Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Stability Analysis of Switched Systems with 'Mixed'-Negative Imaginary Property

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Abstract: This paper discusses the stability of feedback systems in which both plant and controller are switched. Switched systems considered here have all their subsystems satisfying the 'mixed'-negative imaginary property. A definition for dissipativity (for switched systems) is proposed, and dissipative switched systems are shown to be stable (under certain conditions). Switched systems with 'mixed'-negative imaginary property are shown to be dissipative and conditions for stability are derived. As an illustration of the results, a switched controller is designed for a nanopositioning stage, which has a 'mixed'-negative imaginary frequency response function. Simulations show that the closed loop is stable and the designed controller damps the resonances satisfactorily.

1. INTRODUCTION

Highly resonant systems with collocated and compatible sensors and actuators have negative imaginary transfer functions (Preumont [2011]). Systems which are approximately collocated or collocated systems with time delays may not be negative imaginary at all frequencies but behave as a negative imaginary system over a significant bandwidth. Such systems are said to possess 'mixed'-negative imaginary property, *i.e*, they are negative imaginary in certain frequency intervals and have finite gain in the rest. As an example, consider a highly resonant LTI SISO system with transfer function,

$$\hat{M}(s) = \frac{2154}{s^2 + 29s + 3.367 \times 10^3}. \tag{1}$$

The Nyquist plot (Figure 1) of (1) shows that it is negative imaginary (*i.e.,* the imaginary part of the frequncy response is negative at all frequencies). But with a time delay of 6.0675×10^{-3} added to the system, it loses the negative imaginary property beyond a frequency value of 88.3 rad/sec (see Figure 1). Thus the resonant system with delay satisfies the 'mixed'-negative imaginary property.

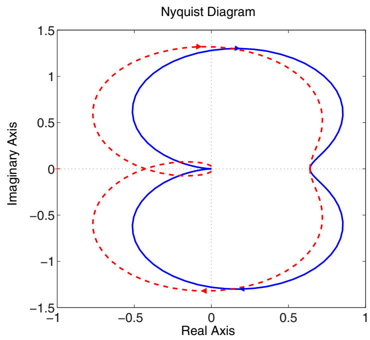

Fig. 1. Nyquist plots of $\hat{M}(s)$ without delay (solid line) and with delay (dotted line)

Feedback systems with 'mixed'-negative imaginary property have been considered in Patra and Lanzon [2011], where the stability of closed loop was proved in the frequency domain. This restricts the concept of 'mixed' property to linear time-invariant systems. In this paper, the time domain version of the 'mixed'-negative imaginary property is obtained which makes it possible to extend the theory to nonlinear systems.

Many physical systems require switched controllers for their satisfactory performance (see Bazaei et al. [2011], Sankaranarayanan et al. [2008]). A switched system is a collection of a finite number of dynamical systems called subsystems. The dynamics of a switched system evolves by switching from one subsystem to another. The stability of switched systems has been studied extensively by many authors (Liberzon and Morse [1999], Branicky [1994]). In Zefran et al. [2001], stability of passive switched systems and their feedback interconnections was analysed. But switched systems with negative imaginary property or 'mixed'-negative imaginary property have not been considered before.

This paper focuses on switched systems with subsystems being 'mixed'-negative imaginary. Stability of a 'mixed' switched system in positive feedback with another 'mixed' switched system is investigated. Since systems with 'mixed' property are known to be dissipative, the notion of dissipativity is used to prove stability. A switched controller is designed for a highly resonant MIMO system and the closed loop stability is verified through simulations.

Section 2 states the required notations. Section 3 and Section 4 explain the concept of 'mixed'-negative imaginary property in the frequency domain and time domain respectively. Section 5 discusses dissipative switched systems. The final stability results on feedback interconnection of 'mixed' switched systems are derived in section 6. Section 7 illustrates an example with supporting simulations.

2. NOTATIONS

2.1 Notations in frequency domain

Consider the real frequency domain Lebesgue space, $\mathscr{L}_2(j\mathbb{R})$ with an inner product $\left\langle \hat{f}, \hat{g} \right\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}^*(j\omega)\hat{f}(j\omega)d\omega$, where $\hat{f}, \hat{g} \in \mathscr{L}_2(j\mathbb{R})$ and the superscript $*$ denotes the complex conjugate. The norm of $\hat{f} \in \mathscr{L}_2(j\mathbb{R})$ is given by $\| \hat{f} \| = \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}^*(j\omega)\hat{f}(j\omega)d\omega \right\}^{1/2}$. Let $\mathscr{R}$ denote the set of proper real rational transfer function matrices.

2.2 Notations in time domain

Consider the real time domain Lebesgue space, $\mathcal{L}_2[0,\infty)$ with the inner product $\langle f, g \rangle = \int_0^{\infty} g'(t)f(t)dt$, where $f, g \in \mathcal{L}_2[0,\infty)$ and the superscript $'$ denotes the transpose. The norm of $f \in \mathcal{L}_2[0,\infty)$ is given by $\| f \| = \left\{ \int_0^{\infty} f'(t)f(t)dt \right\}^{1/2}$. For a linear operator $A : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$, its adjoint $A^{\sim} : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ is defined by $\langle Ap, q \rangle = \langle p, A^{\sim}q \rangle, \quad \forall\ p, q \in \mathcal{L}_2[0,\infty)$. The truncation operator P_T is defined as

$$(P_T f)(t) = \begin{cases} f(t), & 0 \le t \le T \\ 0, & t > T, \end{cases} \qquad \text{for } T \in [0,\infty).$$

Let the truncated function be denoted by $f_T := P_T f$. Also define a scalar function $\langle f, g \rangle_T = \langle f_T, g_T \rangle$. Let $\mathcal{L}_{2e}$ be the extension of the Lebesgue space $\mathcal{L}_2[0,\infty)$ defined by $\mathcal{L}_{2e} = \{f : f_T \in \mathcal{L}_2[0,\infty), \quad \forall\ T \in [0,\infty)\}$. In the current context a system is defined as an operator from $\mathcal{L}_{2e}$ to $\mathcal{L}_{2e}$.

3. 'MIXED' PROPERTY IN FREQUENCY DOMAIN

Consider a causal LTI system with transfer function matrix $\hat{M} \in \mathscr{R}$. If $j(\hat{M}(j\omega) - \hat{M}^*(j\omega)) \ge 0, \ \forall\ \omega$ then the system is said to possess negative imaginary property.
Divide the frequency domain $-\infty < \omega < \infty$ into intervals $\{I_s^{(i)}\}_{i=-N_1}^{N_1}$ and $\{I_p^{(i)}\}_{i=-N_2}^{N_2}$ such that $I_s = \bigcup_{i=-N_1}^{N_1} I_s^{(i)}$, $I_p = \bigcup_{i=-N_2}^{N_2} I_p^{(i)}$ and $\mathbb{R} = I_s \bigcup I_p$. Let I_p be the intervals where $\hat{M}$ is negative imaginary and I_s be the intervals where $\hat{M}$ is not negative imaginary. Let $k > 0$ be such that $\max_{\omega \in I_s} \| \hat{M} \| < k$. Then $\hat{M}$ is said to have a k small gain in the intervals I_s. By abuse of language, this is referred to as small gain in the intervals I_s. Since $\hat{M}$ is negative imaginary in I_p and small gain in I_s, $\hat{M}$ has the 'mixed'-negative imaginary property. Note that the zero frequency is always included in set I_s, *i.e*, in the small gain frequency intervals and hence k has to be greater than or equal to $\bar{\sigma}(\hat{M}(0))$ which is the dc gain of the system. This is because the value of transfer function evaluated at the zero frequency is real.

4. 'MIXED' PROPERTY IN TIME DOMAIN

The concept of 'mixed' property is well understood in the frequency domain. But then the analysis would be limited to only linear systems. In order to extend the notion of 'mixed' property to general nonlinear systems, time domain definitions are required. For linear systems, time domain definitions are equivalent to frequency domain definitions (refer to Patra and Lanzon [2011] for frequency domain definitions) through Fourier transform. The conversion of 'mixed'-negative imaginary property from frequency domain to time domain is done (definition 3) in this paper (refer to Appendix A). Though the conversion is for linear systems, the time domain definitions can be generalised to nonlinear systems.

Definition 1. A causal system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to be negative imaginary if $\exists \quad \delta \ge 0, \ \ \epsilon \ge 0$ such that

$$\left\langle \dot{M}u, u \right\rangle_T \ \ge \ \delta \| u \|_T^2 + \ \epsilon \| \dot{M}u \|_T^2,$$

$$\forall\ u \in \mathcal{L}_{2e} \quad \text{and} \quad \forall\ T \in [0,\infty).$$

Note that $\dot{M}u \triangleq \frac{\partial y}{\partial t}$, where $y = M(u)$.

Definition 2 (Griggs et al. [2009]). A causal system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to have finite gain if $\exists$ a k $(0 < k < \infty)$such that

$$\| Mu \|_T \ \le \ k \| u \|_T, \quad \forall\ u \in \mathcal{L}_{2e}, \ \forall\ T \in [0,\infty).$$

Definition 3. Let $\Gamma : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ and $\mathrm{B} : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ be causal, bounded, linear operators such that $\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B} = \mathrm{I}$.
A causal system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to have 'mixed'-negative imaginary property if $\exists \quad \delta \ge 0, \ \ \epsilon \ge 0$ and k such that

$$\begin{aligned} &-\epsilon \left\langle \dot{\Gamma}Mu, \dot{\Gamma}Mu \right\rangle_T - \frac{1}{k}\langle \mathrm{B}Mu, \mathrm{B}Mu \rangle_T - \left\langle \Gamma Mu, \dot{\Gamma}u \right\rangle_T \\ &+k \langle \mathrm{B}u, \mathrm{B}u \rangle_T - \delta \langle \Gamma u, \Gamma u \rangle_T \ge 0, \end{aligned}$$

$$\forall\ u \in \mathcal{L}_{2e} \quad \text{and} \quad \forall\ T \in [0,\infty), \quad \text{where} \quad 0 < k < \infty.$$

If $\Gamma = 0$, M has finite gain and if $\mathrm{B} = 0$, M is negative imaginary.

Definition 4 (Hill and Moylan [1980]). A causal system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to be dissipative with respect to the triple (Q, S, R) if

$$\langle y, Qy \rangle_T + 2\langle y, Su \rangle_T + \langle u, Ru \rangle_T \ \ge \ 0,$$

$\forall\ u \in \mathcal{L}_{2e}, \forall\ T \in [0,\infty)$. Here $y = Mu$, Q and R are self adjoint, that is, $Q' = Q$ and $R' = R$.

Proposition 1. Consider a causal system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$. Let $\Gamma : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ and $\mathrm{B} : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ be causal, bounded, linear, operators such that $\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B} = \mathrm{I}$. If M has 'mixed'-negative imaginary property, then the system is dissipative with respect to the triple (Q, S, R) where

$$\begin{aligned} Q &= -(\epsilon\dot{\Gamma}^{\sim}\dot{\Gamma} + \frac{1}{k}\mathrm{B}^{\sim}\mathrm{B})\mathrm{I}, \\ S &= -\frac{1}{2}\Gamma^{\sim}\dot{\Gamma}\mathrm{I}, \quad \text{and} \\ R &= (k\mathrm{B}^{\sim}\mathrm{B} - \delta\Gamma^{\sim}\Gamma)\mathrm{I}. \end{aligned} \tag{2}$$

Here $\epsilon \ge 0$, $\delta \ge 0$ and $0 < k < \infty$. Note that $k \ge \bar{\sigma}(\hat{M}(0))$.

Proof. Follows from Definition 3.

5. SWITCHED SYSTEMS

A switched system H consists of a finite number of subsystems, $\{M_i : \mathcal{L}_{2e} \to \mathcal{L}_{2e}\}$ with state space representations:

$$\begin{aligned} \dot{x} &= f_i(x,u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \\ y &= h_i(x,u), \quad y \in \mathbb{R}^m, \end{aligned}$$

where $i \in I = \{1, 2, \dots, N\}$ is a finite index set and each $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. Also, $f_i(0, 0) = 0$ and $h_i(0, 0) = 0, \forall\, i \in I$. The system switches from one subsystem to another in accordance to a switching logic. It is assumed that there are finitely many switches in any finite time interval.

Definition 5. A switched system H is said to be dissipative if the following two conditions are satisfied:

(1) Each subsystem $M_i : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is dissipative with respect to the triple (Q_i, S_i, R_i), *i.e*,

$$\begin{aligned}&\langle y, Q_i y\rangle_{[s,r]} + 2\,\langle y, S_i u\rangle_{[s,r]} + \langle u, R_i u\rangle_{[s,r]}\\ \geq\;& V_i(x(r)) - V_i(x(s)), \qquad \forall\, u \in \mathcal{L}_{2e},\end{aligned} \tag{3}$$

where $\{x(t); s \leq t \leq r\}$ are the states when the subsystem M_i is active, and V_i is a positive semidefinite function called the storage function (refer to Theorem 4 in Hill and Moylan [1980]) associated with M_i. Here, $\forall\;\, x, y \in \mathcal{L}_{2e}$, $\langle x, y\rangle_{[s,r]} = \displaystyle\int_s^r y'(t)x(t)dt$.

(2) The storage functions of each subsystem have the property:

$$\begin{aligned}V_i(x(t_{i,k})) \;&\leq\; V_i(x(t_{i,k-1})) + \sum_{p=1}^{L}(\langle y, Q_{n_p} y\rangle_{[t_{p_1},t_{p_2}]}\\ &+2\,\langle y, S_{n_p} u\rangle_{[t_{p_1},t_{p_2}]} + \langle u, R_{n_p} u\rangle_{[t_{p_1},t_{p_2}]}),\\ \forall\, u \in \mathcal{L}_{2e},&\end{aligned} \tag{4}$$

Here $t_{i,k-1}$ and $t_{i,k}$ are the $(k-1)^{th}$ and k^{th} time at which i^{th} subsystem becomes active respectively. L is the number of switchings from $t_{i,k-1}$ to $t_{i,k}$ (including the switching at $t_{i,k-1}$) such that $\displaystyle\bigcup_{p=1}^{L}[t_{p_1}, t_{p_2}] = [t_{i,k-1}, t_{i,k}]$.

In the interval $[t_{p_1}, t_{p_2}]$, one of the subsystems (say, n_p^{th} subsystem) of H will be active, *i.e*, n_p can be any number belonging to the index set $I = \{1, 2, \dots, N\}$.

Note: In Zefran et al. [2001], the condition on storage function, for passive switched system, was defined as $V_i(x(t_{i,k})) \;\leq\; V_i(x(t_{i,k-1})) + \displaystyle\int_{t_{i,k-1}}^{t_{i,k}} u^T y dt$. Similarly, for negative imaginary switched system, the condition can be defined as $V_i(x(t_{i,k})) \;\leq\; V_i(x(t_{i,k-1})) + \displaystyle\int_{t_{i,k-1}}^{t_{i,k}} u^T \dot{y} dt$. Equation 4 gives a more comprehensive definition for a dissipative or 'mixed' switched system which incorporates both the properties of small gain and negative imaginary.

Proposition 2. Consider a dissipative switched system H according to Definition 5. Let the storage functions $V_i(x)$ be positive definite. Then the origin $x = 0$ of the zero input system $(u(t) = 0)$ is Lyapunov stable if $Q_i \leq 0, \forall\;\, i \in I$.

Proof. Substituting $u = 0$ in property (3) gives,

$$\langle y, Q_i y\rangle_{[s,r]} \;\geq\; V_i(x(r)) - V_i(x(s)).$$

If $Q_i \leq 0$, then $V_i(x(r)) \;\leq\; V_i(x(s))$, where $r \geq s$. $\therefore\ \dfrac{dV_i}{dt} \leq 0$. Hence, the storage functions, V_i, act as Lyapunov functions. When $u = 0$, property (4) gives

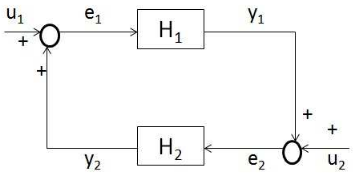

Fig. 2. Positive feedback interconnection of H_1 and H_2

$$\begin{aligned}V_i(x(t_{i,k})) \;&\leq\; V_i(x(t_{i,k-1})) + \sum_{p=1}^{L}\langle y, Q_{n_p} y\rangle_{[t_{p_1},t_{p_2}]}\\ \Leftrightarrow\quad V_i(x(t_{i,k})) \;&\leq\; V_i(x(t_{i,k-1})).\ (\because Q_{n_p} \leq 0,\ \ \forall\; n_p \in I)\end{aligned}$$

Therefore the system is stable according to proposition 1 in Zefran et al. [2001].

6. FEEDBACK SYSTEM

Theorem 3. The positive feedback interconnection of two dissipative switched systems H_1 and H_2 (Figure 2) is also dissipative.

Proof. Let $\{M_i^1\}$ and $\{M_i^2\}$ be the subsystems of H_1 and H_2 respectively. By property (3),

$$\begin{aligned}&\langle y_a, Q_i^a y_a\rangle_{[s,r]} + 2\,\langle y_a, S_i^a e_a\rangle_{[s,r]} + \langle e_a, R_i^a e_a\rangle_{[s,r]} \;\;\geq\\ &V_i^a(x_a(r)) - V_i^a(x_a(s)), \qquad \forall\, i \in I_a,\;\ \forall\; e_a \in \mathcal{L}_{2e},\end{aligned}$$

where $\{x_a(t); s \leq t \leq r\}$ are the states when the subsystem M_i^a is active and $a = 1, 2$. $I_1 = \{1, 2, \dots, N_1\}$ and $I_2 = \{1, 2, \dots, N_2\}$ are the index sets of subsystems of H_1 and H_2 respectively.

The equations for the interconnected system with positive feedback are $e_1 = u_1 + y_2$, $e_2 = u_2 + y_1$. In vector form, $e = u + Fy$, where $F = \begin{bmatrix}0 & \mathrm{I}\\ \mathrm{I} & 0\end{bmatrix}$ Also $e = [e_1\ \ e_2]^T$, $u = [u_1\ \ u_2]^T$ and $y = [y_1\ y_2]^T$. Define, $\forall\; i \in I_1$ and $\forall\; j \in I_2$,

$$\tilde{Q}_{(i,j)} = \begin{bmatrix}Q_i^1 & 0\\ 0 & Q_j^2\end{bmatrix}, \tilde{S}_{(i,j)} = \begin{bmatrix}S_i^1 & 0\\ 0 & S_j^2\end{bmatrix}, R_{(i,j)} = \begin{bmatrix}R_i^1 & 0\\ 0 & R_j^2\end{bmatrix}$$

$$\begin{aligned}Q_{(i,j)} &= \tilde{Q}_{(i,j)} + F'R_{(i,j)}F + \tilde{S}_{(i,j)}F + F'\tilde{S}^*_{(i,j)}\\ S_{(i,j)} &= \tilde{S}_{(i,j)} + F'R_{(i,j)}\\ V_{(i,j)}(x(t)) &= V_i^1(x_1(t)) + V_j^2(x_2(t)),\end{aligned}$$

where $x(t)$ represents the state vector corresponding to the closed loop system. Assume that the i^{th} subsystem of H_1 and the j^{th} subsystem of H_2 are active in a particular time interval.

Adding the dissipativity inequalities of H_1 and H_2,

$$\begin{aligned}&\langle y_1, Q_i^1 y_1\rangle_{[s,r]} + 2\,\langle y_1, S_i^1 e_1\rangle_{[s,r]} + \langle e_1, R_i^1 e_1\rangle_{[s,r]}\\ &+\langle y_2, Q_j^2 y_2\rangle_{[s,r]} + 2\,\langle y_2, S_j^2 e_2\rangle_{[s,r]} + \langle e_2, R_j^2 e_2\rangle_{[s,r]}\\ &\geq V_i^1(x_1(r)) - V_i^1(x_1(s)) + V_j^2(x_2(r)) - V_j^2(x_2(s)).\end{aligned}$$

$$\begin{aligned}\Leftrightarrow &\left\langle y, \tilde{Q}_{(i,j)} y\right\rangle_{[s,r]} + 2\left\langle y, \tilde{S}_{(i,j)} e\right\rangle_{[s,r]} + \langle e, R_{(i,j)} e\rangle_{[s,r]}\\ &\geq V_{(i,j)}(x(r)) - V_{(i,j)}(x(s)).\end{aligned}$$

$$\Leftrightarrow \left\langle y, \tilde{Q}_{(i,j)} y \right\rangle_{[s,r]} + 2\left\langle y, \tilde{S}_{(i,j)} u + \tilde{S}_{(i,j)} F y \right\rangle_{[s,r]} + \left\langle u + Fy, R_{(i,j)} u + R_{(i,j)} F y \right\rangle_{[s,r]} \geq V_{(i,j)}(x(r)) - V_{(i,j)}(x(s)).$$

$$\Leftrightarrow \left\langle y, Q_{(i,j)} y \right\rangle_{[s,r]} + 2\left\langle y, S_{(i,j)} u \right\rangle_{[s,r]} + \left\langle u, R_{(i,j)} u \right\rangle_{[s,r]} \geq V_{(i,j)}(x(r)) - V_{(i,j)}(x(s)).$$

Hence the interconnected system is dissipative in each region (i, j) with respect to the triple $(Q_{(i,j)}, S_{(i,j)}, R_{(i,j)})$ and $V_{(i,j)}$ is the corresponding storage function where i denotes subsystem of H_1 which is active and j denotes subsystem of H_2 which is active. By property (4),

$$V_i^a(x_a(t_{i,k})) \leq V_i^a(x_a(t_{i,k-1})) + \sum_{p=1}^{L_a}(\left\langle y_a, Q^a_{n_p} y_a \right\rangle_{[t_{p_1},t_{p_2}]} + 2\left\langle y_a, S^a_{n_p} e_a \right\rangle_{[t_{p_1},t_{p_2}]} + \left\langle e_a, R^a_{n_p} e_a \right\rangle_{[t_{p_1},t_{p_2}]}),$$

$\forall\, i \in I_a, \forall\, e_a \in \mathcal{L}_{2e}$, where $\{x_a(t); s \leq t \leq r\}$ are the states when the subsystem M_i^a is active and $a = 1, 2$. Let $t_{(i,j),k}$ denote the k^{th} time at which the mode (i, j) becomes active. Then, $\forall\, i \in I_1$ and $\forall\, j \in I_2$,

$$V_{(i,j)}(x(t_{(i,j),k})) = V_i^1(x_1(t_{(i,j),k})) + V_j^2(x_2(t_{(i,j),k}))$$

$$\begin{aligned}
&\leq V_i^1(x_1(t_{(i,j),k-1})) + \sum_{p=1}^{L_1}(\left\langle y_1, Q^1_{n_p} y_1 \right\rangle_{[t_{p_1},t_{p_2}]} \\
&\quad + 2\left\langle y_1, S^1_{n_p} e_1 \right\rangle_{[t_{p_1},t_{p_2}]} + \left\langle e_1, R^1_{n_p} e_1 \right\rangle_{[t_{p_1},t_{p_2}]}) \\
&\quad + V_j^2(x_2(t_{(i,j),k-1})) + \sum_{q=1}^{L_2}(\left\langle y_2, Q^2_{n_q} y_2 \right\rangle_{[t_{q_1},t_{q_2}]} \\
&\quad + 2\left\langle y_2, S^2_{n_q} e_2 \right\rangle_{[t_{q_1},t_{q_2}]} + \left\langle e_2, R^2_{n_q} e_2 \right\rangle_{[t_{q_1},t_{q_2}]}) \\
&= V_{(i,j)}(x(t_{(i,j),k-1})) + \sum_{z=1}^{L}(\left\langle y, \tilde{Q}_{(n_p,n_q)} y \right\rangle_{[t_{z_1},t_{z_2}]} \\
&\quad + 2\left\langle y, \tilde{S}_{(n_p,n_q)} e \right\rangle_{[t_{z_1},t_{z_2}]} + \left\langle e, R_{(n_p,n_q)} e \right\rangle_{[t_{z_1},t_{z_2}]}) \\
&= V_{(i,j)}(x(t_{(i,j),k-1})) + \sum_{z=1}^{L}(\left\langle y, Q_{(n_p,n_q)} y \right\rangle_{[t_{z_1},t_{z_2}]} \\
&\quad + 2\left\langle y, S_{(n_p,n_q)} u \right\rangle_{[t_{z_1},t_{z_2}]} + \left\langle u, R_{(n_p,n_q)} u \right\rangle_{[t_{z_1},t_{z_2}]}).
\end{aligned}$$

Here, L is the number of switchings from $t_{(i,j),k-1}$ to $t_{(i,j),k}$ (including the switching at $t_{(i,j),k-1}$) such that $\bigcup_{z=1}^{L}[t_{z_1}, t_{z_2}] = [t_{(i,j),k-1}, t_{(i,j),k}]$. In the interval $[t_{z_1}, t_{z_2}]$, one of the modes (say, $(n_p, n_q)^{th}$ mode) will be active, *i.e*, n_p can be any number belonging to the index set $I_1 = \{1, 2, \ldots, N_1\}$ and n_q can be any number belonging to the index set $I_2 = \{1, 2, \ldots, N_2\}$. Note that $p \in \{1, \ldots, L_1\}$ and $q \in \{1, \ldots, L_2\}$. Hence, according to definition 5, the closed loop system is dissipative.

Theorem 4. If H_1 and H_2 are dissipative, then the positive feedback interconnected system (Figure 2) is stable if $Q_{(i,j)} \leq 0, \quad \forall\, i \in I_1$ and $\forall\, j \in I_2$.

Proof. Follows from proposition 2 and Theorem 3.

Theorem 5. Consider the feedback interconnection of two switched systems H_1 and H_2. If the subsystems $\{M_i^a\}$ of H_a, $\forall\, i \in I_a$ and for $a = 1, 2$, possess the 'mixed'-negative imaginary property, then the interconnected system (Figure 2) with positive feedback is Lyapunov stable provided the following conditions are satisfied.

(1) $k_i^1 k_j^2 \leq 1, \forall\, i \in I_1$ and $\forall\, j \in I_2$.

(2) Property (4) is satisfied for H_1 and H_2, *i.e*,

$$V_i^a(x_a(t_{i,k})) \leq V_i^a(x_a(t_{i,k-1})) + \sum_{p=1}^{L_a}(\left\langle y_a, Q^a_{n_p} y_a \right\rangle_{[t_{p_1},t_{p_2}]} + 2\left\langle y_a, S^a_{n_p} e_a \right\rangle_{[t_{p_1},t_{p_2}]} + \left\langle e_a, R^a_{n_p} e_a \right\rangle_{[t_{p_1},t_{p_2}]}),$$
$$\forall\, i \in I_a, \quad \forall\, e_a \in \mathcal{L}_{2e} \quad \text{and} \quad a = 1, 2.$$

Proof. Assume the conditions 1 and 2 are true. Each subsystem M_i^a of H_a is dissipative with respect to the triple (Q_i^a, S_i^a, R_i^a) (from proposition 1) where,

$$\begin{aligned}
Q_i^a &= -(\epsilon_i^a \dot{\Gamma}^{\sim} \dot{\Gamma} + \frac{1}{k_i^a} \mathrm{B}^{\sim} \mathrm{B})\mathrm{I}, \\
S_i^a &= -\frac{1}{2} \Gamma^{\sim} \dot{\Gamma} \mathrm{I}, \quad \text{and} \\
R_i^a &= (k_i^a \mathrm{B}^{\sim} \mathrm{B} - \delta_i^a \Gamma^{\sim} \Gamma)\mathrm{I}, \quad \text{for} \quad a = 1, 2.
\end{aligned}$$

Let V_i^a be the storage function corresponding to M_i^a, $\forall\ i \in I_a$ and $a = 1, 2$ (by Theorem 4 in Hill and Moylan [1980]). Then H_1 and H_2 are dissipative according to definition 5.
By definition, $Q_{(i,j)} = \tilde{Q}_{(i,j)} + F' R_{(i,j)} F + \tilde{S}_{(i,j)} F + F' \tilde{S}^*_{(i,j)}$. Substituting the values of $\tilde{Q}_{(i,j)}$, $\tilde{S}_{(i,j)}$, $R_{(i,j)}$ and F gives,

$$Q_{(i,j)} = \begin{bmatrix} Q_i^1 + R_j^2 & S_j^1 + S_i^{2*} \\ S_j^{1*} + S_i^2 & Q_j^2 + R_i^1 \end{bmatrix} = \begin{bmatrix} -q_1 \mathrm{I} & 0 \\ 0 & -q_2 \mathrm{I} \end{bmatrix},$$

$$\text{where} \quad q_1 = (\frac{1}{k_i^1} - k_j^2)\mathrm{B}^{\sim}\mathrm{B} + \epsilon_i^1 \dot{\Gamma}^{\sim} \dot{\Gamma} + \delta_j^2 \Gamma^{\sim} \Gamma,$$
$$q_2 = (\frac{1}{k_j^2} - k_i^1)\mathrm{B}^{\sim}\mathrm{B} + \epsilon_j^2 \dot{\Gamma}^{\sim} \dot{\Gamma} + \delta_i^1 \Gamma^{\sim} \Gamma.$$

We have, $Q_{(i,j)} \leq 0, \quad \forall\, i \in I_1$ and $\forall\, j \in I_2$ (from condition 1). Therefore, according to Theorem 4, the closed loop system is Lyapunov stable.

Remark: The subsystems of H_1 and H_2 which are active during a given time interval should have a common property(small gain or negative imaginary) at each frequency. This is required because Γ and B are the same for H_1 and H_2.

7. EXAMPLE

This section provides a numerical illustration of the results mentioned above. Here, a two-input-two-output MIMO system with the transfer-function $G(s) = e^{-\tau s} \begin{bmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{bmatrix}$ where,

$$\begin{aligned}
G_{xx}(s) &= \tfrac{10^{(174/20)}}{(s^2+2943s+3.367\times10^9)} \\
G_{xy}(s) &= \tfrac{10^{(340/20)}}{(s^2+2943s+3.367\times10^9)(s^2+5399s+3.825\times10^9)} \\
G_{yx}(s) &= \tfrac{10^{(340/20)}}{(s^2+2943s+3.367\times10^9)(s^2+5399s+3.825\times10^9)} \\
G_{yy}(s) &= \tfrac{10^{(175/20)}}{(s^2+5399s+3.825\times10^9)} \quad \text{and} \quad \tau = 6.0676 \times 10^{-5},
\end{aligned}$$

is considered. The above mentioned transfer-functions are models of a Nanopositioning stage (Yong et al. [2013]). Nanopositioning stages are used in Atomic Force Microscopes (AFMs)

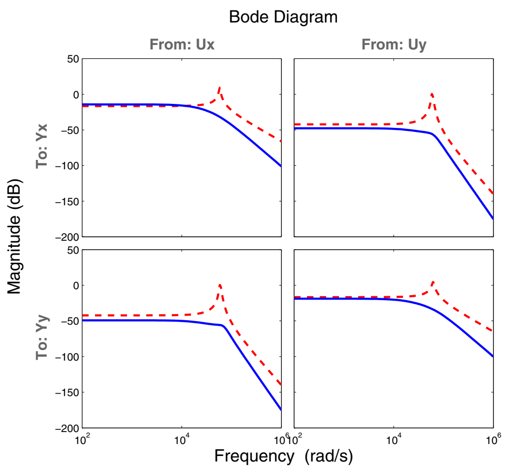

Fig. 3. Open loop (dashed) and closed loop (solid) magnitude plots for G with controller K_{irc}

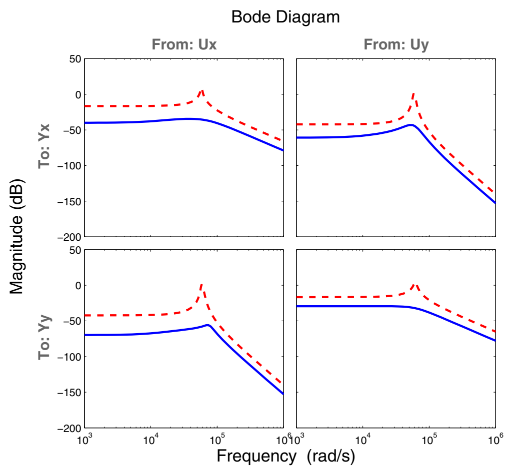

Fig. 4. Open loop (dashed) and closed loop (solid) magnitude plots for G with controller K_{ppf}

for scanning material samples. Flexure guided Nanopositioning stages have two independent piezoelectric stacks for providing displacements along x and y axis. Typically, a material sample (that needs to be scanned) is placed on the nanopositioning stage, and the piezoelectric stacks are actuated such that the stage traces a raster pattern. This is done by applying a triangular waveform input, u_x, on the x axis stack and a slowly increasing ramp or a stair case function input, u_y, on the y axis stack. As the stage has two independent piezo stacks they have different resonances along each axis. Furthermore, Nanopositioning stages also provide a large bandwidth of actuation, leading to significant delays in the displacement sensor outputs, when dealing with high frequency signals.

Let G_{ideal} be the transfer function without including the delay in the system, i.e, $G(s) = e^{-\tau s}G_{ideal}(s)$. Note $G_{ideal}(s)$ is a highly resonant negative imaginary system. Hence, integral resonant control (IRC) (Bhikkaji and Moheimani [2008]) and positive position feedback (PPF) control (Bhikkaji et al. [2007]) are two suitable choices for damping the resonances. A switched controller consisting of two subsystems, *i.e.*, an IRC and a PPF controller, can also be designed. Controllers designed for $G_{ideal}(s)$ will not be effective when applied to $G(s)$, since the time delay would not only destroy the negative imaginary property, it could also render the closed loop unstable. Due to the delay, $G(s)$ will have negative imaginary property in certain frequency intervals only. In other words, the system is 'mixed'-negative imaginary.

The objective is to design a switched controller which effectively damps the resonances in $G(s)$ and at the same time makes the closed loop system stable (achieved by following Theorem 5). The IRC and PPF controllers designed for $G(s)$ are given by, $K_{irc}(s) = \begin{bmatrix} \frac{1.7653\times10^4}{s+2.453\times10^4} & 0 \\ 0 & \frac{1.7605\times10^4}{s+2.416\times10^4} \end{bmatrix}$ and $K_{ppf}(s) = \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix}$, respectively where

$$K_{xx} = \frac{1.247\times10^8 s + 6.299\times10^{12}}{s^2+5.482\times10^8 s+7.253\times10^{12}}$$
$$K_{yy} = \frac{1.111\times10^8 s + 5.549\times10^{12}}{s^2+4.861\times10^8 s+6.362\times10^{12}}$$

K_{irc} and K_{ppf} provide good damping when used in positive feedback with the plant G (see Figures 3 and 4). The obtained

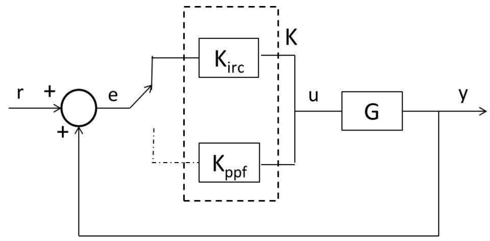

Fig. 5. Positive feedback interconnection of G and K

controllers are negative imaginary by design. Hence we have a plant G which is 'mixed'-negative imaginary and a switched controller K(in positive feedback with G) with two subsystems which are negative imaginary (Refer Figure 5).

The plant G is negative imaginary in the frequency intervals $(0, 5.186\times10^4)\cup(5.924\times10^4,\infty)$ and $(-5.186\times10^4,0)\cup(-\infty,-5.924\times10^4)$ and it has a small gain of 3.2497 in the frequency intervals $[5.186\times10^4, 5.924\times10^4]\cup\{0\}$ and $[-5.924\times10^4,-5.186\times10^4]$. Controllers K_{irc} and K_{ppf} are negative imaginary at all frequencies (except zero frequency) and have a small gain of 0.3077 and 0.0366, respectively in the frequency intervals $[5.186\times10^4, 5.924\times10^4]$ and $[-5.924\times10^4,-5.186\times10^4]$. Hence $k_1^1 = 3.2497$, $k_1^2 = 0.3077$ and $k_2^2 = 0.0366$, and it can be seen that $k_1^1k_1^2 = 1$ and $k_1^1k_2^2 < 1$. (At zero frequency, $k_1^1 = 0.1558$, $k_1^2 = 0.7288$ and $k_2^2 = 0.8723$. Hence, the inequalities $k_1^1k_1^2 \le 1$ and $k_1^1k_2^2 \le 1$ are valid at zero frequency also). This satisfies condition 1 of Theorem 5.

The Lyapunov functions of G, K_{irc} and K_{ppf} are denoted as V_1^1, V_1^2 and V_2^2 respectively. A discrete switching logic is used to switch between the two controllers K_{irc} and K_{ppf}. Let $\eta(t)$ denote the controller which is active at any time t (if $\eta(t) = 1$ then K_{irc} is active and if $\eta(t) = 2$ then K_{ppf} is active). The switching rule used is

$$\eta(t^+) = \begin{cases} \eta(t^-), & \text{if } t - t_w < T_s \text{ or } V^2_{3-\eta(t)}(x_2(t)) \\ & \qquad > V^2_{3-\eta(t)}(x_2(t_v)) \\ 3-\eta(t^-), & \text{otherwise.} \end{cases}$$

Here, T_s is the minimum time for which a subsystem should be active. t_w and t_v denote the last and second last time the system switched between the subsystems. The above switching rule explicitly enforces condition 2 of Theorem 5 for the controller.

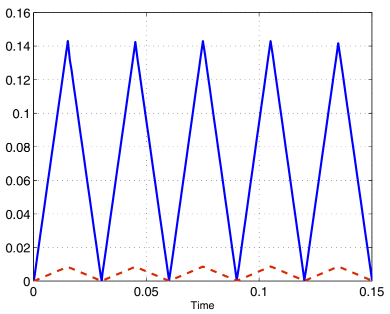

Fig. 6. Output waveforms - triangular waveform (solid line) and ramp (dotted line)

For the plant G, condition 2 is automatically satisfied because it is not switching. Hence, the closed loop system in Figure 5 is Lyapunov stable according to Theorem 5.

The stability of the system was verified by simulation. The closed loop system in Figure 5 was implemented in Simulink with a triangular waveform of frequency 30 Hz and a ramp of slope 0.01 as the two inputs to the system. Figure 6 shows the output waveforms obtained from the system. The system is able to track the given inputs which shows that the system is stable.

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Appendix A. CONVERSION FROM FREQUENCY DOMAIN TO TIME DOMAIN

Consider two causal, bounded, linear and time invariant operators $\Gamma : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ and $\mathrm{B} : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ such that $\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B} = \mathrm{I}$. Let $\gamma(t)$ and $\beta(t)$ be the impulse responses of Γ and B respectively. Then, $\gamma_a(t) * \gamma(t) + \beta_a(t) * \beta(t) = \delta(t)$, where $\gamma_a(t)$ and $\beta_a(t)$ denote the impulse responses of $\Gamma^{\sim}$ and $\mathrm{B}^{\sim}$ respectively. Taking Fourier transforms, we get $\hat{\gamma}(-j\omega)\hat{\gamma}(j\omega) + \hat{\beta}(-j\omega)\hat{\beta}(j\omega) = 1$ (Griggs et al. [2009]).

Consider a causal LTI system with transfer function matrix $\hat{M} \in \mathscr{R}$ which possesses the 'mixed'-negative imaginary property. Then, according to Griggs et al. [2007], $\hat{M}$ is dissipative (in frequency domain), *i.e*,

$$\left\langle \hat{y}, \hat{Q}\hat{y} \right\rangle + 2\left\langle \hat{y}, \hat{S}\hat{u} \right\rangle + \left\langle \hat{u}, \hat{R}\hat{u} \right\rangle \;\geq\; 0, \;\; \forall\;\; \hat{u} \in \mathscr{L}_2(j\mathbb{R}), \tag{A.1}$$

$$\begin{aligned} \text{where} \quad \hat{Q}(\omega) &= -(\epsilon\omega^2\alpha(\omega) + \frac{1}{k}(1-\alpha(\omega)))\mathrm{I}, \\ \hat{S}(\omega) &= -\frac{1}{2}j\omega\alpha(\omega)\mathrm{I}, \quad \text{and} \\ \hat{R}(\omega) &= (k(1-\alpha(\omega)) - \delta\alpha(\omega))\mathrm{I}. \end{aligned} \tag{A.2}$$

Here the values of k, ϵ and δ are the same as mentioned in Proposition 1. $\alpha(\omega)$ is a real even function of frequency, which is equal to one at frequencies where $\hat{M}$ is negative imaginary and is equal to zero at frequencies where $\hat{M}$ has small gain.

Substituting (A.2) in (A.1) gives,

$$\begin{aligned} &-\left\langle \hat{M}\hat{u}, (\epsilon\omega^2\alpha(\omega) + k^{-1}(1-\alpha(\omega)))\hat{M}\hat{u} \right\rangle - \\ &\left\langle \hat{M}\hat{u}, j\omega\alpha(\omega)\hat{u} \right\rangle + \langle \hat{u}, (k(1-\alpha(\omega)) - \delta\alpha(\omega))\hat{u} \rangle \;\geq\; 0. \end{aligned} \tag{A.3}$$

Setting $\alpha(\omega) = \hat{\gamma}(-j\omega)\hat{\gamma}(j\omega)$ and $1-\alpha(\omega) = \hat{\beta}(-j\omega)\hat{\beta}(j\omega)$ in (A.3) gives,

$$\begin{aligned} &-\left\langle \hat{M}\hat{u}, (\epsilon\omega^2\hat{\gamma}(-j\omega)\hat{\gamma}(j\omega) + k^{-1}\hat{\beta}(-j\omega)\hat{\beta}(j\omega))\hat{M}\hat{u} \right\rangle \\ &-\left\langle \hat{M}\hat{u}, j\omega\hat{\gamma}(-j\omega)\hat{\gamma}(j\omega)\hat{u} \right\rangle + \\ &\left\langle \hat{u}, (k\hat{\beta}(-j\omega)\hat{\beta}(j\omega) - \delta\hat{\gamma}(-j\omega)\hat{\gamma}(j\omega))\hat{u} \right\rangle \;\geq\; 0. \end{aligned}$$

Taking inverse Fourier transform gives,

$$\begin{aligned} &-\left\langle Mu, (\epsilon\dot{\Gamma}^{\sim}\dot{\Gamma} + k^{-1}\mathrm{B}^{\sim}\mathrm{B})Mu \right\rangle - \left\langle Mu, \Gamma^{\sim}\dot{\Gamma}u \right\rangle \\ &+ \langle u, (k\mathrm{B}^{\sim}\mathrm{B} - \delta\Gamma^{\sim}\Gamma)u \rangle \;\geq\; 0. \\ \Leftrightarrow &-\epsilon\left\langle \dot{\Gamma}Mu, \dot{\Gamma}Mu \right\rangle - \frac{1}{k}\langle \mathrm{B}Mu, \mathrm{B}Mu \rangle - \left\langle \Gamma Mu, \dot{\Gamma}u \right\rangle \\ &+k\langle \mathrm{B}u, \mathrm{B}u \rangle - \delta\langle \Gamma u, \Gamma u \rangle \;\geq\; 0. \end{aligned}$$

The above inequality gives the time domain version of the 'mixed'-negative imaginary property (Definition 3).