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On Numerical Solution of Differential Games in Classes of Mixed Strategies

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Abstract: A zero-sum linear-convex differential game with a quality index that estimates a set of deviations of a motion trajectory at given instants of time from given target points is considered. A case when the saddle point condition in a small game, also known as Isaac's condition, does not hold is studied. The game is posed in classes of mixed control strategies of players. A numerical method for computing the game value and optimal strategies is elaborated. Results of numerical experiments in model examples are given.

Keywords: Differential games, game value, saddle point, mixed strategies.

1. INTRODUCTION

We consider a zero-sum differential game (see, e.g., Krasovskii (1985); Krasovskii and Krasovskii (1995)). A dynamical system is described by ordinary differential equations that are linear with respect to the phase vector. The system is subjected to bounded control actions of two antagonistic players. The control process is evaluated within a fixed time interval. The quality index of the control process (the game payoff) is a norm of a set of deviations of the motion trajectory from given targets at given instants of time. In the case when the saddle point condition in a small game (see, e.g., Krasovskii (1985)), also known as Isaac's condition, holds, the game has a value and a saddle point in classes of pure control strategies (Krasovskii and Krasovskii (1995)). In Lukoyanov (1997, 1998) a procedure for computing the game value by constructing upper convex hulls of auxiliary functions from the method of stochastic program synthesis (Krasovskii (1985)) was proposed. On the basis of this procedure and the extremal shift (see, e.g., Krasovskii (1985); Krasovskii and Krasovskii (1995)) in Kornev (2012) a numerical method for solving differential games under such conditions was elaborated.

This paper is devoted to the case when the saddle point condition in a small game does not necessarily hold. Under this assumption the considered differential game has a game value and a saddle point in classes of mixed strategies (see, e.g., Krasovskii and Krasovskii (1995)). Below we show that after the introduction of the so-called leader system the methods from Lukoyanov (1997, 1998); Kornev (2012) become applicable to solving games in mixed strategies. In the process of construction of the player's optimal control strategy we apply the extremal shift to provide necessary guarantees for the quality of the motion generated by the leader's deterministic control, while the proximity of the motions of the original system and the leader system is obtained by means of stochastic constructions from Krasovskii and Choi (2001); Krasovskii and Krasovskii (2012).

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2. PROBLEM STATEMENT

Consider a differential game for a dynamical system, described by the following dynamic equation

$$\dot{x} = A(t)x + f\big(t,u,v\big), \quad t_0 \le t < \vartheta, \\ x \in \mathbb{R}^n, \quad u \in \mathbb{U} \subset \mathbb{R}^{n_u}, \quad v \in \mathbb{V} \subset \mathbb{R}^{n_v}, \tag{1}$$

initial condition

$$x(t_0) = x_0 \in \mathbb{R}^n, \tag{2}$$

and quality index

$$\gamma = \mu_1\Big(D_1\big(x(\vartheta_1) - c_1\big), \ldots, D_N\big(x(\vartheta_N) - c_N\big)\Big). \tag{3}$$

Here x is a phase vector; t is time; $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$; $A(t)$ and $f(t,u,v)$ are jointly continuous matrix function and vector function; u and v are values of control actions of the first and the second player; t_0 and ϑ are fixed instants of time; sets $\mathbb{U}$ and $\mathbb{V}$ are compact; $\vartheta_i \in (t_0, \vartheta]$: $\vartheta_{i+1} > \vartheta_i,\ i = \overline{1, N-1},\ \vartheta_N = \vartheta$, are given instants of time of the motion quality evaluation; D_i are constant matrices with dimensions $d_i \times n$ $(1 \le d_i \le n)$; $c_i \in \mathbb{R}^n$ are target vectors; $\mu_1(g_1, \ldots, g_N)$ is a norm in a space of N-tuples $(g_1, \ldots, g_N)$ composed of d_i–dimensional vectors $g_i,\ i = \overline{1, N}$.

It is assumed that there exist such norms $\mu_i\big(g_i, \ldots, g_N\big)$ and $\sigma_i(g_i, \mu)$, for which, for $i = \overline{1, N-1}$,

$$\mu_i(g_i, \ldots, g_N) = \sigma_i(g_i, \mu_{i+1}(g_{i+1}, \ldots, g_N)). \tag{4}$$

In such a case (see Lukoyanov (1998)) quality index γ is positional (Krasovskii and Krasovskii, 1995, p. 43).

Realizations $u(\cdot) = \big\{u(t) \in \mathbb{U},\ t_0 \le t < \vartheta\big\}$ and $v(\cdot) = \big\{v(t) \in \mathbb{V},\ t_0 \le t < \vartheta\big\}$ are admissible if they are Borel-measurable. The system's motion $x(\cdot)$ generated by such realizations is an absolutely continuous function $\{x(t) \in \mathbb{R}^n,\ t_0 \le t \le \vartheta\}$ that satisfies initial condition (2) and together with $u = u(t)$ and $v = v(t)$ satisfies equation (1) for almost every t.

The aim of the first player is to make quality index (3) as small as possible. The aim of the second player is opposite.

Note that we do not assume that the saddle point condition in a small game holds for system (1), so there might exist such a vector $l^* \in \mathbb{R}^n$, for which

$$\min_{u\in\mathbb{U}} \max_{v\in\mathbb{V}} \langle l^*, f(t,u,v)\rangle \ne \max_{v\in\mathbb{V}} \min_{u\in\mathbb{U}} \langle l^*, f(t,u,v)\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of vectors. In such a case, differential game (1)–(3) might not have a value and a saddle point in pure strategies $u(t,x,\varepsilon)$, $v(t,x,\varepsilon)$, and it is appropriate to consider a formalization of the game in classes of mixed strategies (see, e.g., Krasovskii and Krasovskii (1995)). Here ε is an accuracy parameter, whose value is assigned by the player at the beginning of the process of forming the control actions and remains constant.

Since compact sets $\mathbb{U}$ and $\mathbb{V}$ can be approximated by finite sets, further we assume that $\mathbb{U}$ and $\mathbb{V}$ are finite initially:

$$\mathbb{U} = \{u^{[r]} \in \mathbb{R}^{n_u} : r = \overline{1,L}\},\ \mathbb{V} = \{v^{[s]} \in \mathbb{R}^{n_v} : s = \overline{1,M}\}.$$

Put

$$\mathbb{P} = \Big\{(p_1,\dots,p_L) \in \mathbb{R}^L : p_r \ge 0,\ r = \overline{1,L},\ \sum_{r=1}^{L} p_r = 1\Big\},$$

$$\mathbb{Q} = \Big\{(q_1,\dots,q_M) \in \mathbb{R}^M : q_s \ge 0,\ s = \overline{1,M},\ \sum_{s=1}^{M} q_s = 1\Big\}.$$

Following constructions from Krasovskii and Krasovskii (1995), let us describe differential game (1)–(3) in mixed strategies from the point of view of the first player. Along with original x-object (1) consider an auxiliary y-model with a phase vector $y \in \mathbb{R}^n$. This y-model will be used as a leader (Krasovskii and Subbotin, 1988, p. 327) in the process of forming of the first player's control actions. The motion of the y-model is described by the equation

$$\dot y = A(t)y + \sum_{r=1}^{L}\sum_{s=1}^{M} f(t,u^{[r]},v^{[s]}) p_r^* q_s^*, \quad t_0 \le t < \vartheta, \quad y(t_0) = y_0 \in \mathbb{R}^n, \quad p^* \in \mathbb{P}, \quad q^* \in \mathbb{Q}. \tag{5}$$

A mixed strategy S^u of the first player is a triple $\{p_u(\cdot),\ p_u^*(\cdot),\ q_u^*(\cdot)\}$ of functions

$$p_u = p_u(t,x,y,\varepsilon) \in \mathbb{P}, \quad p_u^* = p_u^*(t,x,y,\varepsilon) \in \mathbb{P},$$
$$q_u^* = q_u^*(t,x,y,\varepsilon) \in \mathbb{Q},$$
$$t \in [t_0,\vartheta],\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^n,\ \varepsilon > 0,$$

which are measurable with respect to x, y for fixed t, ε.

Within the formalization of differential game (1)–(3) in classes of mixed strategies players form their control realizations using probabilistic mechanisms. It is assumed that further constructions are based on a sufficiently rich probabilistic space $\Pi = \{\Omega,\ \mathcal{F},\ P\}$, where $\Omega = \{\omega\}$ is a sample space, $\mathcal{F}$ is a σ-algebra, $P = P(B)$, $B \in \mathcal{F}$ is a probability measure. For explanations on formal construction of the appropriate space Π (see, e.g., (Krasovskii and Krasovskii, 1995, p. 250), and also (Krasovskii and Subbotin, 1988, p. 402)).

A control law $\mathcal{U}$ of the first player is a triple $\{S^u,\ \varepsilon,\ \Delta_\delta\}$, where Δ_δ is a partition of the time interval $[t_0,\vartheta]$:

$$\Delta_\delta = \{t_j : 0 < t_{j+1} - t_j \le \delta,\ j = \overline{0,k-1},\ t_k = \vartheta\}. \tag{6}$$

From given positions $\{t_0,x_0\}$ of the x-object and $\{t_0,y_0\}$ of the y-model the control law $\mathcal{U}$ in a pair with some admissible stochastic control realization $v_\omega(\cdot) = \{v_\omega(t) \in \mathbb{V},\ t_0 \le t < \vartheta,\ \omega \in \Omega\}$ of the second player generates a stochastic motion $\{x_\omega(\cdot), y_\omega(\cdot)\}$ of the $\{x$-object, y-model$\}$ complex, which is defined as a solution of the following step-by-step equations:

$$\begin{aligned}
&\dot x_\omega(t) = A(t)x_\omega(t) + f\big(t, u_\omega^{(j)}, v_\omega(t)\big),\\
&\dot y_\omega(t) = A(t)y_\omega(t) + \sum_{r=1}^{L}\sum_{s=1}^{M} \Big(f(t,u^{[r]},v^{[s]})\cdot\\
&\qquad \cdot p_{ur}^*\big(t_j, x_\omega(t_j), y_\omega(t_j), \varepsilon\big) q_{us}^*\big(t_j, x_\omega(t_j), y_\omega(t_j), \varepsilon\big)\Big),\\
&x_\omega(t_0) = x_0,\ y_\omega(t_0) = y_0,\ t_j \le t < t_{j+1},\ j = \overline{0,k-1},
\end{aligned} \tag{7}$$

where value $u_\omega^{(j)} \in \mathbb{U}$ is determined as a result of a random trial under the condition

$$P\Big(u_\omega^{(j)} = u^{[r]} \mid x_\omega(t_j),\ y_\omega(t_j)\Big) = p_{ur}\big(t_j, x_\omega(t_j), y_\omega(t_j), \varepsilon\big).$$

The symbol $P(\cdot|\cdot)$ denotes the conditional probability. It is assumed that on each step $t_j \le t < t_{j+1}$ the realization $v_\omega(\cdot)$ is stochastically independent of the realization $u_\omega(\cdot) = \{u_\omega(t) = u_\omega^{(j)}, t_j \le t < t_{j+1}, j = \overline{0,k-1}, \omega \in \Omega\}$:

$$P\Big(v_\omega(t) \in B \mid x_\omega(t_j),\ y_\omega(t_j),\ u_\omega(t_j)\Big) = P\Big(v_\omega(t) \in B \mid x_\omega(t_j),\ y_\omega(t_j)\Big),\ B \subset \mathbb{V}.$$

A guaranteed result of the control law $\mathcal{U}$, for fixed positions $\{t_0,x_0\}$ and $\{t_0,y_0\}$ and a number $0 < \beta < 1$, is a value

$$\rho(\mathcal{U}; t_0, x_0, y_0; \beta) = \sup_{v_\omega(\cdot)} \min\big\{\alpha \big| P\big(\gamma \le \alpha\big) \ge \beta\big\}, \tag{8}$$

where the value γ of quality index (3) is calculated for the motion $x_\omega(\cdot)$.

A guaranteed result of the strategy S^u, for a fixed initial position $\{t_0,x_0\}$, is a value

$$\rho(S^u; t_0, x_0) = \lim_{\beta\to 1} \overline{\lim_{\varepsilon\to 0}} \lim_{\eta\to 0} \sup_{|x_0 - y_0| \le \eta} \lim_{\delta\to 0} \sup_{\Delta_\delta} \rho(\mathcal{U} = \{S^u, \varepsilon, \Delta_\delta\}; t_0, x_0, y_0; \beta).$$

A strategy S_0^u is called optimal if

$$\rho(S_0^u; t_0, x_0) = \min_{S^u} \rho(S^u; t_0, x_0) = \rho_u^0(t_0, x_0).$$

The value $\rho_u^0(t_0,x_0)$ is the optimal guaranteed result of the first player.

A control law $\mathcal{U}$ is (ζ,β)-optimal if

$$\rho(\mathcal{U}; t_0, x_0, y_0 = x_0; \beta) \le \rho_u^0(t_0, x_0) + \zeta. \tag{9}$$

Similarly, considering differential game (1)–(3) from the point of view of the second player, we introduce an auxiliary z-model. Its motion is described by the following equation

$$\dot{z} = A(t)z + \sum_{r=1}^{L}\sum_{s=1}^{M} f(t, u^{[r]}, v^{[s]})p_r^* q_s^*, \quad t_0 \le t < \vartheta,$$
$$z(t_0) = z_0 \in \mathbb{R}^n, \quad p^* \in \mathbb{P}, \quad q^* \in \mathbb{Q}. \qquad (10)$$

A mixed strategy S^v of the second player is a triple $\{q_v(\cdot),\, p_v^*(\cdot),\, q_v^*(\cdot)\}$ of functions

$$q_v = q_v(t, x, z, \varepsilon) \in \mathbb{Q}, \quad p_v^* = p_v^*(t, x, z, \varepsilon) \in \mathbb{P},$$
$$q_v^* = q_v^*(t, x, z, \varepsilon) \in \mathbb{Q},$$
$$t \in [t_0, \vartheta],\ x \in \mathbb{R}^n,\ z \in \mathbb{R}^n,\ \varepsilon > 0,$$

which are measurable with respect to x, z for fixed t, ε.

A control law $\mathcal{V}$ of the second player is a triple $\{S^v,\, \varepsilon,\, \Delta_\delta\}$. From given positions $\{t_0, x_0\}$ of the x-object and $\{t_0, z_0\}$ of the z-model together with some admissible stochastic control realization $u(\cdot) = \{u_\omega(t) \in \mathbb{U},\ t_0 \le t < \vartheta,\ \omega \in \Omega\}$ of the first player the control law $\mathcal{V}$ generates a stochastic motion $\{x_\omega(\cdot), z_\omega(\cdot)\}$ of the $\{x$-model, z-object$\}$ complex, which is defined as a solution of the step-by-step equations

$$\dot{x}_\omega(t) = A(t)x_\omega(t) + f\big(t, u_\omega(t), v_\omega^{(j)}\big),$$
$$\dot{z}_\omega(t) = A(t)z_\omega(t) + \sum_{r=1}^{L}\sum_{s=1}^{M} \Big(f(t, u^{[r]}, v^{[s]})\cdot$$
$$\cdot p_{v\,r}^*\big(t_j, x_\omega(t_j), z_\omega(t_j), \varepsilon\big) q_{v\,s}^*\big(t_j, x_\omega(t_j), z_\omega(t_j), \varepsilon\big)\Big),$$
$$x_\omega(t_0) = x_0,\ z_\omega(t_0) = z_0,\ t_j \le t < t_{j+1},\ j = \overline{0, k-1}, \qquad (11)$$

where the value $v_\omega^{(j)} \in \mathbb{V}$ is determined as a result of a random trial under the condition

$$P\Big(v_\omega^{(j)} = v^{[s]} \mid x_\omega(t_j),\ z_\omega(t_j)\Big) = q_{v\,s}\big(t_j, x_\omega(t_j), z_\omega(t_j), \varepsilon\big).$$

It is assumed that on each step $t_j \le t < t_{j+1}$ the realization $u_\omega(\cdot)$ is stochastically independent of the realization $v_\omega(\cdot) = \{v_\omega(t) = v_\omega^{(j)}, t_j \le t < t_{j+1}, j = \overline{0, k-1}, \omega \in \Omega\}$:

$$P\Big(u_\omega(t) \in B \mid x_\omega(t_j),\ z_\omega(t_j),\ v_\omega(t_j)\Big) =$$
$$P\Big(u_\omega(t) \in B \mid x_\omega(t_j),\ z_\omega(t_j)\Big), \quad B \subset \mathbb{U}.$$

A guaranteed result of the control law $\mathcal{V}$, for fixed $\{t_0, x_0\}$, $\{t_0, z_0\}$ and $0 < \beta < 1$, is the following value

$$\rho(\mathcal{V}; t_0, x_0, z_0; \beta) = \inf_{u_\omega(\cdot)} \max\big\{\alpha \big| P(\gamma \ge \alpha) \ge \beta\big\}. \qquad (12)$$

Note that in accordance with definitions (8) and (12), for any admissible mixed control laws $\mathcal{U}$ and $\mathcal{V}$, for any value $0.5 < \beta < 1$, the following inequality holds

$$\rho(\mathcal{U}; t_0, x_0, y_0; \beta) \ge \rho(\mathcal{V}; t_0, x_0, z_0 = y_0; \beta). \qquad (13)$$

A guaranteed result of the strategy S^v is

$$\rho(S^v; t_0, x_0) = \lim_{\beta\to 1} \lim_{\varepsilon\to 0} \lim_{\eta\to 0} \inf_{|x_0 - z_0| \le \eta}$$
$$\lim_{\delta\to 0} \inf_{\Delta_\delta} \rho(\mathcal{V} = \{S^v, \varepsilon, \Delta_\delta\}; t_0, x_0, z_0; \beta).$$

A strategy S_0^v is called optimal if

$$\rho(S_0^v; t_0, x_0) = \max_{S^v} \rho(S^v; t_0, x_0) = \rho_v^0(t_0, x_0).$$

The value $\rho_v^0(t_0, x_0)$ is the optimal guaranteed result of the second player.

A control law $\mathcal{V}$ is (ζ, β)-optimal if

$$\rho(\mathcal{V}; t_0, x_0, z_0 = x_0; \beta) \ge \rho_v^0(t_0, x_0) - \zeta. \qquad (14)$$

It is known (Krasovskii and Krasovskii, 1995, p. 257) that differential game (1)–(3) has a value

$$\rho(t_0, x_0) = \rho_u^0(t_0, x_0) = \rho_v^0(t_0, x_0) \qquad (15)$$

and a saddle point $\{S_0^u, S_0^v\}$ in classes of mixed strategies.

The aim of this paper is to elaborate a numerical method for approximate computation of the game value $\rho(t_0, x_0)$ and of the (ζ, β)-optimal control laws.

3. AUXILIARY DIFFERENTIAL GAME

Consider a differential game for y-model (5) with quality index like (3):

$$\gamma_y = \mu_1\Big(D_1\big(y(\vartheta_1) - c_1\big), \ldots, D_N\big(y(\vartheta_N) - c_N\big)\Big). \qquad (16)$$

In this auxiliary game p^* is treated as control actions of the first player, who is aimed to minimize quality index (16), and q^* is control actions of the second player, who is aimed to maximize this quality index.

System (5) satisfies the saddle point condition in a small game, that is why auxiliary game (5), (16) has a value $\rho^*(t_0, y_0)$ and a saddle point $\{p_0^*(t, y, \varepsilon), q_0^*(t, y, \varepsilon)\}$ in classes of pure strategies (Krasovskii, 1985, p. 228–234). According to Lukoyanov (1997, 1998), value $\rho^*(t_0, y_0)$ can be calculated approximately by means of the following procedure. This procedure is based on a partition Δ_δ like (6). All instants of time ϑ_i from quality index (16) are included in this partition.

Denote

$$h(t) = \begin{cases} 0, & \text{if } t < \vartheta_1,\\ \max\{i = \overline{1, N} \mid \vartheta_i \le t\}, & \text{otherwise,}\end{cases}$$

$$\Delta\psi_j(m) = \int_{t_j}^{t_{j+1}} \max_{q^*\in\mathbb{Q}} \min_{p^*\in\mathbb{P}} \langle m,$$
$$\Psi[\vartheta, t] \sum_{r=1}^{L}\sum_{s=1}^{M} f(t, u^{[r]}, v^{[s]}) p_r^* q_s^* \rangle\, dt, \qquad (17)$$
$$m \in \mathbb{R}^n, \quad j = \overline{0, k-1}.$$

Here $\Psi[\vartheta, t]$ is a fundamental solution matrix of the equation $\dot{x} = A(t)x$ such that $\Psi[t, t] = E$, where E is the identity matrix.

Step by step, in the reverse order, starting from the last point of the partition Δ_δ, we define sets G_j of vectors $m \in \mathbb{R}^n$ and scalar functions $\varphi_j(m)$, $m \in G_j$, $j = \overline{0, k}$, by the following recurrent relations.

For $j = k$, we set

$$G_k = \{m = 0\}, \qquad \varphi_k(m) = 0,\ m \in G_k.$$

For the current j, if t_{j+1} is not equal to any instant of time ϑ_i of the motion quality evaluation, i.e., $h(t_j) = h(t_{j+1})$, we set

$$G_j = G_{j+1}, \qquad \varphi_{j+1}^*(m) = \varphi_{j+1}(m),\ m \in G_j,$$

otherwise, when $t_{j+1} = \vartheta_h$, $h = h(t_j) + 1$, we set

$$G_j = \left\{ m \,\middle|\, \begin{array}{l} m = \nu m_* + \Psi^\top[\vartheta_h, \vartheta] D_h^\top l,\\ \nu \ge 0,\ m_* \in G_{j+1},\\ l \in \mathbb{R}^{d_h},\ \sigma_h^*(l, \nu) \le 1 \end{array} \right\},$$
$$\varphi_{j+1}^*(m) = \max_{m_*, \nu, l} \big[\nu\varphi_{j+1}(m_*) - \langle l, D_h c_h\rangle\big], \quad m \in G_j. \qquad (18)$$

Finally, in any case, for $m \in G_j$, we set

$$\psi_j(m) = \Delta\psi_j(m) + \varphi_{j+1}(m),\ \varphi_j(m) = \{\psi_j\}^*_{G_j}(m). \quad (19)$$

Here the superscript "$\top$" denotes the matrix transposition; $\sigma^*_h(\cdot)$ is a norm dual to $\sigma_h(\cdot)$ from (4); the maximum is calculated over all such $m_* \in G_{j+1}$, $\nu \geq 0$, $l \in \mathbb{R}^{d_h}$, $\sigma^*_h(l,\nu) \leq 1$, that satisfy the equality $m = \nu m_* + \Psi^\top[\vartheta_h,\vartheta]D_h^\top l$; symbol $\{\psi_j\}^*_{G_j}$ denotes the upper convex hull of the function ψ_j on the set G_j, i.e., the minimal concave function that majorizes $\psi_j(m)$ for $m \in G_j$.

Consider the system of values

$$e_j(y) = \max_{m\in G_j}\big[\langle m, \Psi[\vartheta,t_j]y\rangle + \varphi_j(m)\big],\ j=\overline{0,k},\ y \in \mathbb{R}^n. \quad (20)$$

It is known (Lukoyanov (1998)), that for any bounded subset $Y_0 \subset \mathbb{R}^n$ and any number $\xi > 0$, there exists a number $\delta > 0$ such that for any partition Δ_δ like (6), the following inequality holds

$$|\rho^*(t_0,y_0) - e_0(y_0)| \leq \xi, \quad y_0 \in Y_0. \quad (21)$$

Put

$$\varepsilon(t) = \sqrt{\varepsilon + \varepsilon(t-t_0)}\exp\big\{\lambda_A(t-t_0)\big\},\ \varepsilon > 0,\ t \in [t_0,\vartheta],$$
$$\lambda_A = \sup_{t\in[t_0,\vartheta],\ \|x\|=1} \|A(t)x\| < +\infty. \quad (22)$$

Here and below $\|\cdot\|$ denotes the Euclidean vector norm.

On the basis of system of values (20) by means of extremal shift to accompanying points (see, e.g., Krasovskii (1985); Krasovskii and Krasovskii (1995)) we define a function $p^{*\mathrm{e}}_u(\ \cdot\ ;\Delta_\delta)$:

$$p^{*\mathrm{e}}_u = p^{*\mathrm{e}}_u(t,y,\varepsilon;\Delta_\delta) \in \arg\min_{p^*\in\mathbb{P}}\ \max_{q^*\in\mathbb{Q}}$$
$$\Big\langle \frac{\Psi^\top[\vartheta,t]m_u}{\sqrt{1+\|\Psi^\top[\vartheta,t]m_u\|^2}}\varepsilon(t),\ \sum_{r=1}^{L}\sum_{s=1}^{M} f(t,u^{[r]},v^{[s]})p^*_r q^*_s\Big\rangle, \quad (23)$$

where

$$m_u \in \arg\max_{m\in G_j}\Big[\langle m,\Psi[\vartheta,t]y\rangle + \varphi_j(m) - \varepsilon(t)\sqrt{1+\|\Psi^\top[\vartheta,t]m\|^2}\Big],$$
$$t_j \leq t < t_{j+1},\ j=\overline{0,k-1},\quad y\in\mathbb{R}^n,\quad \varepsilon>0.$$

For auxiliary game (5), (16), consider a control law $\big\{p^{*\mathrm{e}}_u(\ \cdot\ ;\Delta_\delta),\varepsilon,\Delta_\delta\big\}$, that, basing on the partition Δ_δ used to construct (20), forms a piecewise constant control realization of the first player according to the rule

$$p^*(t) = p^{*\mathrm{e}}_u\big(t_j, y(t_j),\varepsilon;\Delta_\delta\big),\quad t_j\leq t<t_{j+1},\quad j=\overline{0,k-1}.$$

As stated in Kornev (2012), this control law is ζ-optimal. It means that for any bounded subset $Y_0\subset\mathbb{R}^n$, for any number $\zeta>0$, there exist a number $\varepsilon^*_y>0$ and a function $\delta_y(\varepsilon)>0$, $0<\varepsilon\leq\varepsilon^*_y$, such that, for any $y_0\in Y_0$, any value $0<\varepsilon\leq\varepsilon^*_y$, and any partition Δ_δ, $\delta\leq\delta_y(\varepsilon)$, the control law $\big\{p^{*\mathrm{e}}_u(\ \cdot\ ;\Delta_\delta),\varepsilon,\Delta_\delta\big\}$ ensures the inequality

$$\gamma_y \leq \rho^*(t_0,y_0)+\zeta/2, \quad (24)$$

for any measurable $q^*(\cdot) = \big\{q^*(t)\in\mathbb{Q}, t_0\leq t<\vartheta\big\}$.

On the other hand, considering the identical differential game for z-model (2) and quality index γ_z like (16), basing on values (20), define a function $q^{*\mathrm{e}}_v(\ \cdot\ ;\Delta_\delta)$:

$$q^{*\mathrm{e}}_v = q^{*\mathrm{e}}_v(t,z,\varepsilon;\Delta_\delta) \in \arg\max_{q^*\in\mathbb{Q}}\ \min_{p^*\in\mathbb{P}}$$
$$\Big\langle \frac{\Psi^\top[\vartheta,t]m_v}{\sqrt{1+\|\Psi^\top[\vartheta,t]m_v\|^2}}\varepsilon(t),\ \sum_{r=1}^{L}\sum_{s=1}^{M} f(t,u^{[r]},v^{[s]})p^*_r q^*_s\Big\rangle, \quad (25)$$

where

$$m_v \in \arg\max_{m\in G_j}\Big[\langle m,\Psi[\vartheta,t]z\rangle + \varphi_j(m) + \varepsilon(t)\sqrt{1+\|\Psi^\top[\vartheta,t]m\|^2}\Big],$$
$$t_j \leq t < t_{j+1},\ j=\overline{0,k-1},\quad z\in\mathbb{R}^n,\quad \varepsilon>0.$$

Consider a corresponding control law $\big\{q^{*\mathrm{e}}_v(\ \cdot\ ;\Delta_\delta),\varepsilon,\Delta_\delta\big\}$, that forms a control realization for the second player according to the rule

$$q^*(t) = q^{*\mathrm{e}}_v\big(t_j, z(t_j),\varepsilon;\Delta_\delta\big),\quad t_j\leq t<t_{j+1},\quad j=\overline{0,k-1}.$$

For any bounded subset $Z_0\subset\mathbb{R}^n$, for any number $\zeta>0$, there exist a number $\varepsilon^*_z>0$ and a function $\delta_z(\varepsilon)>0$, $0<\varepsilon\leq\varepsilon^*_z$, such that, for any $z_0\in Z_0$, any value $0<\varepsilon\leq\varepsilon^*_z$, and any partition Δ_δ, $\delta\leq\delta_z(\varepsilon)$, the control law $\big\{q^{*\mathrm{e}}_v(\ \cdot\ ;\Delta_\delta),\varepsilon,\Delta_\delta\big\}$ ensures the inequality

$$\gamma_z \geq \rho^*(t_0,z_0)-\zeta/2, \quad (26)$$

for any measurable $p^*(\cdot) = \big\{p^*(t)\in\mathbb{P},\ t_0\leq t<\vartheta\big\}$.

4. OBJECT-MODEL PROXIMITY

In original differential game (1)–(3) the first player during the process of forming the stochastic motion $\{x_\omega(\cdot),y_\omega(\cdot)\}$ of complex (7) can ensure an appropriate proximity of $x_\omega(\cdot)$ and $y_\omega(\cdot)$ by means of a proper choice of functions $p_u(\cdot)$ and $q^*_u(\cdot)$ in its mixed strategy S^u.

On the other hand, by means of a proper choice of $q_v(\cdot)$ and $p^*_v(\cdot)$ in the mixed strategy S^v, the second player can ensure proximity of $x_\omega(\cdot)$ and $z_\omega(\cdot)$ for the corresponding motion $\{x_\omega(\cdot),z_\omega(\cdot)\}$ of complex (11). Namely, the following lemmas are valid (see Krasovskii and Choi (2001); Krasovskii and Krasovskii (2012)). Denote

$$\lambda(t,x,y) = \|x-y\|^2\exp\{-2\lambda_A(t-t_0)\},$$

where constant λ_A is taken from (22).

Lemma 1. Let functions $p_u = p_u(t,x,y,\varepsilon)$ and $q^*_u = q^*_u(t,x,y,\varepsilon)$ from the mixed strategy of the first player $S^u = \{p_u(\cdot),p^*_u(\cdot),q^*_u(\cdot)\}$ be defined from the following condition

$$\langle x-y, \sum_{r=1}^{L}\sum_{s=1}^{M} f(t,u^{[r]},v^{[s]})p_{u_r}q^*_{u_s}\rangle$$
$$= \min_{p\in\mathbb{P}}\max_{q\in\mathbb{Q}}\big\langle x-y,\sum_{r=1}^{L}\sum_{s=1}^{M} f(t,u^{[r]},v^{[s]})p_r q_s\big\rangle. \quad (27)$$

Then, for any bounded subset $X_0\subset\mathbb{R}^n$ and any numbers $\lambda^*>0$ and $0<\beta<1$, there exist such values $\lambda_0>0$ and $\delta>0$, that, for any initial $x_0\in X_0$ and $y_0\in\mathbb{R}^n$, which satisfy the condition $\lambda(t_0,x_0,y_0)\leq\lambda_0$, for any value $\varepsilon>0$, any partition Δ_δ (6), and any stochastic motion $\{x_\omega(\cdot),y_\omega(\cdot)\}$ of complex (7), generated by the control law $\mathcal{U}=\{S^u,\varepsilon,\Delta_\delta\}$, the following inequality holds

$$P\Big(\lambda(t,x_\omega(t),y_\omega(t))\leq\lambda^*,\ t\in[t_0,\vartheta]\Big)\geq\beta,$$

for any function $p^*_u(\cdot)$ in the strategy S^u and any admissible stochastic control realization $v_\omega(\cdot)$ of the second player.

Lemma 2. Let functions $q_v = q_v(t,x,z,\varepsilon)$ *and* $p_v^* = p_v^*(t,x,z,\varepsilon)$ *from the mixed strategy of the second player* $S^v = \{q_v(\cdot), p_v^*(\cdot), q_v^*(\cdot)\}$ *be defined from the following condition*

$$\Big\langle z-x, \sum_{r=1}^{L}\sum_{s=1}^{M} f(t,u^{[r]},v^{[s]})p_{vr}^* q_{vs}\Big\rangle = \max_{q\in\mathbb{Q}}\min_{p\in\mathbb{P}}\Big\langle z-x, \sum_{r=1}^{L}\sum_{s=1}^{M} f(t,u^{[r]},v^{[s]})p_r q_s\Big\rangle. \quad (28)$$

Then, for any bounded subset $X_0 \subset \mathbb{R}^n$ *and any numbers* $\lambda^* > 0$ *and* $0 < \beta < 1$*, there exist such values* $\lambda_0 > 0$ *and* $\delta > 0$*, that, for any initial* $x_0 \in X_0$ *and* $z_0 \in \mathbb{R}^n$*, which satisfy the condition* $\lambda(t_0,x_0,z_0) \le \lambda_0$*, for any value* $\varepsilon > 0$*, any partition* Δ_δ *(6), and any stochastic motion* $\{x_\omega(\cdot), z_\omega(\cdot)\}$ *of complex (11), generated by the control law* $\mathcal{V} = \{S^v, \varepsilon, \Delta_\delta\}$*, the following inequality holds*

$$P\Big(\lambda(t,x_\omega(t),z_\omega(t)) \le \lambda^*,\ t\in[t_0,\vartheta]\Big) \ge \beta,$$

for any function $q_v^*(\cdot)$ *in the strategy* S^v *and any admissible stochastic control realization* $u_\omega(\cdot)$ *of the first player.*

5. GAME VALUE AND (ζ,β)-OPTIMAL CONTROL LAWS CONSTRUCTION

Define a mixed control law $\mathcal{U}^\varepsilon_{\Delta_\delta} = \{S^u, \varepsilon, \Delta_\delta\}$, $S^u = \{p_u(\cdot), p_u^*(\cdot), q_u^*(\cdot)\}$ of the first player. Assume that all instants of time ϑ_i from quality index (3) are included in the partition Δ_δ like (6), the functions $p_u(\cdot)$ and $q_u^*(\cdot)$ satisfy condition (27), and the function $p_u^*(\cdot)$ is determined from condition of the extremal shift (23) as follows

$$p_u^* = p_u^*(t,x,y,\varepsilon) = p_u^{*\mathrm{e}}(t,y,\varepsilon;\Delta_\delta),$$
$$t\in[t_0,\vartheta], x\in\mathbb{R}^n, y\in\mathbb{R}^n, \varepsilon>0.$$

Similarly, define a mixed control law $\mathcal{V}^\varepsilon_{\Delta_\delta} = \{S^v, \varepsilon, \Delta_\delta\}$, $S^v = \{q_v(\cdot), p_v^*(\cdot), q_v^*(\cdot)\}$ of the second player. Assume that the functions $q_v(\cdot)$ and $p_v^*(\cdot)$ satisfy condition (28), and the function $q_v^*(\cdot)$ is determined from condition (25):

$$q_v^* = q_v^*(t,x,z,\varepsilon) = q_v^{*\mathrm{e}}(t,z,\varepsilon;\Delta_\delta),$$
$$t\in[t_0,\vartheta], x\in\mathbb{R}^n, z\in\mathbb{R}^n, \varepsilon>0.$$

Theorem 3. For any bounded subset $X_0 \subset \mathbb{R}^n$ *and any numbers* $\zeta > 0$ *and* $0.5 < \beta < 1$*, there exist a number* $\varepsilon^* > 0$ *and a function* $\delta(\varepsilon) > 0$, $0 < \varepsilon \le \varepsilon^*$ *such that, for any* $x_0 \in X_0$*, any value* $0 < \varepsilon \le \varepsilon^*$*, and any partition* Δ_δ*,* $\delta \le \delta(\varepsilon)$*, the control laws* $\mathcal{U}^\varepsilon_{\Delta_\delta}$ *and* $\mathcal{V}^\varepsilon_{\Delta_\delta}$ *are* (ζ,β)*-optimal and the following inequality holds*

$$|\rho(t_0,x_0) - e_0(x_0)| \le \zeta.$$

Proof. Let $X_0 \subset \mathbb{R}^n$; $\zeta > 0$; $0.5 < \beta < 1$. Since game (1)–(3) has a saddle point $\{S_0^u, S_0^v\}$, there exist control laws $\mathcal{U}_0$ and $\mathcal{V}_0$ such that, for any $x_0 \in X_0$ the following holds

$$\begin{aligned}\rho(\mathcal{U}_0;t_0,x_0,y_0=x_0;\beta) &\le \rho(t_0,x_0)+\zeta,\\ \rho(\mathcal{V}_0;t_0,x_0,z_0=x_0;\beta) &\ge \rho(t_0,x_0)-\zeta.\end{aligned} \quad (29)$$

Using Lemma 1, continuity of quality index (3) and ζ-optimality of the auxiliary control law $\{p_u^{*\mathrm{e}}(\,\cdot\,;\Delta_\delta), \varepsilon, \Delta_\delta\}$, find a number $\varepsilon_y^* > 0$ and a function $\delta'_y(\varepsilon) > 0, 0 < \varepsilon \le \varepsilon_y^*$, such that the following relations hold

$$P(|\gamma_x - \gamma_y| \le \zeta/2) \ge \beta, \qquad \gamma_y \le \rho^*(t_0,y_0=x_0)+\zeta/2,$$

for any $x_0 \in X_0$, $0 < \varepsilon \le \varepsilon_y^*$, and $0 < \delta \le \delta'_y(\varepsilon)$.

From here, in accordance with (8), for the control law $\mathcal{U}^\varepsilon_{\Delta_\delta}$ we obtain

$$\beta \le P(|\gamma_x-\gamma_y|\le\zeta/2) \le P(\gamma_x \le \rho^*(t_0,y_0=x_0)+\zeta),$$
$$\rho(\mathcal{U}^\varepsilon_{\Delta_\delta};t_0,x_0,y_0=x_0;\beta) \le \rho^*(t_0,y_0=x_0)+\zeta. \quad (30)$$

Consider a case, when in differential game (1)–(3) control actions of the first player are chosen by the control law $\mathcal{U}^\varepsilon_{\Delta_\delta}$ and control actions of the second player are chosen by $\mathcal{V}_0$. From inequalities (29), (13) and (30) we conclude

$$\begin{aligned}\rho(t_0,x_0)-\zeta &\le \rho(\mathcal{V}_0;t_0,x_0,z_0=x_0;\beta)\\ &\le \rho(\mathcal{U}^\varepsilon_{\Delta_\delta};t_0,x_0,y_0=x_0;\beta) \le \rho^*(t_0,y_0=x_0)+\zeta.\end{aligned}$$

Similarly, using Lemma 2, continuity of quality index (3) and ζ-optimality of the control law $\{q_v^{*\mathrm{e}}(\,\cdot\,;\Delta_\delta), \varepsilon, \Delta_\delta\}$, find a number $\varepsilon_z^* > 0$ and a function $\delta'_z(\varepsilon) > 0$, $0 < \varepsilon \le \varepsilon_z^*$ such that, for any $x_0 \in X_0$, $0 < \varepsilon \le \varepsilon_z^*$, and $0 < \delta \le \delta'_z(\varepsilon)$, the guaranteed result of the control law $\mathcal{V}^\varepsilon_{\Delta_\delta}$ satisfies

$$\begin{aligned}\rho^*(t_0,z_0=x_0)-\zeta &\le \rho(\mathcal{V}^\varepsilon_{\Delta_\delta};t_0,x_0,z_0=x_0;\beta)\\ &\le \rho(\mathcal{U}_0;t_0,x_0,y_0=x_0;\beta) \le \rho(t_0,x_0)+\zeta.\end{aligned}$$

Thus $|\rho(t_0,x_0) - \rho^*(t_0,y_0=x_0)| \le 2\zeta$, hence

$$\rho(t_0,x_0) = \rho^*(t_0,y_0=x_0).$$

For $\xi = \zeta$, find a number $\delta' > 0$ such that inequality (21) holds. Put

$$\varepsilon^* = \min\{\varepsilon_y^*,\ \varepsilon_z^*\},$$
$$\delta(\varepsilon) = \min\{\delta'_y(\varepsilon),\ \delta'_z(\varepsilon),\ \delta'\},\ 0<\varepsilon\le\varepsilon^*.$$

Then, for any $x_0 \in X_0$, $0 < \varepsilon \le \varepsilon^*$ and $0 < \delta \le \delta(\varepsilon)$, we have

$$|\rho(t_0,x_0) - e_0(x_0)| \le \zeta,$$
$$\rho(\mathcal{U}^\varepsilon_{\Delta_\delta};t_0,x_0,y_0=x_0;\beta) \le \rho(t_0,x_0)+\zeta,$$
$$\rho(\mathcal{V}^\varepsilon_{\Delta_\delta};t_0,x_0,z_0=x_0;\beta) \ge \rho(t_0,x_0)-\zeta.$$

Taking (9), (14), and (15) into account, we conclude that control laws $\mathcal{U}^\varepsilon_{\Delta_\delta}$ and $\mathcal{V}^\varepsilon_{\Delta_\delta}$ are (ζ,β)-optimal.

6. SOFTWARE IMPLEMENTATION

Details of the software implementation of the procedure used to construct values (20) together with estimations of time complexities of the algorithms in use can be found in Kornev (2012); Kornev and Lukoyanov (2012).

In the implementation the so-called "pixel" approximation is used for compact sets. Each compact set is covered by a finite uniform ε-net. All elements of the set, which belong to the ε-neighborhood of a node from this net, are identified with that node and constitute the corresponding pixel.

To calculate $\Delta\psi_j(m)$ from (17) we solve auxiliary matrix games in mixed strategies by means of algorithms from Williams (1966).

7. NUMERICAL EXPERIMENTS

Consider two differential games for a dynamical system, described by the following dynamic equation

$$\ddot{x} = \frac{4\,u}{1+e^{\,8(t-2)}} + \frac{(u+v)^2}{2} + \frac{2\,v}{1+e^{\,8(3-t)}}, \quad 0 \le t < 4,$$
$$u\in\mathbb{U}=\{-1,1\},\ v\in\mathbb{V}=\{-1,1\}, \quad (31)$$

initial condition

$$x(0) = 0, \quad \dot{x}(0) = 0, \tag{32}$$

and quality indices

$$\gamma^{(1)} = \sqrt{x^2(1) + x^2(4)}, \tag{33}$$

$$\gamma^{(2)} = |x(1)| + |x(2) - 0.5| + |x(4)|. \tag{34}$$

The following values of the precision parameters were used in the experiment: the diameter of the partition Δ_δ of the control interval $[0, 4]$ was $\delta = 0.001$, pixel sizes $\Delta_m = \Delta_l = \Delta_\nu = 0.01$, $\Delta_\phi = \pi/360$. The a priori calculated value $e_0^{(1)}(x_0)$, which approximates the value of game (31)–(33), was equal to 0.3553, and the value $e_0^{(2)}(x_0)$ for game (31), (32), (34) was equal to 1.050. Three control simulations were performed. In the first one, the control actions of the both players were formed according to the control laws $\mathcal{U}^{\varepsilon\,(1)}_{\Delta_\delta}$, $\mathcal{V}^{\varepsilon\,(1)}_{\Delta_\delta}$ in the first game and $\mathcal{U}^{\varepsilon\,(2)}_{\Delta_\delta}$, $\mathcal{V}^{\varepsilon\,(2)}_{\Delta_\delta}$ in the second game. Under these conditions, the resulting values of the quality indices (33) and (34) were

$$\begin{aligned} \gamma_0^{(1)} &= \sqrt{(0.0237)^2 + (0.3551)^2} \approx 0.3559 \\ &\approx e_0^{(1)}(x_0) = 0.3553, \\ \gamma_0^{(2)} &= |0.005| + |-0.045 - 0.5| + |-0.516| \approx 1.066 \\ &\approx e_0^{(2)}(x_0) = 1.050. \end{aligned}$$

In the second simulation, the control actions of the first player were formed according to $\mathcal{U}^{\varepsilon\,(1)}_{\Delta_\delta}$ and $\mathcal{U}^{\varepsilon\,(2)}_{\Delta_\delta}$, respectively, and the control actions of the second player were random. Under these conditions the resulting values of the quality indices (33) and (34) were

$$\begin{aligned} \gamma_u^{(1)} &= \sqrt{(0.0237)^2 + (-0.0049)^2} \approx 0.0242 \\ &< e_0^{(1)}(x_0) = 0.3553, \\ \gamma_u^{(2)} &= |0.005| + |-0.046 - 0.5| + |0.034| \approx 0.585 \\ &< e_0^{(2)}(x_0) = 1.050. \end{aligned}$$

In the third simulation, the control actions of the first player were random, the control actions of the second player were formed according to control laws $\mathcal{V}^{\varepsilon\,(1)}_{\Delta_\delta}$ and $\mathcal{V}^{\varepsilon\,(2)}_{\Delta_\delta}$, respectively. The resulting values of the quality indices were

$$\begin{aligned} \gamma_v^{(1)} &= \sqrt{(0.4935)^2 + (8.2750)^2} \approx 8.2897 \\ &> e_0^{(1)}(x_0) = 0.3553, \\ \gamma_v^{(2)} &= |0.505| + |2.063 - 0.5| + |8.731| \approx 10.799 \\ &> e_0^{(2)}(x_0) = 1.050. \end{aligned}$$

The motion trajectories, that were obtained in the first (thick line), the second (thin line), and the third simulations (dashed line), are shown in the Fig. 1. The first plot is related to the game with the quality index $\gamma^{(1)}$, the second one is for the case of the quality index $\gamma^{(2)}$. Small circles on the trajectories show the state of the system at the instants of time of the motion quality evaluation.

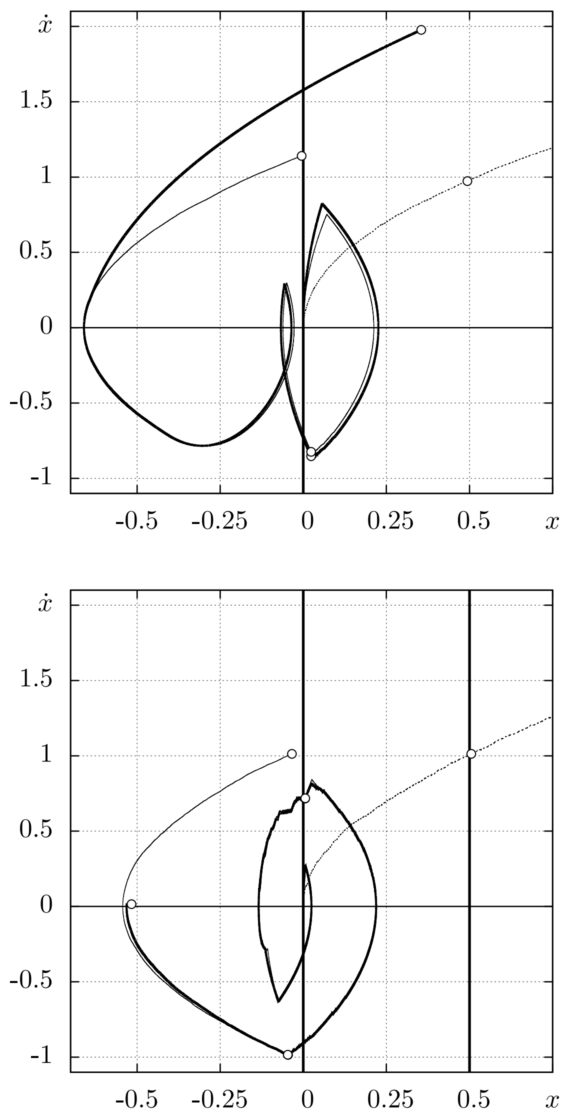

Fig. 1. The obtained motion trajectories.

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