reach exact average consensus under undirected tree communication topologies, if the accuracy of logarithmic quantizer is small enough (Dimarogonas *et al*., 2010). The logarithmic quantizer is an infinite-level quantizer. When only a finite number of quantization levels is available, only the so-called near consensus can be obtained. One may consider applying a dynamic scaling approach to make the subset arbitrarily small. This motivates the development of dynamic quantization using a finite number of quantization levels (Baldan *et al*., 2009, Dong *et al*., 2013, Liu *et al*., 2012). This strategy is inspired by the quantized stabilization technique of linear and nonlinear systems proposed in (Brockett *et al*., 2000, Liberzon 2003), which is called zooming in - zooming out strategy.

Although dynamic quantization has been implemented in nonlinear and networked control systems, so far, there is no direct application of Liberzon's dynamic quantization strategy into the consensus problem of MAS. In this paper we adapt the dynamic quantization method proposed in the field of control under communication constraints to the average consensus problem. Compared with the previous work, less conservative dynamic adjusting strategy leads to faster convergence to the average consensus. The MAS under consideration consists of limited number of single integrator or double integrator agents that share quantized information under undirected communication topologies.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Algebraic Graph Theory

Consider an MAS with N agents, the communication topology among the agents are modeled by a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with a set of N nodes $\mathcal{V} = \{1,2,\cdots,N\}$, a set of M edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{A} = (a_{ij} \geq 0) \in R^{N\times N}$. A node $i \in \mathcal{V}$ represents the agent i. An undirected edge is denoted by an unordered pair of nodes $(i,j) \in \mathcal{E}$ if and only if there is a communication link between

i and j, and there is no self-edge (i,i) in the graph, i.e., $(i,i) \notin \mathcal{E}$. The adjacency elements associated with the edges are positive, i.e., $(i,j) \in \mathcal{E} \Leftrightarrow a_{ij} = a_{ji} > 0$, otherwise $a_{ij} = a_{ji} = 0$, $a_{ii} = 0$ for all $v_i \in \mathcal{V}$ because $(i,i) \notin \mathcal{E}$. Therefore, for an undirected graph, $\mathcal{A}$ is symmetric. The neighbor set of node i is denoted by $N_i = \{j \in \mathcal{V}: (i,j) \in \mathcal{E}, j \neq i\}$. The degree of node i is defined by $d_i = \sum_{j=1}^{N} a_{ij}$. Let $\mathcal{D} = diag(d_1, d_2, \cdots, d_N)$. The Laplacian matrix of $\mathcal{G}$ is defined by $\mathcal{L} = (l_{ij}) = \mathcal{D} - \mathcal{A} \in R^{N \times N}$ with $l_{ii} = d_i = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$. For a connected undirected graph $\mathcal{G}$, $\mathcal{L}$ is a positive semidefinite matrix with a single zero eigenvalue and its corresponding eigenvector $\mathbf{1} = [1,1,\cdots,1]^T$, i.e., $\mathcal{L}\mathbf{1} = \mathbf{0}$. By assigning each edge a direction, the incidence matrix $\mathcal{B} = (b_{ij}) \in R^{N \times M}$ is defined as a $\{0, \pm 1\}$-matrix with rows and columns indexed by the vertices and edges of $\mathcal{G}$, respectively, such that $b_{ij} = 1$ or $b_{ij} = -1$ if the node i is the head or tail of the edge (i,j) respectively, otherwise $b_{ij} = 0$. Then $\tilde{x} = \mathcal{B}^T x \in R^M$ denotes the stack edge vector of relative states of neighboring agents, and $\mathcal{L} = \mathcal{B}\mathcal{B}^T$. A sequence of edges $(i, i_1), (i_1, i_2), \cdots, (i_{r-1}, j)$ is called a path of length r between nodes i and j in the graph $\mathcal{G}$ with $r+1$ distinct nodes. A graph is called connected if there is a path between any pair of distinct nodes i and j. For a connected graph $\mathcal{G}$, when all agents' states are equal, $\tilde{x} = 0$ and $\mathcal{L}x = \mathcal{B}\mathcal{B}^T x = \mathcal{B}\tilde{x} = 0$.

2.2 Consensus dynamics of MAS

We consider a team of N autonomous agents, each of which is governed by the following single integrator

$$\dot{x}_i = u_i,\ i = 1,2,\cdots,N \tag{1}$$

or double integrator

$$\dot{x}_i = v_i,\ \dot{v}_i = u_i,\ \ i = 1,2,\cdots,N \tag{2}$$

where $x_i, v_i, u_i \in R$ denote the position, velocity and control input of agent i, respectively. Suppose $x = [x_1,\ x_2, \cdots, x_N]^T$, $v = [v_1,\ v_2, \cdots, v_N]^T$, $u = [u_1,\ u_2, \cdots, u_N]^T$. Then MAS (1) and (2) can be expressed in the stack vector form as $\dot{x} = u$ or $\dot{x} = v, \dot{v} = u$. For any initial condition $x(0) = [x_1(0),\ x_2(0), \cdots, x_N(0)]^T$, $v(0) = [v_1(0),\ v_2(0), \cdots, v_N(0)]^T$, the control aim is to construct distributed feedback controllers u_i, $i = 1,2,\cdots,N$ such that the MASs achieve the average consensus asymptotically, i.e.,

$$\lim_{t\to\infty} x(t) = x_{ave}(0) \tag{3}$$

for single-integrator dynamics or

$$\lim_{t\to\infty} v(t) = v_{ave}(0) \tag{4}$$

for double-integrator dynamics, where $x_{ave}(t) = \mathbf{1}^T x(t)/N$ and $v_{ave}(t) = \mathbf{1}^T v(t)/N$ are the average value of all agents' states.

2.3 Dynamic uniform quantizer

Assume that each agent i has only quantized communication of state $q(x_i)$ or relative state $q(x_i - x_j)$, where the quantizer $q: R \to S$ is a piecewise constant function, where S is a finite subset of R. A uniform quantizer q_u is defined by

$$q_u(x_i) = \left\lfloor \frac{x_i}{\Delta} + \frac{1}{2} \right\rfloor \Delta \tag{5}$$

Note that the bound of quantization error is $|q_u(x_i) - x_i| \leq \Delta/2$. For a finite-level uniform quantizer with the quantization range $\mathcal{M}$, $|q_u(x_i) - x_i| \leq \Delta/2$ for $|x_i| \leq \mathcal{M}$, and $|q_u(x_i)| = \mathcal{M} - \Delta/2$ for $|x_i| > \mathcal{M}$. Moreover if the state vector x is the variable to be quantized, $q_u(x) = [q_u(x_1),\ q_u(x_2), \cdots, q_u(x_N)]^T$. $|q_u(x) - x| \leq \sqrt{N}\Delta/2$ if $|x_i| \leq \mathcal{M}$, and $|q_u(x) - x| \leq \sqrt{N}(\mathcal{M} - \Delta/2)$ if $|x_i| > \mathcal{M}$.

A dynamic quantizer $q_\mu: R \to S$ is a variation of the uniform quantizer through introducing a zooming parameter μ as

$$q_\mu(x_i) = \mu q_u\left(\frac{x_i}{\mu}\right) = \left\lfloor \frac{x_i}{\mu\Delta} + \frac{1}{2} \right\rfloor \mu\Delta \tag{6}$$

with the quantization range $\mu\mathcal{M}$ and error bound $\mu\Delta/2$: $|q_\mu(x_i) - x_i| \leq \mu\Delta/2$ for $|x_i| \leq \mu\mathcal{M}$, and $|q_\mu(x_i)| = \mu(\mathcal{M} - \Delta/2)$ for $|x_i| > \mu\mathcal{M}$. For the state vector x, $q_\mu(x) = [q_\mu(x_1), q_\mu(x_2), \cdots, q_\mu(x_N)]^T$. $|q_\mu(x) - x| \leq \sqrt{N}\mu\Delta/2$ if $|x_i| \leq \mu\mathcal{M}$, and $|q_\mu(x) - x| \leq \sqrt{N}(\mathcal{M} - \Delta)$ if $|x_i| > \mu\mathcal{M}$. The basic idea of dynamic quantization is to dynamically update μ according to the location of the state x for obtaining the quantized information with a required resolution.

2.4 Problem formulation

For the MAS (1) and (2), the asymptotic average consensus (3) and (4) can be achieved by the feedback controllers

$$u_i = -\sum_{j\in N_i} a_{ij}\,(x_i - x_j) \text{ or } u = -\mathcal{L}x \tag{7}$$

and

$$u_i = -\sum_{j\in N_i} a_{ij}\,(x_i - x_j) - \sum_{j\in N_i} a_{ij}\,(v_i - v_j) \text{ or } u = -\mathcal{L}x - \mathcal{L}v \tag{8}$$

respectively. Then the closed-loop MASs become

$$\dot{x} = -\mathcal{L}x \tag{9}$$

and

$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}0 & I\\ -\mathcal{L} & -\mathcal{L}\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix} \text{ or } \dot{z} = \begin{bmatrix}0 & I\\ -\mathcal{L} & -\mathcal{L}\end{bmatrix} z \tag{10}$$

where $z = [x^T,\ v^T]^T$.

The main objective of this paper is to design a dynamic uniform quantizer q_μ such that the asymptotic average consensus (3) and (4) can be still preserved under the certainty equivalent controllers when only quantized relative state is available for feedback.

$$u_i = -\sum_{j\in N_i} a_{ij}\, q_\mu(x_i - x_j) \text{ or } u = -\mathcal{B}q_\mu(\mathcal{B}^T x) \tag{11}$$

and

$$u_i = -\sum_{j\in N_i} a_{ij}\, q_\mu(x_i - x_j) - \sum_{j\in N_i} a_{ij}\, q_\mu(v_i - v_j) \text{ or } u = -\mathcal{B}q_\mu(\mathcal{B}^T x) - \mathcal{B}q_\mu(\mathcal{B}^T v) \quad (12)$$

3. DYNAMIC QUANTIZATION OF A LINEAR SYSTEM

Consider a linear stabilizable system

$$\dot{x} = Ax + Bu \quad x \in R^n,\ u \in R^m \quad (13)$$

there exists a state feedback law $u = Kx$ such that $A + BK$ is negative definite. Then the closed-loop system $\dot{x} = (A + BK)x$ is stable, and there exists positive definite matrices P and Q such that

$$(A + BK)^T P + P^T(A + BK) = -Q \quad (14)$$

When only quantized measurements $q_\mu(x)$ are available, consider the certainty equivalent quantized feedback control law

$$u = Kq_\mu(x) \quad (15)$$

Then the closed-loop system is given by

$$\dot{x} = Ax + BKq_\mu(x) = (A + BK)x + BK\mu\left(q_u\left(\frac{x}{\mu}\right) - \frac{x}{\mu}\right) = (A + BK)x + BKe \quad (16)$$

where $e = \mu(q_u(x/\mu) - x/\mu)$ is the quantization error and $|e| \le \sqrt{n}\mu\Delta$.

For a Lyapunov function $V = x^T Px$, we have

$$\dot{V} = -x^T Qx + 2x^T PBKe \le -|x|\lambda_{min}(Q)(|x| - \theta\mu\Delta) \quad (17)$$

where $\theta = 2\|PBK\|\sqrt{n}/\lambda_{min}(Q) > 0$. Assume that $\mathcal{M}$ is larger enough than Δ such that $\sqrt{\lambda_{min}(P)}\,\mathcal{M} \ge \sqrt{\lambda_{max}(P)}\theta\Delta(1 + \varepsilon)$ for arbitrary $\varepsilon > 0$. Because $|x| \ge (1 + \varepsilon)\theta\mu\Delta$ holds in the region between the ellipsoids $\mathcal{R}_1 := \{x: x^T Px \le \lambda_{min}(P)\mathcal{M}^2\mu^2\}$ and $\mathcal{R}_2 := \{x: x^T Px \le \lambda_{max}(P)\theta^2\Delta^2(1 + \varepsilon)^2\mu^2\}$, this means $\theta\mu\Delta \le |x|/(1 + \varepsilon)$. Therefore

$$\dot{V} \le -\lambda_{min}(Q)\frac{\varepsilon}{1+\varepsilon}|x|^2 \le -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}\frac{\varepsilon}{1+\varepsilon}V \quad (18)$$

Let α denote $\lambda_{min}(Q)\varepsilon/\big(\lambda_{max}(P)(1 + \varepsilon)\big)$, then V will decay at least at the exponential rate α, i.e., $V \le V(0)e^{-\alpha t}$. It can be seen that $\dot{V}$ is negative outside the ellipsoid $\mathcal{R}_2$ centered at the origin. Then the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$ are invariant regions for system (16). Moreover, all solutions of (16) starting in $\mathcal{R}_1$ enter $\mathcal{R}_2$ in finite time, and this time is upper bound by

$$T = \frac{1}{\alpha} ln \frac{\lambda_{min}(P)\mathcal{M}^2}{\lambda_{max}(P)\theta^2\Delta^2(1+\varepsilon)^2} \quad (19)$$

According to Liberzon's design strategy, when the initial state is in the ellipsoid $\mathcal{R}_1$ with the initial zooming variable μ_0, the zooming-in phase starts with the update interval T and the zooming-in rule is

$$\mu = \Omega^k \mu_0,\ \Omega = \frac{\sqrt{\lambda_{max}(P)}\theta\Delta(1+\varepsilon)}{\sqrt{\lambda_{min}(P)}\mathcal{M}} < 1 \quad (20)$$

for $t \in [kT, (k + 1)T]$ where k is the number of update times. Then it is guaranteed that x converges to zero at least exponentially.

Remark 1. The main difference from Liberzon's strategy is the estimation of convergence rate from $\mathcal{R}_1$ to $\mathcal{R}_2$. We use the exponential decay rate which is state-dependent: fastest on $\mathcal{R}_1$ and slowest on $\mathcal{R}_2$, to estimate the travel time from $\mathcal{R}_1$ to $\mathcal{R}_2$, while the latter uses the slowest rate on $\mathcal{R}_2$ to estimate the time so as to result in a longer time $T = (\lambda_{min}(P)\mathcal{M}^2 - \lambda_{max}(P)\theta^2\Delta^2(1 + \varepsilon)^2)/(\theta^2\Delta^2(1 + \varepsilon)\varepsilon\lambda_{min}(Q))$. Therefore the estimated convergence time is less conservative than Liberzon's strategy.

4. ASYMPTOTIC AVERAGE CONSENSUS OF MAS WITH DYNAMIC QUANTIZERS

4.1 Single-integrator agents

Consider the single-integrator MAS (1), under the dynamics (10), the closed-loop MAS becomes

$$\dot{x} = -\mathcal{B}q_\mu(\mathcal{B}^T x) = -\mathcal{B}\mathcal{B}^T x - \mathcal{B}\big(q_\mu(\mathcal{B}^T x) - \mathcal{B}^T x\big) = -\mathcal{L}x - \mathcal{B}e \quad (21)$$

where the quantization error $e = q_\mu(\mathcal{B}^T x) - \mathcal{B}^T x$ and $|e| \le \sqrt{M}\mu\Delta$. $-\mathcal{L}$ is negative semi-definite with a single zero eigenvalue. Therefore, in order to be compatible with dynamic uniform quantization strategy presented in Section 3, we will apply state transformation to (21) such that the reduced Laplacian matrix $\overline{\mathcal{L}}$ is positive definite.

Define that $\overline{x}_i = x_i - x_N, i = 1,2,\cdots, N - 1$, we have

$$\overline{x} = [\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_{N-1}]^T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -1 \end{bmatrix}_{(N-1)\times N} x = \mathcal{T}x \quad (22)$$

The transformation means that the Nth agent is chosen as the reference agent, so the relative information between the agents are preserved. Then (21) becomes

$$\dot{\overline{x}} = -\mathcal{T}\mathcal{L}\mathcal{T}^+\overline{x} - \mathcal{T}\mathcal{B}e = -\overline{\mathcal{L}}\overline{x} - \overline{\mathcal{B}}e \quad (23)$$

where $\mathcal{T}^+$ is the pseudo inverse of $\mathcal{T}$, $\overline{\mathcal{L}} = \mathcal{T}\mathcal{L}\mathcal{T}^+ \in R^{(N-1)\times(N-1)}$ and $\overline{\mathcal{B}} = \mathcal{T}\mathcal{B} \in R^{(N-1)\times M}$.

$$\mathcal{T}^+ = \begin{bmatrix} \frac{N-1}{N} & \frac{1}{N} & \frac{1}{N} \\ \frac{1}{N} & \ddots & \frac{1}{N} \\ \vdots & \ddots & \frac{N-1}{N} \\ \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix}_{N\times(N-1)},$$

$$\overline{\mathcal{L}} = \begin{bmatrix} l_{11} - l_{N1} & \cdots & l_{1(N-1)} - l_{N(N-1)} \\ \vdots & \cdots & \vdots \\ l_{(N-1)1} - l_{N1} & \cdots & l_{(N-1)(N-1)} - l_{N(N-1)} \end{bmatrix}_{(N-1)\times(N-1)}$$

$$\overline{\mathcal{B}} = \begin{bmatrix} b_{11} - b_{N1} & \cdots & b_{1M} - b_{NM} \\ \vdots & \cdots & \vdots \\ b_{(N-1)1} - b_{N1} & \cdots & b_{(N-1)M} - b_{NM} \end{bmatrix}_{(N-1)\times M}$$

Now the consensus problem of MAS (1) is transformed to a stability problem of the reduced MAS (23), i.e.,

$$\lim_{t\to\infty} \overline{x}(t) = 0 \tag{24}$$

Lemma 1. $\overline{\mathcal{L}}$ has the same eigenvalues as $\mathcal{L}$ but the zero eigenvalue.

Proof: Define the nonsingular matrix

$$\mathcal{S} = \begin{bmatrix} \mathcal{T} \\ \frac{1}{\sqrt{N}}\mathbf{1}^T \end{bmatrix} \tag{25}$$

It can be verified that

$$\mathcal{S}^{-1} = \begin{bmatrix} \mathcal{T}^+ & \frac{1}{\sqrt{N}}\mathbf{1} \end{bmatrix} \tag{26}$$

Clearly the eigenvalues of $\mathcal{S}\mathcal{L}\mathcal{S}^{-1}$ are same as those of $\mathcal{L}$. Incorporating the fact $\mathbf{1}^T\mathcal{L} = \mathbf{0}^T$ and $\mathcal{L}\mathbf{1} = \mathbf{0}$, we have

$$\mathcal{S}\mathcal{L}\mathcal{S}^{-1} = \begin{bmatrix} \mathcal{T}\mathcal{L}\mathcal{T}^+ & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \overline{\mathcal{L}} & 0 \\ 0 & 0 \end{bmatrix} \tag{27}$$

Since (27) is a block matrix, the eigenvalues of (27) are the solutions of

$$det(\lambda I - \mathcal{S}\mathcal{L}\mathcal{S}^{-1}) = \lambda det(\lambda I - \overline{\mathcal{L}}) = 0 \tag{28}$$

Therefore $\overline{\mathcal{L}}$ has the same eigenvalues of $\mathcal{L}$ but the zero eigenvalue.

From Lemma 1, we can conclude that $-\overline{\mathcal{L}}$ is negative definite, then the reduced MAS (23) is compatible with the linear stable system (16) and the dynamic quantization strategy can be applied in MAS. For the positive definite symmetric matrices $\overline{P}$ and $\overline{Q}$, define a Lyapunov function

$$\overline{V} = \overline{x}^T\overline{P}\overline{x} \tag{29}$$

Following the similar procedure in section 3, we have

$$\dot{\overline{V}} = -\overline{x}^T\overline{Q}\overline{x} - 2\,\overline{x}^T\overline{P}\overline{\mathcal{B}}e \le -\lambda_{min}(\overline{Q})|\overline{x}|^2 + 2|\overline{x}|\|\overline{P}\overline{\mathcal{B}}\||e| \le -|\overline{x}|\lambda_{min}(\overline{Q})(|\overline{x}| - \overline{\theta}\mu\Delta) \tag{30}$$

where $\overline{\theta} = 2\|\overline{P}\overline{\mathcal{B}}\|\sqrt{M}/\lambda_{min}(\overline{Q})$. In the region between the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$, $\overline{\theta}(1+\varepsilon)\mu\Delta \le |\overline{x}| \le \mu\mathcal{M}$ holds, therefore

$$\dot{\overline{V}} \le -\lambda_{min}(\overline{Q})\frac{\varepsilon}{1+\varepsilon}|\overline{x}|^2 \le -\frac{\lambda_{min}(\overline{Q})}{\lambda_{max}(\overline{P})}\frac{\varepsilon}{1+\varepsilon}\overline{V} = -\overline{\alpha}\overline{V} \tag{31}$$

Similar to the statement in section 3, the upper bound of the time that the system states travel from $\overline{\mathcal{R}}_1$ to $\overline{\mathcal{R}}_2$ can be estimated as

$$\overline{T} = \frac{1}{\overline{\alpha}} ln \frac{\lambda_{min}(\overline{P})\mathcal{M}^2}{\lambda_{max}(\overline{P})\overline{\theta}^2\Delta^2(1+\varepsilon)^2} \tag{32}$$

and the zooming rate

$$\overline{\Omega} = \sqrt{\lambda_{max}(\overline{P})}\overline{\theta}\Delta(1+\varepsilon) \Big/ \sqrt{\lambda_{min}(\overline{P})}\mathcal{M} \tag{33}$$

Remark 2: Since the initial states of MAS are known, for a dynamic quantizer with fixed $\mathcal{M}$ (which has to satisfy the inequality $\sqrt{\lambda_{min}(\overline{P})}\,\mathcal{M} \ge \sqrt{\lambda_{max}(\overline{P})}\overline{\theta}\Delta(1+\varepsilon)$), we can choose a suitable μ_0 such that the MAS starts in the ellipsoid $\mathcal{R}_1$. Therefore the original open-loop zooming-out stage is avoided.

4.2 Double-integrator agents

Consider the double-integrator MAS (2), under the controller (12), the closed-loop MAS becomes

$$\begin{aligned} \ddot{x} = -\mathcal{B}q_\mu(\mathcal{B}^Tx) - \mathcal{B}q_\mu(\mathcal{B}^Tv) \\ &= -\mathcal{B}\mathcal{B}^Tx - \mathcal{B}\mathcal{B}^Tv - \mathcal{B}(q_\mu(\mathcal{B}^Tx) - \mathcal{B}^Tx) \\ &- \mathcal{B}(q_\mu(\mathcal{B}^Tv) - \mathcal{B}^Tv) \\ &= -\mathcal{L}x - \mathcal{L}v - \mathcal{B}e_x - \mathcal{B}e_v \end{aligned} \tag{34}$$

or in matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{L} & -\mathcal{L} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ \mathcal{B} \end{bmatrix}(e_x + e_v) \text{ or } \dot{z} = \mathbb{L}z - \mathbb{B}e_z \tag{35}$$

where the quantization error $e_x = q_\mu(\mathcal{B}^Tx) - \mathcal{B}^Tx$, $e_v = q_\mu(\mathcal{B}^Tv) - \mathcal{B}^Tv$ and $|e_x|, |e_v| \le \sqrt{M}\mu\Delta$, $e_z = e_x + e_v$. As is well known, the MAS matrix $\mathbb{L}$ has two zero eigenvalues and all other eigenvalues with negative real parts. Therefore, in order to make the dynamic uniform quantization strategy presented in Section 3 feasible, we will apply state transformation to (35) such that the reduced Laplacian matrix $\overline{\mathbb{L}}$ is Hurwitz.

Define a new state transformation matrix $\mathbb{T}$ such that

$$\overline{z} = \mathbb{T}z = \begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T} \end{bmatrix} z \tag{36}$$

where $\mathcal{T}$ is defined in Section 4.1. Apply this transformation to the MAS (35), a reduced MAS is obtained:

$$\begin{bmatrix} \dot{\overline{x}} \\ \dot{\overline{v}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\overline{\mathcal{L}} & -\overline{\mathcal{L}} \end{bmatrix}\begin{bmatrix} \overline{x} \\ \overline{v} \end{bmatrix} - \begin{bmatrix} 0 \\ \overline{\mathcal{B}} \end{bmatrix}(\overline{e}_x + \overline{e}_v) \text{ or } \dot{\overline{z}} = \overline{\mathbb{L}}\overline{z} - \overline{\mathbb{B}}\overline{e}_z \tag{37}$$

Lemma 2. $\overline{\mathbb{L}}$ has the same eigenvalues as $\mathbb{L}$ but the two zero eigenvalues.

Proof: Define the nonsingular matrix

$$\mathbb{S} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{bmatrix} \tag{38}$$

where $\mathcal{S}$ is define in Section 4.1. The inverse of $\mathbb{S}$ is

$$\mathbb{S}^{-1} = \begin{bmatrix} \mathcal{S}^{-1} & 0 \\ 0 & \mathcal{S}^{-1} \end{bmatrix} \tag{39}$$

Clearly the eigenvalues of $\mathbb{S}\mathbb{L}\mathbb{S}^{-1}$ are same as those of $\mathbb{L}$.. Similar with the proof of Lemma 1, we have

$$\mathbb{S}\mathbb{L}\mathbb{S}^{-1} = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & \cdots & 0 \\ -\mathcal{T}\mathcal{L}\mathcal{T}^{+} & \vdots & -\mathcal{T}\mathcal{L}\mathcal{T}^{+} & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & \cdots & 0 \\ -\bar{\mathcal{L}} & \vdots & -\bar{\mathcal{L}} & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{40}$$

The eigenvalues of (39) are the solutions of the characteristic polynomial

$$det(\lambda I - \mathbb{S}\mathbb{L}\mathbb{S}^{-1}) = \lambda^2 det(\lambda I - \bar{\mathbb{L}}) = 0 \tag{41}$$

Therefore $\bar{\mathbb{L}}$ has the same eigenvalues of $\mathbb{L}$ but the two zero eigenvalue.

Then the reduced MAS (37) is compatible with the linear stable system (16) and the dynamic quantization strategy can be applied in MAS. Following the similar procedure in section 4.1, we can achieve the update interval

$$\bar{\bar{T}} = \frac{1}{\bar{\alpha}} ln \frac{\lambda_{min}(\bar{\bar{P}})\mathcal{M}^2}{\lambda_{max}(\bar{\bar{P}})\bar{\bar{\theta}}^2\Delta^2(1+\varepsilon)^2} \tag{42}$$

where $\bar{\bar{\theta}} = 4\|\bar{\bar{P}}\bar{\mathbb{B}}\|\sqrt{M}/\lambda_{min}(\bar{\bar{Q}})$, and the zooming rate

$$\bar{\bar{\Omega}} = \sqrt{\lambda_{max}(\bar{\bar{P}})}\bar{\bar{\theta}}\Delta(1+\varepsilon) \Big/ \sqrt{\lambda_{min}(\bar{\bar{P}})}\mathcal{M} \tag{43}$$

5. SIMULATION RESULTS

The single-integrator and double-integrator MAS with graph $\mathcal{G}$ shown in Fig. 1 are considered. Then the initial zooming variable μ_0 of the dynamic quantizer (6) with Δ= 0.5 is chosen such that the initial conditions after transformation lie in ellipsoid $\mathcal{R}_1$. The calculation of the zooming rate Ω and the update interval T is direct from (32), (33), (42), and (43).

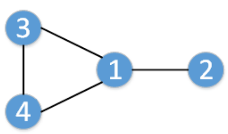


Fig. 1. Communication graph $\mathcal{G}$

5.1 Single-integrator MAS

According to the communication graph in Fig. 1, the incidence matrix, the Laplacian matrix and its eigenvalues

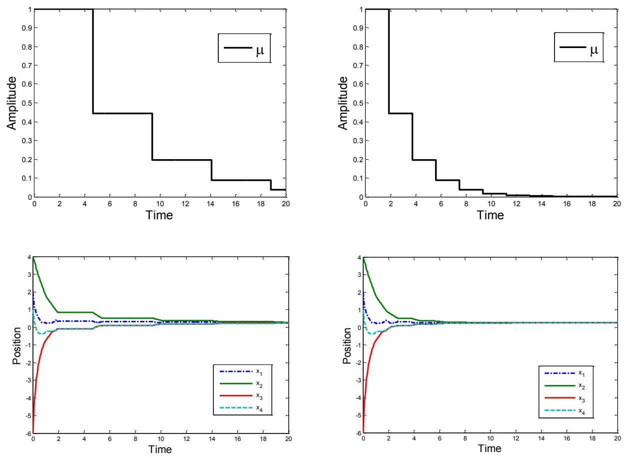
$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \bar{\mathcal{B}} = \mathcal{T}\mathcal{B} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3, \lambda_4 = 4$$

and the reduced Laplacian matrix and its eigenvalues:

$$\bar{\mathcal{L}} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \lambda_1 = 4, \lambda_2 = 1, \lambda_3 = 3$$

We observe that the eigenvalue $\lambda_1 = 0$ is eliminated and other eigenvalues are preserved after the transformation, and $-\bar{\mathcal{L}}$ is Hurwitz.



(a) (b)

Fig. 2. Quantized consensus of single-integrator MAS (a) original Liberzon's approach, (b) the improved approach

and $\lambda_{max}(\bar{P}) = 4.39$, $\lambda_{min}(\bar{P}) = 0.94$, $\lambda_{min}(\bar{Q}) = 7.65$.

$$\bar{\theta} = 2\|\bar{P}\bar{\mathcal{B}}\|\sqrt{M}/\lambda_{min}(\bar{Q}) = 3.07$$

Under first-order dynamics with initial condition $x(0) = [2\ 4\ -6\ 1]^{\mathrm{T}}$, and the quantization level $\mathcal{M} = 15$, which together with $\varepsilon = 1$, satisfies the inequality (19). In order to ensure the transformed initial condition lies in the ellipsoid $\mathcal{R}_1$, that is , $\bar{x}^T(0)\bar{P}\bar{x}(0) \le \lambda_{min}(\bar{P})\mathcal{M}^2\mu_0^2$, μ_0 must be larger than 0.87, so is chosen to be 1. The resulted zooming interval of the scheme proposed in this paper is $\bar{T} = \frac{1}{\bar{\alpha}} ln\left(\lambda_{min}(\bar{P})\mathcal{M}^2 / \left(\lambda_{max}(\bar{P})\bar{\theta}^2\Delta^2(1+\varepsilon)^2\right)\right) = 1.87$s, much smaller than 4.70 seconds, that of Liberzon's original approach. As a result, the convergence is faster.

5.2 Double-integrator MAS

The transformed system matrix

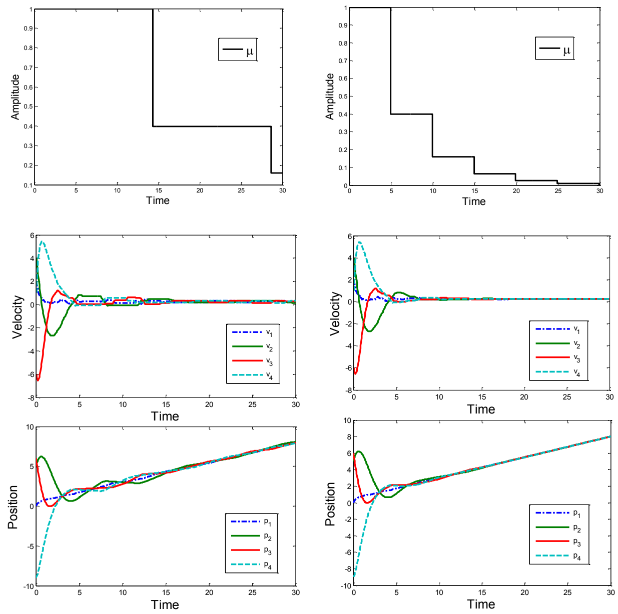
$$\bar{\mathbb{L}} = \begin{bmatrix} 0 & I \\ -\bar{\mathcal{L}} & -\bar{\mathcal{L}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 1 & 0 & -4 & 1 & 0 \\ 0 & -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & -3 & 0 & 0 & -3 \end{bmatrix}$$

Use LMI to calculate matrix $\bar{\bar{P}}$ and $\bar{\bar{Q}}$,

$$\lambda_{max}(\bar{\bar{P}}) = 7.13 , \lambda_{min}(\bar{\bar{P}}) = 0.48,$$

$$\bar{\bar{Q}} = 3.51 I^{6\times6},\ \lambda_{min}(\bar{\bar{Q}}) = 3.51$$

$$\bar{\bar{\theta}} = 13.25$$



(a) (b)

Fig. 3. Quantized consensus of double-integrator MAS (a) original Liberzon's approach, (b) the improved approach

Under second-order dynamics with initial condition $x(0) = [0\ 5\ 6 -9]^{\mathrm{T}}$, $v(0) = [2\ 4\ -6\ 1]^{\mathrm{T}}$ and $\mathcal{M} = 255$, which, when we have chosen $\varepsilon = 3$, is larger than its smallest available value 101.8. A value of μ_0 larger than 0.27 will satisfy the requirement of the initial values, for simplicity we still let $\mu_0 = 1$. Follow a process similar to that of first-order MAS, we obtain the zoom-in interval $\bar{\bar{T}} = 4.98$ seconds, much smaller than 14.32 seconds of the present method.

6. CONCLUSIONS

In this paper we propose a method to make single-integrator and double-integrator MASs compatible with Liberzon's dynamic quantizer design approach using system model transformation. Asymptotic average consensus is reached with finite quantization level and communication effort. We also improve the zooming frequency of the existing approach by closer estimating the convergence rate, resulting in faster consensus.

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REFERENCES

Baldan, G. and Zampieri, S. (2009). An efficient quantization algorithm for solving average-consensus problems. *Proceedings of European Control Conference in Hungary*, 761-766.

Brockett, R. W. and Liberzon, D. (2000). Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control,* **45(7),** 1279-1289.

Cao M., Morse, A. S., and Anderson, B.D.O. (2008). Reaching a consensus in a dynamically changing environment: a graphical approach. *SIAM Journal on Control and Optimization,* **47(2),** 575-600.

Carli, R., Bullo, F. and Zampieri, S. (2010). Quantized average consensus via dynamic coding/decoding schemes. *International Journal of Robust and Nonlinear Control,* **20(2),** 156-175.

Ceragioli, F., Persis, C.D. and Frasca, P. (2011). Discontinuities and hysteresis in quantized average consensus. *Automatica,* **47(9),** 1916-1928.

Cortés, J. (2008). Distributed algorithms for reaching consensus on general functions. *Automatica,* **44(3),** 726-737.

Dimarogonas, D.V. and Johansson, K.H. (2010). Stability analysis for multi-agent systems using the incidence matrix: quantized communication and formation control. *Automatica,* **46(4),** 695-700.

Dong R. and Geng Z. (2013). Design and analysis of quantizer for multi-agent systems with a limited rate of communication data. *Communications in Nonlinear Science and Numerical Simulation,* **18(2),** 282-290.

Young, G.F., L. Scardovi and N.E. Leonard (2010). Robustness of noisy consensus dynamics with directed communication. *American Control Conference,* 6312-6317.

Guo M. and D.V. Dimarogonas (2013). Consensus with quantized relative state measurements. *Automatica,* **49,** 2531-2537.

Hui Q. (2011). Quantised near-consensus via quantised communication links. *International Journal of Control,* **84(5),** 931-946.

Kashay, T. Basar, and R. Srikant (2007). Quantized consensus. *Automatica,* **43,** 1192-1203.

Li S., Du H., and Lin X. (2011). Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics. *Automatica,* **47(8),** 1706–1712.

Liberzon, D. (2003). Quantized feedback stabilization of linear systems. *Automatica,* **39(9),** 1543-1554.

Liu H., Cao M. and Persis, C.D. (2012). Quantization effects on synchronized motion of teams of mobile agents with second-order dynamics. *Systems & Control Letters,* **61(12),** 1157-1167.

Liu T., Jiang Z.P., and Hill, D.J. (2012). Small-gain based output-feedback controller design for a class of nonlinear systems with actuator dynamic quantization. *IEEE Transactions on Automatic Control,* **57(5),** 1195-1201

Ren W. and Cao Y. (2011). Distributed coordination of multi-agent networks: emergent problems, models, and issues. *Springer.*

Wang L. and Xiao F. (2010). Finite-time consensus problems for networks of dynamic agents. *IEEE Transactions on Automatic Control,* **55(4),** 950-955.