Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Inverted Pendulum Stabilization Via a Pyragas-Type Controller: Revisiting the Triple Zero Singularity

Islam Boussaada *,** Irinel-Constantin Morărescu *** Silviu-Iulian Niculescu *

* Laboratoire des Signaux et Systèmes (L2S) Supélec-CNRS-Université Paris Sud, 3 rue Joliot-Curie 91192 Gif-sur-Yvette cedex (France) (e-mail: islam.boussaada@gmail.com, Silviu.Niculescu@lss.supelec.fr)
** Laboratoire de Modélisation et Calcul Scientifique (LMCS) Institut Polytechnique des Sciences Avancées, 7 rue Maurice Grandcoing 94200 Ivry-sur-Seine (France)
*** Université de Lorraine, CNRS, CRAN, UMR 7039, 2 avenue de la forêt de Haye, 54516 Vandoeuvre-lès-Nancy Cedex (France) (e-mail: constantin.morarescu@univ-lorraine.fr)

Abstract: The use of Pyragas-Type controller proved interest in the stabilization of unstable periodic orbits. The stabilization problem of a balancing inverted pendulum on an horizontally moving cart by the use of such a controller is considered. The main objective of the paper is to propose delayed control law containing only proportional gains able to stabilize the inverted pendulum by avoiding the existence of a triple zero eigenvalue at the origin. We analyze the center dynamics described by a three dimensional system of ordinary differential equations (ODEs) with a codimension-three triple zero bifurcation. Furthermore, the stability analysis of the corresponding linear time invariant system with two delays describing the behavior around the equilibrium is also proposed. This analysis is done in order to characterize the possible local bifurcations. Finally, the proposed control scheme is numerically illustrated and discussed.
Time-Delay, Stability, Delayed Feedback, Control, Center Manifold Theorem, Normal Forms, Local Bifurcation, Inverted Pendulum.

1. INTRODUCTION

In this paper, a well known and classical engineering problem which is the stabilization of a balancing inverted pendulum on an horizontally moving cart is considered, see for instance Atay [1999], Eker and Aström [1996], Landry et al. [2005], Shiriaev et al. [2000], Sieber and Krauskopf [2004]. This typical unstable nonlinear system is often considered as a benchmark to discuss new ideas in the field of nonlinear control and in the general theory of dynamical systems. This is certainly due to the richness of its dynamics even if the structure of the physical system remains relatively simple. To cite only one application, one thinks to the modeling of the human balance control, see for instance, Campbell [2009]. It is well known that the inverted pendulum has two equilibria, one is stable and it corresponds to the pendulum pointing downward while the other one is unstable and it corresponds to the upward position of the pendulum. Therefore, the unstable equilibrium of the system can be maintained only in the presence of an appropriate control input.

Several approaches are developed in the open literature to overcome the challenge of swinging up and maintaining the pendulum in the upward position whereas, we are focusing on the design of infinite dimensional controllers taking into account only the delayed state. The starting idea of the present paper is a result proposed in Niculescu and Michiels [2004]. As proven there, a chain of n integrators can be stabilized using n distinct delay blocks, where a delay block is described by two parameters: "gain" and "delay". The interest of considering control laws of the form $\sum_{k=1}^{m} \gamma_k \, y(t-\tau_k)$ lies in the simplicity of the controller as well as in its practical implementation facility. The performances of delayed controllers to overcome the challenge of stabilizing the inverted pendulum are emphasized in the following recent works Atay [1999], Sieber and Krauskopf [2004], Landry et al. [2005].

In Atay [1999], the author pointed out that a simple position feedback is not sufficient to obtain satisfactory performance from the control system, and one needs additional knowledge such as the rate of change of the position. Next, a proportional minus delay controller (PMD) is proposed to obtain asymptotic stability in second-order undamped systems modeling an inverted pendulum Atay [1999]. This strategy shows that the effect of the delay is similar to the derivative feedback in modifying the behavior of the system. It is worth noting here that replacing the derivative with its numerical approximation will not allow to directly apply the results in Sieber and Krauskopf [2004]. Indeed, the behavior of a system (even a linear one) may be different from the behavior of its approximation. In Morărescu et al. [2013], it has been shown that using a polynomial function $(1 - s\frac{\tau}{n})^n$ of arbitrary degree n to approximate an exponential $e^{-s\tau}$ allows finding stabilizing controller gains for the approximated system even they do not necessarily exist for the original one.

To the best of the authors' knowledge, PMD controllers were first introduced by I.H. Suh & Z. Bien in Suh and Bien [1979] where it is shown that the conventional P-controller equipped with an appropriate time-delay performs an averaged derivative action and thus can replace the PD-controller, showing quick responses to input changes but being insensitive to high-frequency noise. Recent idea of J. Sieber and & B. Krauskopf for stabilizing the inverted pendulum by designing a delayed PD controller has been proposed in Sieber and Krauskopf [2004]. Moreover, they established a linearized stability analysis allowing to characterize all the possible local bifurcations additionally to the nonlinear analysis. This analysis involves the center manifold theory and normal forms which are known to be powerful tools for the local qualitative study of the dynamics. The study emphasized the existence of a codimension-three triple zero bifurcation. It is also shown that the stabilization of the inverted pendulum in its upright position cannot be achieved by a PD controller when the delay exceeds some critical value τ_c.

Finally, in Landry et al. [2005], S.A. Campbell et. al. considered a proportional controller to locally maintain the pendulum in the upright position. The authors shown that when this proportional is delayed and the time-delay sampling is not too large, the controller still locally stabilizes the system. Among other results, they show the loss of stability when the delay exceeds a critical value and a supercritical Andronov-Hopf Bifurcation Kuznetsov [1998] occurs generating stable limit cycles.

The main contribution of this paper consists in introducing a Pyragas-type controller allowing the stabilization of the inverted pendulum without the use of derivative measurements. Usually, the use of PD controller needs the knowledge of the velocity history but in general we are only able to have approximate measurements due to technological constraints. In absence of measurement of the derivative, a classical idea is to use an observer to reconstruct the state, but this might degrade the performance to some extent Atay [1999] and it is, in general, computationally involved for delay systems. To avoid such degradation and since the position measurement can be easily obtained by sensors, we restrict ourselves to delayed proportional gains.

Our analysis agrees with the above claim of F.M. Atay Atay [1999] but extends it by proving that the knowledge of the delayed derivative gain considered in the delayed PD controller Sieber and Krauskopf [2004] can be replaced by the knowledge of two delayed state positions. Our main idea can be summarized as follows, we use MDP controllers to reach the configuration of multiple-zero eigenvalue by setting the parameters (delays and gains) values, then we identify the appropriate parameters moving direction allowing to stabilize the inverted pendulum avoiding this singularity.

The use of proposed control law prove that the cubic normal form of the solutions evolution on the center manifold is exactly the same as the one for the delayed PD considered in Sieber and Krauskopf [2004]. In some sense, this can be seen as a discretization of the feedback state derivative. By the way, such a constructive approach has been adopted into different context in the controller design developed in Niculescu and Michiels [2004], Kharitonov et al. [2005]. The stability analysis of the delayed linearized system employs the geometrical interpretation of the corresponding characteristic equation proposed in Morărescu et al. [2007b], Gu et al. [2005], Morărescu et al. [2007a]. An alternative technique for studying the stability of this class of systems is proposed in Sipahi and Olgac [2005]. For more details on the existing techniques, the reader is referred to Insperger and Stepan [2011].

The remaining part of our paper is organized as follows: First, the model of the inverted pendulum on a cart is introduced and some mathematical notions used in the analysis are given. Next, Pyragas-type control strategy is presented and analyzed. The analysis includes the normal form of the cental dynamics and the linear stability analysis pointing out the Andronov-Hopf Bifurcation. A conclusion ends the paper.

2. SETTINGS AND USEFUL NOTIONS

2.1 *Friction free model of an Inverted Pendulum on a Cart*

Consider the friction free model presented in Sieber and Krauskopf [2004] by adopting the same notations in the sequel. Denote the mass of the cart M, the mass of the pendulum m and let the relative mass be $\epsilon = m/(m+M)$.

In the dimensionless form, by neglecting the frictions, the dynamics of the inverted pendulum on a cart in figure 1 is governed by the following ODE, see also Sieber and Krauskopf [2005]:

$$\left(1 - \frac{3\epsilon}{4}\cos^2(\theta)\right)\ddot{\theta} + \frac{3\epsilon}{8}\dot{\theta}^2\sin(2\theta) - \sin(\theta) + D\cos(\theta) = 0, \tag{1}$$

where D represents the control law that is the horizontal driving force.
In Sieber and Krauskopf [2004], the authors consider $D = a\,\theta(t-\tau) + b\,\dot{\theta}(t-\tau)$ and prove that the truncated cubic central dynamics reduces to:

$$\dot{u} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & \beta & \gamma \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ u_1^3 \end{bmatrix},$$

where α, β and γ are small parameters, showing that the triple zero singularity can be avoided.

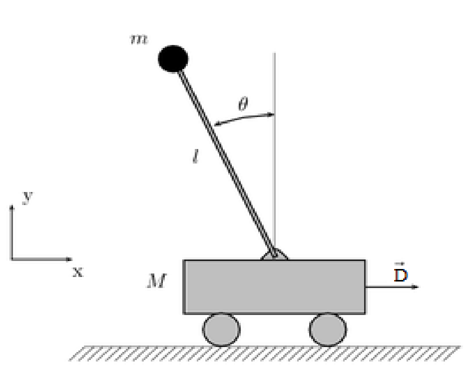

Fig. 1. Inverted Pendulum on a cart

In the forthcoming section, the horizontal control force will be designed just as position feedback as suggested in Pyragas [1992] in the context of stabilizing an unstable limit cycle. In the sequel, it is explicitly proven that the triple zero singularity can be avoided using such a controller.

2.2 Space decomposition for time-delay systems

Consider the general discrete delayed autonomous first-order nonlinear system where its linear and nonlinear quantities are separated as follows:

$$\frac{d}{dt}x(t) = \sum_{k=0}^{n} A_k\, x(t-\tau_k) + \mathcal{F}(x(t),\ldots,x(t-\tau_n)), \tag{2}$$

where A_i are $n \times n$ real valued matrix and the delays τ_k are ordered such that $\tau_i < \tau_j$ when $i < j$ and let $\tau_n = r$ and $\tau_i \geq 0$.

The latter system can be written as:

$$\frac{d}{dt}x = \mathcal{L}x_t + \mathcal{F}(x_t), \tag{3}$$

where $x_t \in C_{r,n} = C([-r,0],\mathbb{R}^n)$, $x_t(\theta) = x(t+\theta)$ denotes the translation operator and $\mathcal{L}$ is a bounded linear operator such that $\mathcal{L}\phi = \sum_{k=0}^{n} A_k\, \phi(-\tau_k)$ and $\mathcal{F}$ is assumed to be a sufficiently smooth function mapping $C_{r,n}$ into $\mathbb{R}^n$ with $\mathcal{F}(0) = D\mathcal{F}(0) = 0$. The linear operator $\mathcal{L}$ can be written in the integral form by $\mathcal{L}\phi = \int_{-r}^{0} d\eta(\theta)\phi(\theta)$ where η is a real valued $n \times n$ matrix.

The linearization of (3) is simply given by

$$\frac{d}{dt}x = \mathcal{L}x_t, \tag{4}$$

for which the solution is given by operator $\mathcal{T}(t)$ defined by $\mathcal{T}(t)(\phi) = x_t(.,\phi)$ such that $x_t(.,\phi)(\theta) = x(t+\theta,\phi)$ for $\theta \in [-r,0]$ is a strongly continuous semigroup with the infinitesimal generator given by $\mathcal{A} = \frac{d\phi}{d\theta}$ with the domain

$$\mathcal{D}om(\mathcal{A}) = \{\phi \in C_{r,n} : \frac{d\phi}{d\theta} \in C_{r,n}, \frac{d\phi}{d\theta} = \mathcal{L}\phi\}.$$

It is also known that $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ and the spectrum of $\mathcal{A}$ consists of complex values $\lambda \in \mathbb{C}$ satisfying the characteristic equation $p(\lambda) = 0$, (see Michiels and Niculescu [2007] for further details).

In the spirit of Diekmann et al. [1995], let us denote by $\mathcal{M}_\lambda$ the eigenspace associated with $\lambda \in \sigma(\mathcal{A})$. We define $C^*_{r,n} = C([-r,0],\mathbb{R}^{n*})$ where $\mathbb{R}^{n*}$ is the space of n-dimensional row vectors and consider the bilinear form on $C^*_{r,n} \times C_{r,n}$ as proposed in Hale and Lunel [1993]:

$$(\psi,\phi) = \phi(0)\,\psi(0) + \int_{-r}^{0}\int_{0}^{\theta} \psi(\tau-\theta)d\eta(\theta)\phi(\tau)d\tau.$$

Let $\mathcal{A}^T$ be the transposed operator of $\mathcal{A}$, i.e., $(\psi,\mathcal{A}\phi) = (\mathcal{A}^T\psi,\phi)$. The following result presented in Hale and Lunel [1993] permits the decomposition of the space $C_{r,n}$:

Theorem 1. (Banach space decomposition). Let Λ be a nonempty finite set of eigenvalues of $\mathcal{A}$ and let $P = span\{\mathcal{M}_\lambda(\mathcal{A}),\, \lambda \in \Lambda\}$ and $P^T = span\{\mathcal{M}_\lambda(\mathcal{A}^T),\, \lambda \in \Lambda\}$. Then P is invariant under $\mathcal{T}(t), t \geq 0$ and there exists a space $\mathcal{Q}$, also invariant under $\mathcal{T}(t)$ such that $C_{r,n} = P \bigoplus Q$. Furthermore, if $\Phi = (\phi_1,\ldots,\phi_m)$ forms a basis of P, $\Psi = col(\psi_1,\ldots,\psi_m)$ is a basis of P^T in $C^*_{r,n}$ such that $(\Phi,\Psi) = Id$, then

$$\begin{aligned} Q &= \{\phi \in C_{r,n} \setminus (\Psi,\phi) = 0\} \text{ and} \\ P &= \{\phi \in C_{r,n} \setminus \exists b \in \mathbb{R}^m : \phi = \Phi b\}. \end{aligned} \tag{5}$$

Also, $\mathcal{T}(t)\Phi = \Phi\, e^{Bt}$, where B is an $m \times m$ matrix such that $\sigma(B) = \Lambda$.

Consider the extension of the space $C_{r,n}$ that contains continuous functions on $[-r,0)$ with possible jump discontinuity at 0, we denote this space BC. A given function $\xi \in BC$ can be written as $\xi = \varphi + X_0\,\alpha$, where $\varphi \in C_{r,n}$, $\alpha \in \mathbb{R}^n$ and X_0 is defined by $X_0(\theta) = 0$ for $-r \leq \theta < 0$ and $X_0(0) = Id_{n\times n}$. Then Hale-Verduyn Lunel bilinear form Hale and Lunel [1993] can be extended to the space $C^*_{r,n} \times BC$ by $(\psi, X_0) = \psi(0)$ and the infinitesimal generator $\mathcal{A}$ extends to an operator $\tilde{\mathcal{A}}$ (defined in C^1) onto the space BC as follows:

$$\tilde{\mathcal{A}}\phi = \mathcal{A}\phi + X_0[\mathcal{L}\phi - \phi']. \tag{6}$$

Under the above consideration one can write equation (3) as an abstract ODE:

$$\dot{x}_t = \tilde{\mathcal{A}}x_t + X_0\mathcal{F}(x_t). \tag{7}$$

Due to the projection $\Pi : BC \to P$ defined by $\Pi(\varphi + X_0\alpha) = \Phi[(\Psi,\varphi) + \Psi(0)\alpha]$ and the state decomposition such that $x_t = \Phi y(t) + z_t$ where $y(t) \in \mathbb{R}^m$ and $z_t \in Q$, then the equation (3) can be splitted into two equations. Our interest lies essentially in the evolution equation for the finite dimensional part of the space i.e. the first equation of the following system:

$$\begin{cases} \dot{y} = By + \Psi(0)\mathcal{F}(\Phi y + z), \\ \dot{z} = \tilde{A}_Q z + (I-\pi)\mathcal{F}(\Phi y + z). \end{cases} \tag{8}$$

For more details and insights, see for instance, Hale and Lunel [1993], Faria and Magalhães [1995]. Assume now that $\mathcal{F}$ depends on some parameter p, and if denote the semiflow generated by (8) as $\mathcal{S}(t,y,z,p)$, then $\mathcal{S}$ is equivalent to the semiflow generated by (3):

Theorem 2. (Properties of the Center Manifold). Let $k > 0$ and $\mathcal{U}_y \times \mathcal{U}_z \times \mathcal{U}_p$ be a small neighborhood of $(0,0,p_0) \in \mathbb{R}^n \times Q \times \mathbb{R}^m$. There exists a graph $\omega : \mathcal{U}_y \times \mathcal{U}_p \to Q$ of smoothness C^k such that the following statements hold.

(1) (Invariance) The manifold $\{(y,z) \in \mathcal{U}_y \times Q : z = \omega(y,p)\}$ is invariant with respect to $\mathcal{S}$ relative to $\mathcal{U}_y \times \mathcal{U}_z$.
(2) (Exponential attraction) Let (y,z) s. t. $\mathcal{S}(y,z,p) \in \mathcal{U}_y \times \mathcal{U}_z$ $\forall t \geq 0$. Then there exists $\tilde{y}$ and $\tilde{t} \geq 0$ such that $\|\mathcal{S}(t+\tilde{t},y,z,p) - \mathcal{S}(t,\tilde{y},\omega(\tilde{y}),p)\| \leq Ke^{\frac{-t_0}{2}}$ for all $t > 0$.

3. MAIN RESULT: PYRAGAS-TYPE CONTROLLER

By Pyragas controller, we understand a controller of the form

$$u(t) := \alpha(\theta(t) - \theta(t-\tau)).$$

It is worth mentioning that such a controller proved interest in the stabilization of unstable periodic orbits, see for instance, Pyragas [1992], Pyragas. Furthermore, in Laplace domain, the corresponding characteristic function includes an additional root at the origin.

In this section we consider the control law
$D(t) = a\ (\theta(t) - \theta(t-\tau_1)) + b\ (\theta(t) - \theta(t-\tau_2)) + c\,\theta(t)$,
and consider the relative mass $\epsilon = 3/4$. Equation (1) can be written as a DDE of the form:

$$\dot{x} = f(x(t), x(t-\tau_1), x(t-\tau_2), \lambda), \tag{9}$$

where $x = (x_1, x_2)^\top = \left(\theta(t), \dot{\theta}(t)\right)^\top$ and $\lambda = (a, b, \tau_1, \tau_2)$. The right hand side $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^2$ is given by:

$$\begin{cases} f_1(.\lambda) = x_2, \\ f_2(.,\lambda) = \dfrac{-\frac{9}{32}\sin(2\,x_1)\,{x_2}^2 + \sin(x_1)}{1 - \frac{9}{16}\,(\cos(x_1))^2} \\ \qquad - \dfrac{\cos(x_1)\,(a\,(x_1 - y_1) + b\,(x_1 - z_1) + c x_1)}{1 - \frac{9}{16}\,(\cos(x_1))^2}, \end{cases} \tag{10}$$

where $y = (y_1, y_2)^\top = (\theta(t-\tau_1), \dot{\theta}(t-\tau_1))^\top$ and $z = (z_1, z_2)^\top = (\theta(t-\tau_2), \dot{\theta}(t-\tau_2))^\top$.

3.1 Linear Stability Analysis

It is always possible to normalize one of the delays by a simple scaling of time, let $\tau_1 = 1$. The linearization of f with respect to its three arguments, x, y and z at the origin is given by

$$\partial_1 f(0,\lambda) = \begin{bmatrix} 0 & 1 \\ \dfrac{16\,(1 - a - b - c)}{7} & 0 \end{bmatrix},$$

$$\partial_2 f(0,\lambda) = \begin{bmatrix} 0 & 0 \\ \dfrac{16}{7}\,a & 0 \end{bmatrix},\ \partial_3 f(0,\lambda) = \begin{bmatrix} 0 & 0 \\ \dfrac{16}{7}\,b & 0 \end{bmatrix},$$

where $f(0,\lambda)$ designate $f(0,0,0,\lambda)$. Then, the characteristic function is given by

$$\Delta(\lambda) = \lambda^2 + \frac{16\,(a+b+c-1)}{7} - \frac{16}{7}\,\mathrm{e}^{-\lambda\,\tau_1} a - \frac{16}{7}\,\mathrm{e}^{-\lambda\,\tau_2} b. \tag{11}$$

The stability analysis follows from Proposition 3.1 in Gu et al. [2005].

Remark 3. The crossing set Ω associated to the system described by the characteristic equation (11) consists of a finite union of intervals. Moreover, when $c \neq 1$, all the intervals are closed while for $c = 1$ all the intervals are closed excepting the first one which has the left end equals zero i.e. a value that does not belong to the crossing set. The stability crossing curves are either open ended or closed as explained in the classification proposed by Gu et al. [2005] (see figure 2 for the case $c = 1$ which is relevant for this study).

In the (a, b, c) parameter space instead of crossing curves we have stability crossing surfaces referred to as $\mathbf{A}$. The corresponding crossing set is again dented by Γ. Thus, $\mathbf{A}$ is the set of (a, b, c) such that $\Delta(z)$ has imaginary solutions while Γ consist of those frequencies ω such that there exists a parameter triple (a, b, c) such that $\Delta(j\omega, a, b, c) = 0$. The stability analysis in (a, b, c) parameter space is summarized as follows:

Proposition 4. The crossing set Γ consist of all frequencies satisfying

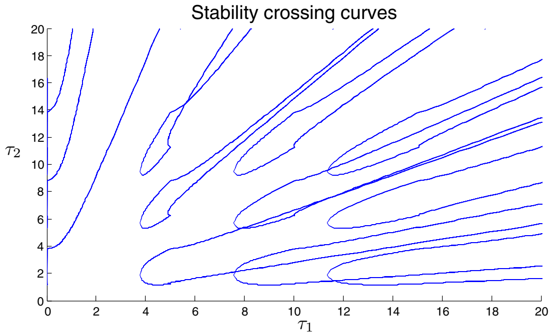

Fig. 2. The crossing curves associated to the first interval $(0, \omega^r]$ for $a = \frac{6}{5}$, $b = \frac{3}{5}$ and $c = 1$. The curves $\mathbf{T}^-_{u,v}$ and $\mathbf{T}^+_{u,v+1}$ are connected at ω^r.

$$0 < \omega < \left|\frac{\pi}{\tau_1 - \tau_2}\right|,$$

and the crossing surfaces are defined $\forall \omega \in \Gamma$ by:

$$\begin{cases} a \in \mathbb{R} \\ b = -\dfrac{\sin(\omega\tau_1)}{\sin(\omega\tau_2)}a \\ c = 1 + \dfrac{7}{16}\omega^2 - a - b + a\cos(\omega\tau_1) + b\cos(\omega\tau_2) \end{cases}.$$

It is easy to see that as ω approaches 0 the parameter c approaches 1 (for illustration see figure 3).

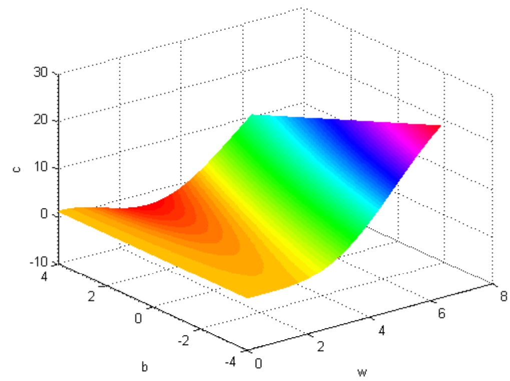

Fig. 3. The stability crossing surface in the (ω, b, c) parameter space for $\tau_1 = 1$, $\tau_2 = \frac{1}{2}$ and $a = 2$

The function Δ with arbitrary gain c has a purely imaginary root iw if:

$$\begin{cases} a = \dfrac{7\,\sin(w\tau_2)\,w^2 + 16\,\sin(w\tau_2) - 16\,\sin(w\tau_2)\,c}{16\,\sin(w\tau_2) - 16\,\sin(w\tau_2 - w) - 16\,\sin(w)}, \\ b = \dfrac{-7\,\sin(w)\,w^2 - 16\,\sin(w) + 16\,\sin(w)\,c}{16\,\sin(w\tau_2) - 16\,\sin(w\tau_2 - w) - 16\,\sin(w)}. \end{cases} \tag{12}$$

We note that substituting a and b into the second derivative of Δ and by setting $w = 0$ one gets the relation $a\,\tau_1 = -b\,\tau_2$ guarateeing a zero eigenvalue of algebraic

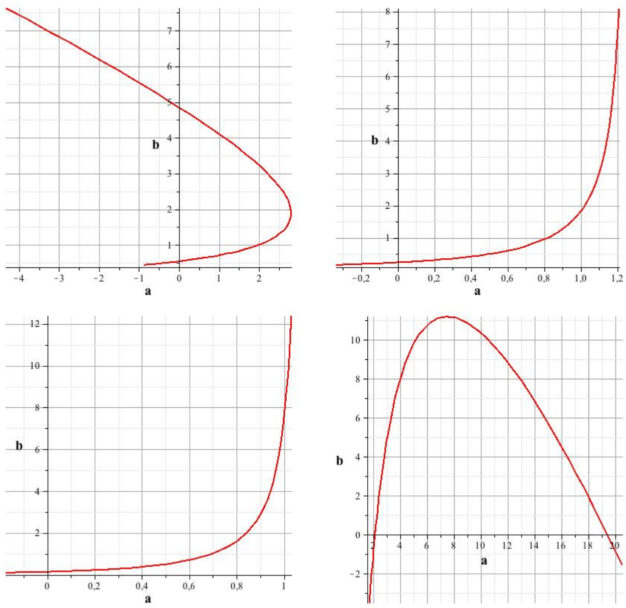

Fig. 4. Hopf curves for (10)-(9) in the gains plan (a,b) with $c = \tau_1 = 1$ and τ_2 such that (top left) $\tau_2 = 2$ (top right) $\tau_2 = 3$ (bottom left) $\tau_2 = 4$ (bottom right) $\tau_2 = \frac{1}{2}$

multiplicity 2. Finally, a zero eigenvalue of algebraic multiplicity 3 is given by

$$a = \frac{7}{8}\,\frac{1}{(\tau_1 - \tau_2)\,\tau_1},\ b = -\frac{7}{8}\,\frac{1}{\tau_2\,(\tau_1 - \tau_2)},\ c = 1.$$

This shows that the configuration of a triple zero eigenvalue can not be achieved when one of the gains a or b vanishes, in other words, two Pyragas controllers are necessary to reach this multiplicity.

Equation (12) defines the curve of Hopf Bifurcation in the (a, b) plane in figure 4 for $c = 1$ and several values of the delay τ_2, thus there are coexistence of Pitchfork and Hopf bifurcation on this curves.

Note also, that when $\tau_1\,\tau_2 \neq 0$ a zero eigenvalue of multiplicity 4 is not possible since the fourth derivative at zero gives $2\,(\tau_1 + \tau_2)$.

3.2 Central Dynamics

We show that a triple zero eigenvalue occurs for arbitrary value of the delay τ_2. Then let us restrict to the case of a fixed value for the delay $\tau_2 = 2$, the parameter point $\lambda_0 = (a_0, b_0, c_0, \tau_1{}^*, \tau_2{}^*) = (-\frac{7}{8}, \frac{7}{16}, 1, 1, 2)$ characterize a triple zero eigenvalue at the origin. System (10)-(9) can be normalized by setting $\tau_1 = 1$ leading to:

$$\begin{cases} f_1(.,\lambda) = x_2, \\ f_2(.,\lambda) = \dfrac{-\frac{9}{32}\sin(2\,x_1)\,x_2{}^2 + \tau_1{}^2\,\sin(x_1)}{1 - \frac{9}{16}\,(\cos(x_1))^2} \\ \qquad - \dfrac{\tau_1{}^2\,\cos(x_1)\,(a\,(x_1 - y_1) + b\,(x_1 - y_1) + c\,x_1)}{1 - \frac{9}{16}\,(\cos(x_1))^2}. \end{cases} \tag{13}$$

Let X be the Banach space $\mathbb{R}^2 \times C([-1, 0], \mathbb{R}^2)$. Consider $D(H) := \{(y, \tilde{y}) \in \mathbb{R}^2 \times C^1([-1, 0], \mathbb{R}^2) \,:\, \tilde{y}(0) = y\} \subset X$, and define the linear operator $H\begin{bmatrix} y \\ \tilde{y} \end{bmatrix}$

$$= \begin{bmatrix} \partial_1 f(0,\lambda)\tilde{y}(0) + \partial_2 f(0,\lambda)\tilde{y}(-1) + \partial_3 f(0,\lambda)\tilde{y}(-2) \\ \partial_s \tilde{y} \end{bmatrix},$$

where the spatial variable in $C^1([-2, 0], \mathbb{R}^2)$ is denoted by s. Let g be the nonlinear part of f i.e.

$$g\left(\begin{bmatrix} y \\ \tilde{y} \end{bmatrix}, \lambda\right) = \begin{bmatrix} g_0(\tilde{y}(0), \tilde{y}(-1), \tilde{y}(-2), \lambda) \\ 0 \end{bmatrix}, \tag{14}$$

where:

$$g_0(\tilde{y}(0), \tilde{y}(-1), \tilde{y}(-2), \lambda) = f(\tilde{y}(0), \tilde{y}(-1), \tilde{y}(-2), \lambda) -$$
$$\Big(\partial_1 f(0,\lambda)\tilde{y}(0) + \partial_2 f(0,\lambda)\tilde{y}(-1) + \partial_3 f(0,\lambda)\tilde{y}(-2)\Big).$$

System (9)-(10) is equivalent to the autonomous evolution equation:

$$\dot{x} = Hx + g(x, \lambda). \tag{15}$$

The decomposition of the Banach space $X = P \bigoplus Q$ such that P is the H-invariant generalized eigenspace associated to triple zero sigularity which is isomorphic to $\mathbb{R}^3$ and Q also H-invariant of infinite dimension. Next, we compute Φ a basis of P satisfying $H\Phi = \Phi J$ where

$$\Phi(s) = [\phi_1, \phi_2, \phi_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & s & \frac{s^2}{2} + 1 \\ 0 & 1 & s \end{bmatrix},\ \text{and } J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We compute the invariant spectral projection $\mathcal{P} : X \to P$ such that $\mathcal{P}x = Res_{z=0}(z\,I - H)^{-1}$. In other words, $\mathcal{P}x = l_1(x)\phi_1 + l_2(x)\phi_2 + l_3(x)\phi_3$ where:

$$\begin{cases} l_1(x) = \frac{7}{12}\,\tilde{y}_1(0) - \frac{131}{144}\,\tilde{y}_2(0) + \frac{131}{72}\displaystyle\int_0^1 \tilde{y}_1(t-1)\,dt \\ \quad - \frac{131}{144}\displaystyle\int_0^2 \tilde{y}_1(t-2)\,dt + 7/6\displaystyle\int_0^1 t\tilde{y}_1(t-1)\,dt \\ \quad - \frac{7}{12}\displaystyle\int_0^2 t\tilde{y}_1(t-2)\,dt - \displaystyle\int_0^1 t^2\tilde{y}_1(t-1)\,dt \\ \quad + 1/2\displaystyle\int_0^2 t^2\tilde{y}_1(t-2)\,dt, \\ l_2(x) = \tilde{y}_1(0) + \frac{7}{12}\,\tilde{y}_2(0) - 7/6\displaystyle\int_0^1 \tilde{y}_1(t-1)\,dt \\ \quad + \frac{7}{12}\displaystyle\int_0^2 \tilde{y}_1(t-2)\,dt + 2\displaystyle\int_0^1 t\tilde{y}_1(t-1)\,dt \\ \quad - \displaystyle\int_0^2 t\tilde{y}_1(t-2)\,dt, \\ l_3(x) = \tilde{y}_2(0) - 2\displaystyle\int_0^1 \tilde{y}_1(t-1)\,dt + \displaystyle\int_0^2 \tilde{y}_1(t-2)\,dt, \end{cases}$$

which allows decomposing equation (15) to:

$$\begin{cases} \dot{v} \;= Jv + \Psi(0)g_0 \\ \dot{\tilde{w}}_0 = \partial_1 f\tilde{w}_0 + \partial_2 f\tilde{w}(-1) + \partial_3 f\tilde{w}(-2) \\ \quad + (I - \Phi(0)\Psi(0))g_0 \\ \dot{\tilde{w}} = \partial_s\tilde{w} - \tilde{\Phi}\Psi(0)g_0, \end{cases}$$

g_0 designate $g_0(\tilde{\Phi}(0)v + \tilde{w}_0, \tilde{\Phi}(-1)v + \tilde{w}(-1), \tilde{\Phi}(-2)v + \tilde{w}(-2))$ and $\tilde{w}_0 = \tilde{w}(0)$ and

$$\Psi(0) = \begin{bmatrix} \frac{7}{12} & -\frac{131}{144} \\ 1 & \frac{7}{12} \\ 0 & 1 \end{bmatrix}, \; \tilde{\Phi}(s) = \begin{bmatrix} 1 & s & 1+\frac{s^2}{2} \\ 0 & 1 & s \end{bmatrix}.$$

By using the Center Manifold Theorem presented in the previous section and the following changes of coordinates:

$$a = -\frac{7}{8} + \frac{7}{8}\,\alpha\, r^2, b = \frac{7}{16} + \frac{7}{16}\,\beta\, r^2, c = 1 + \frac{7}{16}\,\sigma\, r^2,$$

$$\tau_1 = 1 + \frac{1}{2}\,\delta\, r^4, v_1 = r^3 u_1, v_2 = r^5 u_2, v_3 = r^7 u_3, w = r^3 q,$$

we arrive to the expansion of the graph (the center manifold) in power of r which is of order 6 i.e. $q(u, \mu, r) = r^6 q_6(u, \mu, r)$ where $\mu = (\alpha, \beta, \gamma)$ and the expression of the flow on the local center manifold:

$$\dot{u} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha' & \beta' & \gamma' \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ u_1^3 \end{bmatrix} + r^2 R(u, \mu, r),$$

where $\alpha' = \delta\,(\sigma + \beta)$, $\beta' = 3\delta$, $\gamma' = \alpha - \sigma - 3\beta$ and R is the remainder, a smooth function in u, μ and r.

4. CONCLUDING REMARKS

The use of multiple delay blocks was suggested in Niculescu and Michiels [2004], Kharitonov et al. [2005] for stabilizing chains of integrators. In this paper, we presented a configuration of such multi-delayed-proportional controller allowing to stabilize the inverted pendulum by avoiding a triple zero eigenvalue singularity, a singularity already identified in Sieber and Krauskopf [2004] through the use of a delayed PD controller. These results agree with the claim of Atay [1999], that is, the effect of the delay is similar to derivative feedback in modifying the behavior of the system, but extend it to the nonlinear analysis by proving that the cubic truncated normal form of the center manifold dynamics is the same as the one obtained by using a delayed PD regulator. In an extended version of the present investigation Boussaada et al. [2014], another variant of multi-delayed-proportional controller is considered and control loop latency is discussed.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for careful reading of the manuscript and for their valuable comments.

REFERENCES

F. M. Atay. Balancing the inverted pendulum using position feedback. *Appl. Math. Lett.*, 12(5):51–56, 1999.

I. Boussaada, I.-C. Morărescu, and S-I. Niculescu. Inverted pendulum stabilization: Characterization of codimension-three triple zero bifurcation via multiple delayed proportional gains. *Submitted*, pages 1–16, 2014.

S.A. Campbell. Calculating centre manifolds for delay differential equations using maple. In *In Delay Differential Equations: Recent Advances and New Directions.* Springer-Verlag, 2009.

O. Diekmann, S.A. Van Gils, S.M. Verduyn Lunel, and H.O. Walther. *Delay equations*, volume 110 of *Applied Mathematical Sciences, Functional, complex, and nonlinear analysis.* Springer-Verlag, New York, 1995.

J. Eker and K.J. Aström. A nonlinear observer for the inverted pendulum. *The 8th IEEE Conference on Control Application, Dearborn, USA*, page 7pp, 1996.

T. Faria and L.T. Magalhães. Normal forms for retarded functional-differential equations with parameters and applications to Hopf bifurcation. *J. Differential Equations*, 122(2):181–200, 1995.

K. Gu, S.-I. Niculescu, and J. Chen. On stability of crossing curves for general systems with two delays. *J. Math. Anal. Appl.*, 311:231–253, 2005.

J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations*, volume 99 of *Applied Mathematics Sciences.* Springer Verlag, 1993.

T. Insperger and G. Stepan. *Semi-Discretization for Time-Delay Systems: Stability and Engineering Applications.* Applied Mathematics Sciences. Springer, 2011.

V.L. Kharitonov, S-I. Niculescu, J. Moreno, and W. Michiels. Static output feedback stabilization: necessary conditions for multiple delay controllers. *IEEE Trans. on Aut. Cont.*, 50(1):82–86, 2005.

Y. Kuznetsov. *Elements of applied bifurcation theory; Second edition*, volume 112 of *Applied Mathematics Sciences.* Springer, New York, 1998.

M. Landry, S.A. Campbell, K. Morris, and C. O. Aguilar. Dynamics of an inverted pendulum with delayed feedback control. *SIAM J. Appl. Dyn. Syst.*, 4(2):333–351 (electronic), 2005. ISSN 1536-0040.

W. Michiels and S-I. Niculescu. *Stability and stabilization of time-delay systems*, volume 12 of *Advances in Design and Control.* SIAM, Philadelphia, PA, 2007.

I.-C. Morărescu, S.-I. Niculescu, and K.Gu. On the geometry of stability regions of smith predictors subject to delay uncertainty. *IMA J. of Math. Control and Information*, 24(3):411–423, 2007a.

I.-C. Morărescu, S.-I. Niculescu, and K.Gu. Stability crossing curves of shifted gamma-distributed delay systems. *SIAM J. on App. Dyn. Sys.*, 6(2):475–493, 2007b.

I.-C. Morărescu, W. Michiels, and M. Jungers. Synchronization of coupled nonlinear oscillators with gamma-distributed delays. In *Proc. of IEEE ACC*, 2013.

S-I. Niculescu and W. Michiels. Stabilizing a chain of integrators using multiple delays. *IEEE Trans. on Aut. Cont.*, 49(5):802–807, 2004.

K. Pyragas. Control of chaos via an unstable delayed feedback controller. *Phys. Rev. Lett.*, 86:2265–2268.

K. Pyragas. Continuous control of chaos by self-controlling feedback. *Phys. Lett. A*, 170:421–428, 1992.

A. Shiriaev, A. Pogromsky, H. Ludvigsen, and O. Egeland. On global properties of passivity-based control of an inverted pendulum. *Int. J. of Robust Nonlinear Control*, 10:283–300, 2000.

J. Sieber and B. Krauskopf. Bifurcation analysis of an inverted pendulum with delayed feedback control near a triple-zero eigenvalue singularity. *Nonlinearity*, 17:85–103, 2004.

J. Sieber and B. Krauskopf. Extending the permissible control loop latency for the controlled inverted pendulum. *Dynamical Systems*, 20(2):189–199, 2005.

R. Sipahi and N. Olgac. Complete robustness of third-order lti multiple time-delay systems. *Automatica*, 41: 1413–1422, 2005.

I.H. Suh and Z. Bien. Proportional minus delay controller. *IEEE Trans. on Aut. Cont.*, AC24:370–372, 1979.