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The Recursive Bayesian Estimation Problem via Orthogonal Expansions: an Error Bound

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Abstract When solving the non linear non Gaussian filtering problem via orthognal series expansions the involved probability density functions are approximated with truncated series expansions. Inevitable the truncation introduces an error. In this paper an upper bound on the 1-norm of the approximation error in the probability density function of the state vector conditional on the system output measurements, due to the truncation, is derived and numerically evaluated in a simulation example. The bound quantifies the proximity of the obtained approximate solution to the true one. To explore the choice of orthonormal basis as a degree of freedom in the proposed method, a comparison between the Fourier and Legendre bases in a bearings-only tracking problem is performed.

1. INTRODUCTION

Nonlinear non-Gaussian filtering problems arise in numerous signal processing and control applications such as communication, radar and sonar target tracking and satellite navigation, to mention a few.

The problem under consideration is to provide an estimate of the state vector $x_t \in \mathbb{R}^n$, given the measurements $Y_t = \{y_1, y_2, ..., y_t\}$, $y_t \in \mathbb{R}^p$, of the nonlinear discrete-time system

$$x_{t+1} = f(x_t) + w_t, \tag{1}$$

$$y_t = g(x_t) + v_t, \tag{2}$$

with the process and measurement noise $w_t \in \mathbb{R}^n$, $v_t \in \mathbb{R}^p$, respectively, and t denoting discrete time. The probability density functions (pdf:s) $p(w_t)$, $p(v_t)$ are assumed to be *known* but are allowed to have arbitrary form. A (known) input signal u_t is omitted here for brevity, but is straightforward to incorporate in the computations as a deterministic quantity.

In the Recursive Bayesian Estimation (RBE) framework, see e.g. Anderson and Moore (2012), Van Trees (2004), that is the general underlying problem in this setup, the state estimation problem above is solved by a recursive construction of the pdf $p(x_t|Y_t)$, via the recursive formula

$$p(x_t|Y_t) = \frac{p(y_t|x_t)}{p(y_t|Y_{t-1})} \times \int p(x_t|x_{t-1})p(x_{t-1}|Y_{t-1})dx_{t-1},\ t = 1, 2, \ldots, \tag{3}$$

where $p(x_t|Y_t)$ denotes the probability density for the state x_t given the measurements Y_t. For linear systems with white Gaussian process and measurement noise, the analytical solution to the RBE problem is given by the Kalman filter, Kalman (1960). For more general system structures, no analytical closed-form solution is available and approximative approaches have to be resorted to.

There are several commonly used approaches such as grid-based methods Jazwinski (1970), numerical integration methods Ito and Xiong (2000), and Monte-Carlo methods Doucet et al. (2001), Chen (2003), Budhiraja et al. (2007). Another appealing approach with roots in stochastic processes (see e.g. Cambanis (1971)) is to give an approximative solution to the recursive problem by expressing the pdf:s in a orthogonal series expansion.

In the RBE framework, the posterior pdf $p(x_t|Y_t)$ in (3) is then approximated by a truncated orthogonal series expansion

$$p(x_t|Y_t) \approx \hat{p}(x_t|Y_t) = \sum_{n=0}^{N} c_n^{t|t}\phi_n(x),$$

where $\{\phi_n(x)\}$ are the orthogonal basis functions and the coefficients $\{c_n^{t|t}\}$ are recursively computed via the prediction and update equations. Rosén and Medvedev (2013) provides the formulas for solving the problem in a general orthogonal basis while in Brunn et al. (2006), Hekler et al. (2010) particularly study solutions in the Fourier and wavelet bases.

While being of interest from an algorithmic point of view, the main strength of RBE in orthonormal bases comes from the computational properties and particularly its parallelizability. Owing to the orthogonality of the basis functions, the method has been shown to achieve linear speedup in the number of cores on a shared-memory multicore implementation, Rosén and Medvedev (2013). Since all high-performance and much of embedded hardware is nowadays based on parallel processing, parallelizability of an algorithm is highly desirable and crucial for software scalability.

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However, no analysis of how the approximation error propagates through the iterations, for the orthonormal basis approach in RBE, has yet been performed. A worst case scenario would be that the approximation errors, due to the truncations of the expansions, would accumulate in such a way that $\hat{p}(x_t|Y_t)$ is no longer a meaningful approximation of $p(x_t|Y_t)$.

The main contribution of this paper is to provide a bound on the 1-norm for the approximation error in the pdf of the state vector of (1) conditional on the measurements of (2), i.e. $\|e(x_t|Y_t)\|_1 = \|p(x_t|Y_t) - \hat{p}(x_t|Y_t)\|_1$. The derived bound, although not being sharp, serves as a tool to ensure that the estimated pdf represents a sensible approximation to the true pdf throughout the iterations. When solving the RBE with orthogonal series expansions there is an option of which basis functions to employ. A second contribution of the paper is a comparison of the method performance in a bearings-only tracking problem being solved with the Fourier and Legendre basis functions.

The paper structure is as follows. In Sec. 2, background material on the RBE by orthogonal series expansion is given. A bound on the approximation error of the state pdf is derived in Sec. 3. In Sec. 4 numerical experiments in a bearings-only tracking problem are described and a comparison between the Fourier and Legendre solutions is performed. A discussion of the results is given in Sec. 5, followed by the conclusions in Sec. 6.

2. BACKGROUND

Some background material and a method to solve RBE by series expansions are provided in this section. Here, to illustrate the ideas, only the one-dimensional problem will be treated for brevity. The proposed method can though be as extended to the multidimensional case as well.

2.1 Notation

Let $\{\phi_n(x)\}_{n=0}^{\infty}$ denote a set of complex basis functions orthogonal on the domain Ω w.r.t. the inner product

$$< f(x), g(x) > \int_{\Omega} f(x)\overline{g(x)}dx.$$

Any square integrable function $h(x)$ on Ω ($h \in \mathbb{L}_2(\Omega)$), can be expressed in terms of a series expansion in $\{\phi_n(x)\}_{n=0}^{\infty}$ with the coefficients $a_n =< h(x), \phi_n(x) >$, i.e.

$$\begin{aligned} h(x) &= \sum_{n=0}^{\infty} a_n\phi_n(x) \\ &= \underbrace{\sum_{n=0}^{N} a_n\phi_n(x)}_{\hat{h}(x)} + \underbrace{\sum_{n=N+1}^{\infty} a_n\phi_n(x)}_{e_h(x)}. \end{aligned}$$

In practice, the expansion is truncated to some order N. The resulting expansion and the corresponding approximation error are denoted $\hat{h}(x)$ and $e_h(x)$, respectively. When suitable, the notation $\hat{h}_N(x)$ will be utilized to stress the approximation order of the truncated series.

2.2 Solving the RBE via orthogonal series expansions

There exist many methods to solve the recursive Bayesian estimation problem. Two examples are the Kalman filter, which propagates the mean and covariance of the posterior distribution, and the Monte Carlo methods that propagate a sample from the posterior distribution. When solving the recursive Bayesian estimation problem with orthogonal basis functions, the posterior density is approximated as

$$p(x_t|Y_t) \approx \sum_{m=0}^{N} c_m^{t|t}\phi_m(x_t),$$

where the coefficients $\{c_m^{t|t}\}$ are propagated through the iterations. The expression $c_m^{t|t-1}$ shall be interpreted as the m:th coefficient at time step t given the measurements up to time $t-1$. As shown in Rosén and Medvedev (2013), the coefficients $c_m^{t|t}$ can be computed iteratively via the prediction and update equations as

$$c_n^{t|t-1} = \sum_{m=0}^{N} a_{mn} c_m^{t-1|t-1}, \tag{4}$$

$$f_p^t = \sum_{k=0}^{N} b_{pk}\phi_k(y_t), \tag{5}$$

$$c_l^{t|t} = \sum_{n=0}^{N}\sum_{p=0}^{N} f_p^t g_{npl} c_n^{t|t-1}, \tag{6}$$

where g_{nlp} is a coefficient in the orthogonal expansion of the product of the n:th and complex conjugated l:th basis functions

$$\phi_n(x)\overline{\phi_l}(x) = \sum_{p=0}^{N} g_{nlp}\phi_p(x)$$

and a_{mn} and b_{pk} are the coefficients in the truncated expansions

$$\hat{p}(x_t|x_{t-1}) = \sum_{m=0}^{N}\sum_{n=0}^{N} a_{mn}\phi_n(x_t)\overline{\phi_m}(x_{t-1}),$$

$$\hat{p}(y_t|x_t) = \sum_{p=0}^{N}\sum_{k=0}^{N} b_{pk}\phi_k(y_t)\phi_p(x_t),$$

of the pdf:s $p(x_t|x_{t-1})$, $p(y_t|x_t)$ that are implicitly defined by the system model. Notice that the approach above requires far fewer expansion terms to approximate the pdf than e.g. a grid (in grid based methods) or a random sample (Monte Carlo methods). This fact lays the ground to the relatively low computational cost of the method.

2.3 Legendre and Fourier basis functions

In the example presented in Sec. 4, the filtering problem is solved by making use of the Legendre (Koekoek et al. (2012)) and Fourier basis functions (Vretblad (2010)). The Legendre basis functions $\{L_n(x)\}$ are orthogonal over the interval $[-1, 1]$ and recursively defined by Bonnet's formula as

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= x \\ L_{n+1}(x) &= \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x), \end{aligned}$$

$n = 1, 2, ..., N-1$. The rescaled and normalized polynomials, $\tilde{L}_n(x) = \frac{2}{\sqrt{a(2n+1)}} L_n(\frac{x}{a})$, are orthonormal over the interval $[-a, a]$, $a \in \mathbb{R}$. The complex Fourier basis functions $\{\varphi_n(x)\}$, orthonormal over the interval $[a-\pi, a+\pi]$ for any $a \in \mathbb{R}$, are given by

$$\varphi_0(x) = 1$$
$$\varphi_{2n-1}(x) = \overline{\varphi_{2n}}(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$$

$n = 1, 2, .. \frac{N-1}{2}$.

3. AN ERROR BOUND

In this section, an expression for the approximation error in an orthogonal expansion of the recursively computed posterior pdf and a 1-norm bound on it are derived. The non-normalized recursion expressed by (3) can be written in the form

$$g^{t+1}(z) = v(y|z) \int_\Omega f(z|x) g^t(x) dx,\ t = 0, 1, \ldots, \tag{7}$$

where $v(y|z)$, $f(z|x)$ and $g^t(x)$ are pdf:s. For tractability, the equations with the notation in (7) rather than (3) will be studied. In this problem formulation, the pdf $g^t(z)$ corresponds to $p(x_t|Y_t)$ in (3) and is the main target of the approximation. When solving the recursion with orthogonal basis expansions, the truncated expansions $\hat{v}(y|z)$, $\hat{f}(z|x)$ and $\hat{g}^t(x)$ are used in place of the true pdf:s. It is of interest to know how the error caused by the truncation propagates through the iterations. An expression for the approximation error $e_g^{t+1}(z) = g^{t+1}(z) - \hat{g}^{t+1}(z)$ is therefore sought. Assuming that $g(x)$ has the same approximation order in the x-dimension as $f(z|x)$ does, the following two relations hold in virtue of the orthogonality of the basis functions

$$\int_\Omega \hat{f}(z|x) e_g(x) dx = 0,$$
$$\int_\Omega e_f(z|x) g(x) dx = \int_\Omega e_f(z|x) e_g(x) dx.$$

Then it follows that

$$\begin{aligned}
\hat{g}^{t+1}(z) &= \hat{v}(y|z) \int_\Omega \hat{f}(z|x) \hat{g}^t(x) dx \\
&= \hat{v}(y|z) \int_\Omega \hat{f}(z|x) [g^t(x) - e_g^t(x)] dx \\
&= \hat{v}(y|z) \int_\Omega \hat{f}(z|x) g^t(x) dx - \hat{v}(y|z) \int_\Omega \hat{f}(z|x) e_g^t(x) dx \\
&= [v(y|z) - e_v(y|z)] \int_\Omega [f(z|x) - e_f(z|x)] g^t(x) dx \\
&= v(y|z) \int_\Omega f(z|x) g^t(x) dx - v(y|z) \int_\Omega e_f(z|x) g^t(x) dx \\
&\quad - e_v(y|z) \int_\Omega [f(z|x) - e_f(z|x)] g^t(x) dx \\
&= g^{t+1}(z) - v(y|z) \int_\Omega e_f(z|x) e_g^t(x) dx \\
&\quad - e_v(y|z) \int_\Omega f(z|x) g^t(x) dx + e_v(y|z) \int_\Omega e_f(z|x) e_g^t(x) dx \\
&= g^{t+1}(z) - [v(y|z) - e_v(y|z)] \int_\Omega e_f(z|x) e_g^t(x) dx \\
&\quad - e_v(y|z) \int_\Omega f(z|x) g^t(x) dx.
\end{aligned}$$

This gives the expression for the approximation error

$$\begin{aligned}
e_g^{t+1}(z) &= g^{t+1}(z) - \hat{g}^{t+1}(z) \\
&= \hat{v}(y|z) \int_\Omega e_f(z|x) e_g^t(x) dx + \\
&\quad e_v(y|z) \int_\Omega f(z|x) g^t(x) dx.
\end{aligned} \tag{8}$$

From (8) the following result can be derived:

Theorem 1. For $e_g^t(z)$ given by (8), it holds that $\left\| e_g^t(z) \right\|_1 \le \gamma_t$, $t = 0, 1, \ldots,$ where

$$\gamma_t = \begin{cases} r^t Q^t \left\| e_g^0 \right\|_1 + Rq \dfrac{1 - r^t Q^t}{1 - rQ} & \text{if } rQ \ne 1 \\ \left\| e_g^0 \right\|_1 + tRq & \text{if } rQ = 1 \end{cases} \tag{9}$$

and

$$\begin{aligned}
Q &:= \max_y \int |\hat{v}(y|z)| dz \\
q &:= \max_y \int |e_v(y|z)| dz \\
r &:= \max_{x,z} |e_f(z|x)| \\
R &:= \max_{x,z} f(z|x)
\end{aligned}$$

Proof. The triangle inequality yields

$$\begin{aligned}
&\left\| e_g^{t+1}(z) \right\|_1 = \int_\Omega |e_g^{t+1}(z)| dz = \\
&\int_\Omega |\hat{v}(y|z) \int_\Omega e_f(z|x) e_g^t(x) dx \\
&+ e_v(y|z) \int_\Omega f(z|x) g^t(x) dx | dz \le \\
&\int_\Omega [|\hat{v}(y|z)| \int_\Omega |e_f(z|x)| |e_g^t(x)| dx \\
&+ |e_v(y|z)| \int_\Omega |f(z|x)| |g^t(x)| dx] dz \le \\
&\int_\Omega [|\hat{v}(y|z)| r \int_\Omega |e_g^t(x)| dx + |e_v(y|z)| R \int |g^t(x)| dx] dz = \\
&\int_\Omega |\hat{v}(y|z)| dz \cdot r \int_\Omega |e_g^t(x)| dx + \int_\Omega |e_v(y|z)| dz R \le \\
&rQ \left\| e_g^t(z) \right\|_1 + Rq
\end{aligned}$$

i.e.

$$\left\| e_g^{t+1}(z) \right\|_1 \le rQ \left\| e_g^t(z) \right\|_1 + Rq. \tag{10}$$

The right hand side in (10) is a monotonically increasing function in $\left\| e_g^t(z) \right\|_1$. An upper bound γ_t on $\left\| e_g^t(z) \right\|_1$ hence obeys the recursion $\gamma_{t+1} = rQ\gamma_t + Rq$, whose closed-form expression is given by Eq. (9).

Note that q, Q, r and R in (9) only depend on constant quantities that can be computed offline and before the recursion starts.

Corollary 2. If $rQ \le 1$, $\left\| e_g^k(z) \right\|_1$ is asymptotically bounded by $\frac{Rq}{1-rQ}$.

Proof. If $rQ < 1$

$$\lim_{t \to \infty} \gamma_t = \lim_{t \to \infty} r^t Q^t \left\| e_g^0(z) \right\|_1 + Rq \frac{1 - r^t Q^t}{1 - rQ} = \frac{Rq}{1 - rQ} \tag{11}$$

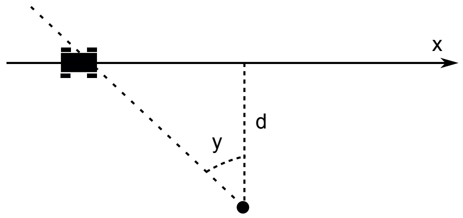

Figure 1. An object with position x_t traveling along a path. Noisy bearing measurements, Y_t, are taken by a sensor (black dot), positioned a distance d from the path.

4. NUMERICAL EXPERIMENTS

A nonlinear non-Gaussian bearings-only tracking problem is studied. It arises in defense and surveillance applications as well as in robotics. It exhibits a severe non-linearity in the measurement equation and is known to require nonlinear filtering to avoid divergence of the estimate, Aidala (1979). For comparison, the filtering problem is solved both with the Fourier basis functions and the Legendre basis functions. Numerical experiments were conducted in order to experimentally verify the error bound derived in the previous section, and also to explore its conservatism.

4.1 The system

An object travelling along a path is detected within the range $x_t \in [-\pi, \pi]$. Noisy bearing measurements y_t of its position x_t are taken by a sensor stationed at a distance $d = 1$ from the road, see Fig. 1. The tracking filter employs the model

$$x_{t+1} = x_t + w_t \tag{12}$$

$$y_t = \tan^{-1}(x_t/d) + v_t, \tag{13}$$

where w_t is normally distributed with the mean $\mu_w = 0$ and standard deviation $\sigma_w = 0.3$. The measurement noise v_k obeys the multi-modal pdf

$$p_v(v) = \frac{p_1}{\sigma_{v_1}\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{v-\mu_{v_1}}{\sigma_{v_1}})^2} + \frac{p_2}{\sigma_{v_2}\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{v-\mu_{v_2}}{\sigma_{v_2}})^2}$$

with $p_1 = 0.5$, $p_2 = 0.5$, $\sigma_{v_1} = 0.3$, $\sigma_{v_2} = 0.3$, $\mu_{v_1} = 0.45$, $\mu_{v_2} = -0.45$. The system was simulated up to time step $T = 40$.

4.2 Solution with orthogonal basis functions

The filtering problem to estimate the pdf $p(x_t|Y_t)$ for system (12)-(13) was solved by using the Legendre and Fourier basis functions, as described in Sec. 2. The estimated pdf:s obtained by the orthogonal series method were cross-validated against the results obtained by applying a particle filter to the same data set, to ensure correct implementation.

The filtering problem was solved for the approximation orders $N = 9 + 4k$, $k = 0, 1, \ldots, 14$. The upper bound $\gamma_t(N)$ on $\|e(x_t|Y_t)\|_1$ was computed according to (9) for each N using both the Fourier and Legendre basis while the empirical values of $||e(x_t|Y_t)||_1$ were evaluated as

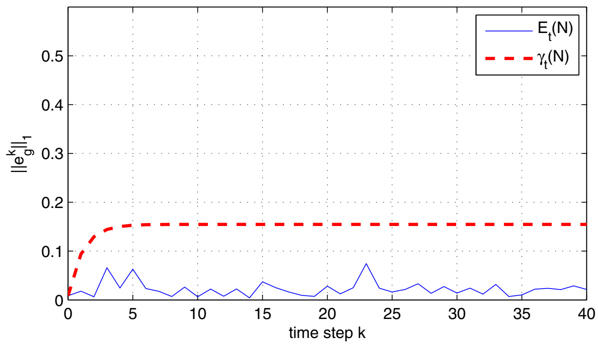

Figure 2. Theoretical bound $\gamma_t(N)$ and empirically measured values of the approximation error in 1-norm, $E_t(N)$ for the solution obtained with the Fourier basis functions and approximation order $N = 25$.

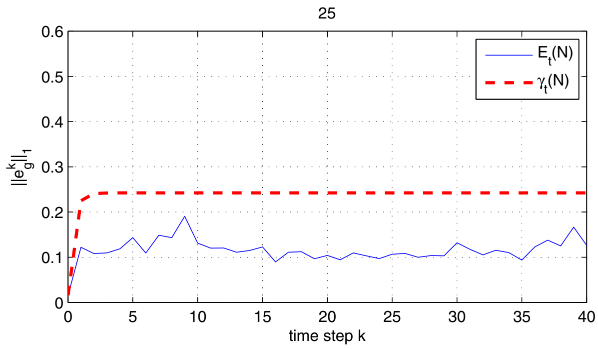

Figure 3. Theoretical bound $\gamma_t(N)$ and empirically measured values of the approximation error in 1 norm, $E_t(N)$ for the solution obtained with the Legendre basis functions and approximation order $N = 25$.

$$\|e(x_t|Y_t)\|_1 \approx E_t(N) = \int_{x_t \in \Omega} |\hat{p}_{65}(x_t|Y_t) - \hat{p}_N(x_t|Y_t)| dx,$$

where $\hat{p}_N(x_t|Y_t)$ denotes the approximation of $p(x_t|Y_t)$ of the approximation order N. As $\hat{p}_{65}(x_t|Y_t)$ can be considered a very close approximation to the true pdf $p(x_t|Y_t)$, $E_t(N)$ can be deamed a good approximation to $\|e(x_t|Y_t)\|_1$.

In Fig. 2 and Fig. 3, the empirical and theoretical bounds $E_t(N)$ and $\gamma_t(N)$ are shown for $N = 25$, using the Fourier basis and the Legendre basis respectively, where $\gamma_t(N)$ denotes the theoretical bound for an approximation order N.

For all N studied, the bound converges to the value given by (11) and the value of $\gamma_t(N)$ is basically constant after time step $t = 10$ in all cases. To illustrate the empirical and and theoretical bound for each N, the steady-state value $\gamma_{30}(N)$, the mean and maximum of the empirical value $E_t(N)$

$$\mu(N) = \frac{1}{30}\sum_{t=11}^{40} E_t(N),$$

$$\rho(N) = \max_{t \in [11,40]} E_t(N)$$

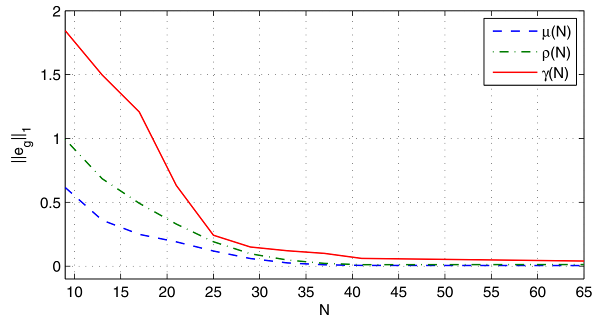

Figure 4. Theoretical bound $\gamma(N)$, the mean $\mu(N)$ and maximum $\rho(N)$ of the empirically measured values of $E_t(N)$, when solving the problem with Legendre basis functions.

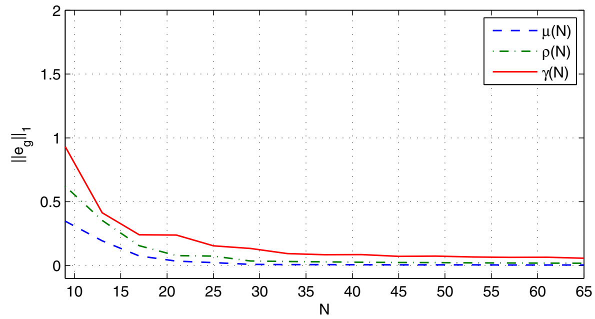

Figure 5. Theoretical bound $\gamma(N)$, the mean $\mu(N)$ and maximum $\rho(N)$ of the empirically measured values of $E_t(N)$, when solving the problem with Fourier basis functions.

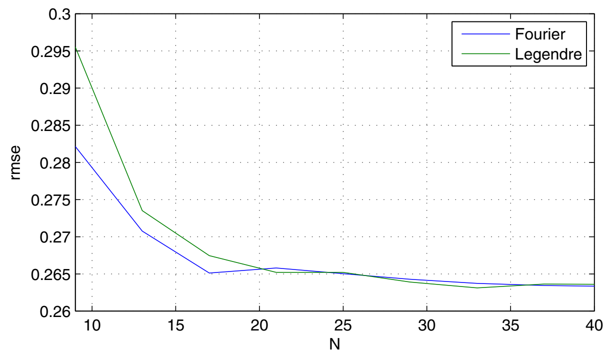

Figure 6. The root mean square error, for the estimation error as a function of the approximation order N.

were computed on the stationary interval $t \in [11, 40]$. The results are shown for the Fourier basis and the Legendre basis in Fig. 4 and Fig. 5, respectively.

Point estimates, $\hat{x}_t = E[x_t|Y_t]$ from the approximated pdf:s were computed. To compare and quantify the estimation quality, the root mean square error

$$E_{rmse}(\hat{x}_{1:T}) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(x_t - \hat{x}_t)^2}$$

was calculated for the estimated states and is shown in Fig. 6 for different approximation orders N, for the Fourier and Legendre basis functions.

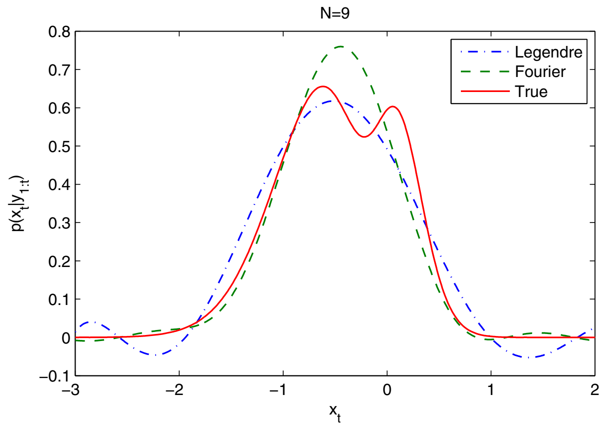

Figure 7. The true pdf $p(x_t|Y_t)$ and $\hat{p}_9(x_t|Y_t)$ for $t = 25$, for the Fourier and Legendre solutions, $N = 9$.

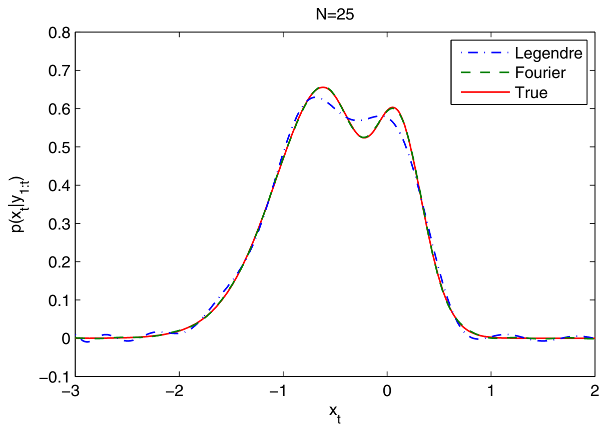

Figure 8. The true pdf $p(x_t|Y_t)$ and $\hat{p}_{25}(x_t|Y_t)$ for $t = 25$, for the Fourier and Legendre solutions, $N = 25$.

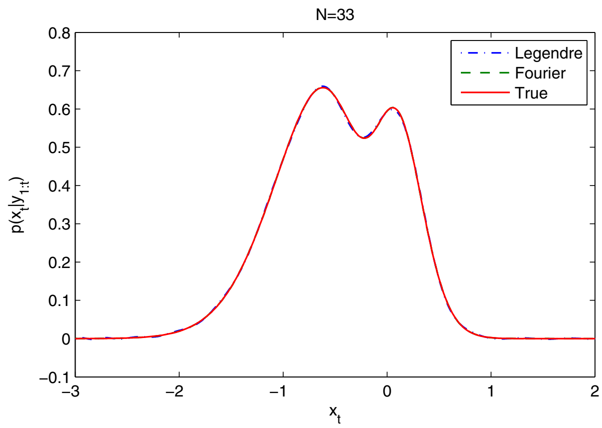

Figure 9. The true pdf $p(x_t|Y_t)$ and $\hat{p}_{33}(x_t|Y_t)$ for $t = 25$, for the Fourier and Legendre solutions, $N = 33$.

For the particular time instant $t = 25$, the true pdf $p(x_t|Y_t)$ and the estimated pdf:s $\hat{p}(x_t|Y_t)$ obtained with the Fourier and Legendre basis functions are shown for $N = 9, 25, 33$ in Fig. 7, Fig. 8 and Fig. 9, respectively.

5. DISCUSSION

In the studied bearings-only tracking problem, it can be concluded that the Fourier basis functions generally give a better approximation to the problem than the Legendre basis functions do, which phenomenon is especially prominent for lower approximation orders N. It can be seen that for low N, ($N = 9$, Fig. 7), both the Fourier and Legendre basis functions fail to capture the multi-modal shape of the true density. Yet the Fourier basis based solution yields a closer approximation than that of Legendre functions, measured in the 1-norm of the approximation error. When N is in the medium range ($N = 25$, Fig. 8), the Fourier basis solution gives an almost perfect approximation, while the Legendre functions still show some difficulties in fully capturing the multi-modality of $p(x_t|Y_t)$. For high approximation orders ($N = 33$, Fig. 9), both the Legendre and Fourier bases produce close to perfect approximations.

However, as can be seen from Fig. 6, a better pdf fit does not necessarily translate into a superior point estimate of the state $\hat{x}_t$. The root mean square error for the Fourier and Legendre solutions are practically the same for $N \geq 20$, even though the Fourier basis solution provides a better fit of the actual underlying pdf.

Another aspect that should be taken into account is the numerical properties of the basis functions. With the Legendre basis functions it is not possible, in the given implementation, to go above $N = 65$ due to numerical problems, while no numerical problems are encountered using the Fourier basis functions. However, as virtually perfect approximation is reached already for $N = 33$, it is not an issue with the Legendre basis solution in this case.

From Fig. 4 and Fig. 5, the bound can be seen to be close to tight for some N values, but more conservative for other N values. For the Legendre case, the bound is conservative for small values of N as a consequence of the poorly approximated pdf:s $p(x_t|x_{t-1})$ and $p(y_t|x_t)$ in some intervals. The bound accounts for the worst case effects of this poor approximation, which scenario does not apparently realize in the final estimate, for the particular problem and implementation at hand.

In the derivation of the bound the inequality

$$\int f(z|x)g(x)dx \leq \max_{x,z} f(z|x)$$

was used. This relationship holds if f and g are pdf:s, but appears in some cases to be a rather conservative bound. By imposing assumptions on e.g. the smoothness of f and g, this bound can be tightened and hence bring about an improvement of the final bound.

6. CONCLUSIONS

A bound on the 1-norm of the approximation error in the estimate of the posterior distribution in the recursive Bayesian estimation problem solved by means of orthogonal series expansions is derived. The bound is instrumental in preventing unlimited growth of the approximation error that renders the approximative solution useless. A comparison between the use of Fourier and Legendre basis functions is also conducted with the conclusion that the Fourier basis provides a more suitable option for the particular bearings-only tracking problem under consideration.

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