Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Stability and Stabilization of Discrete-time Markov Jump Piecewise-affine Systems ⋆

Lixian Zhang * **Yusong Leng** *

* *Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin, China, 150001 (e-mail: lixianzhang@hit.edu.cn, yusong.leng@gmail.com).*

Abstract: In this paper, the problems of stability and stabilization for a class of discrete-time Markov jump nonlinear systems are investigated, where the nonlinearities are approximated by piecewise-affine (PWA) dynamics. The proposed system is general as it can cover both the conventional Markov jump linear system and PWA systems as special cases. A concept of admissible adjacent switching paths (AASPs) set is proposed to reduce the conservatism of stability analysis, paralleling to the region switching pairs in the context of conventional PWA systems. A simultaneous mode-dependent and region-dependent affine controller is designed based on a $\mathcal{S}$-procedure of using ellipsoidal outer approximation such that the closed-loop system is stochastically stable. A numerical example is given to illustrate the effectiveness and potential of the developed theoretical results.

Keywords: Markov jump systems, Piecewise-affine systems, Stability and Stabilization.

1. INTRODUCTION

Markov jump systems have been extensively studied in the field of hybrid systems during past three decades. Essentially, the system is a class of stochastic hybrid system that consists of a finite number of subsystems and a Markov stochastic process (in continuous-time) or Markov chain (discrete-time) dominating the transitions of the subsystems. Research motivations on the systems lie in their powerful ability of modeling many typical engineering systems or processes subject to random abrupt parameter variations characterized by Markov processes, such as manufacturing systems, Boukas and Yang [1996], power systems, Willsky and Levy [1979], aerospace systems, Boukas [2006] and networked control systems (NCS), Zhang et al. [2013], to mention a few. So far, many control issues have been addressed in this field, including control, estimation, model reduction of the systems and on complex dynamics involved with the systems such as uncertainties, time-delays and so on, see for example, Costa et al. [1999], Zhang [2009], Zhang et al. [2003], Shi et al. [1999], Xu et al. [2003]. Recent advances in the area of Markov jump systems include the investigations on uncertain mode transitions and asynchronous control of the systems, etc., Zhang et al. [2003], Zhang and Lam [2010], Wu et al. [2014]. It is worth noting that most of achieved results on the systems are in linear context, few results on analysis and design for Markov jump nonlinear systems are available Lin et al. [2009], Lin et al. [2011], even though almost all practical applications are nonlinear.

As an important means of approximating nonlinear systems, piecewise-affine (PWA) systems have received increasing attention in the academic studies and engineering practice. The systems offer a modeling framework in which original nonlinear systems can be approximately represented, see for the fundamental work in Sontag [1981], with a set of affine (or linear, dependent on the inclusion of the origin or not) models that vary with different system regions. Various control problems of PWA systems, e.g., optimal control, Rantzer and Johansson [2000], robust control, Gao et al. [2009], model predictive control, Lazar [2006], have been broadly investigated. Distinguished from the general switched linear systems, the affine term appeared in the process of linearizing the original nonlinear systems pose a rather intractable difficulty to the analysis and synthesis of the control systems. Up to date, diverse methodologies have been proposed to effectively handle the affine term in the area, and a typical one is the ellipsoidal approximation approach, see for example, Rungtweesuk and Wongsaisuwan [2012], Vandenberghe et al. [1998]. Note that though the PWA systems have shown many significant advantages in the literature in solving control problems for nonlinear systems, it is quite surprise that the systems have never been considered in the context of Markov jump nonlinear systems.

Motivated by the above observations, this paper will investigate the problems of stability and stabilization for a class of discrete-time Markov jump nonlinear systems, in which the nonlinearities are approximated by PWA dynamics in each system mode. First, comparable to the region switching pairs to be determined in conventional PWA systems, a concept of admissible adjacent switching paths (AASPs) set is proposed such that a less conservative stability criterion can be established. Then, a both mode-dependent and region-dependent affine controller guaranteeing the stochastic stability of closed-loop system is obtained with the aid of ellipsoidal outer approximation structured $\mathcal{S}$-procedure. The results are valid when the un-

⋆ This work was supported by National Natural Science Foundation of China (61322301, 61021002).

derlying systems reduce to conventional PWA systems or arbitrarily switched PWA systems with all the transition probabilities in Markov chain being unknown completely. The remainder of this paper is organized as follows. In Section II, the mathematical model of the systems, some preliminary results, and problem formulation are given. Section III is devoted to establishing the stability criterion and the mode-dependent and region-dependent affine controller for the underlying system. A numerical example is provided in Section IV and the paper is concluded in Section V.

Notation: The notation used in this paper is fairly standard. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space, the notation $\|\cdot\|$ refers to the Euclidean vector norm. For notation $(\Psi, \mathcal{F}, \Pr)$, Ψ represents the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and Pr is the probability measure on $\mathcal{F}$. $\mathbb{E}[\cdot]$ and $\det[X]$ stand for the mathematical expectation and determinant of matrix X, respectively. In addition, $diag\{\cdots\}$ stands for a block-diagonal matrix and in symmetric block matrices or long matrix expressions, symbol $*$ was used as an ellipsis for the terms that are introduced by symmetry, for example, $X^TX = X^T(*)$ and $X^TYX = X^TY(*)$.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider a class of discrete-time Markov jump nonlinear systems defined on a complete probability space $(\Psi, \mathcal{F}, \Pr)$:

$$x(k+1) = f_{r(k)}(x(k), u(k))$$

where $x(k) \in \mathbb{R}^{n_x}, u(k) \in \mathbb{R}^{n_u}$ are the system state and control input, respectively. The jumping process $\{r(k), k \geq 0\}$, governing the switching among different subsystems, is described by a discrete-time homogeneous Markov chain, which takes values in a finite set $\mathcal{M} \triangleq \{1, 2, \ldots, M\}$ with transition probabilities $\Pr(r_{k+1} = n | r_k = m) = \pi_{mn}, \pi_{mn} \geq 0, \forall m \times n \in \mathcal{M} \times \mathcal{M}$, and $\sum_{n=1}^{M} \pi_{mn} = 1$. Furthermore, the Markov transition probabilities matrix is defined by $\Pi = [\pi_{mn}]$.

Suppose the nonlinearities be approximated by PWA dynamics with corresponding polyhedral partitions of the state space: $\mathcal{R}_{m,i} \triangleq \left\{x | L_{m,i}^T x < l_{m,i}\right\}$, and $\bigcup_{i \in \mathcal{I}_m} \mathcal{R}_{m,i} = \mathbb{R}^{n_x}$, $\mathcal{R}_{m,i} \cap \mathcal{R}_{m,j} = \varnothing, \forall i \neq j \in \mathcal{I}_m \times \mathcal{I}_m, \forall m \in \mathcal{M}$. $\mathcal{I}_m \triangleq \{1, 2, \ldots, I_m\}$, $\forall m \in \mathcal{M}$ is the index set of the regions associated with the mth subsystem. Then, within each region, the dynamics are affine of the form: $\forall r(k) = m, x(k) \in \mathcal{R}_{m,i}$,

$$(\Xi_{m,i}) : x(k+1) = A_{m,i}x(k) + a_{m,i} + B_{m,i}u_{m,i}(k) \quad (1)$$

The system matrices $A_{m,i}$, $B_{m,i}$ and the affine term $a_{m,i}$ are appropriately dimensioned real-valued matrices and vector, respectively. For further use, $\mathcal{I}_m$ is partitioned as $\mathcal{I}_m = \mathcal{I}_m^0 \cup \mathcal{I}_m^1, \forall m \in \mathcal{M}$, where $\mathcal{I}_m^0$ denotes the set of indexes for regions that include the origin and $\mathcal{I}_m^1$ otherwise. Suppose $a_{m,i} = 0, \forall i \in \mathcal{I}_m^0, \forall m \in \mathcal{M}$.

For later development, the following definition is needed.

Definition 1. Given a target region $\mathcal{R}_{n,j}$, the one-step controllable region to $\mathcal{R}_{n,j}$ from region $\mathcal{R}_{m,i}$ is defined as: $\forall i \times j \in \mathcal{I}_m \times \mathcal{I}_n, \forall m \times n \in \mathcal{M} \times \mathcal{M}$,

$$\mathcal{R}_{m,i}^{n,j} \triangleq \{x \in \mathcal{R}_{m,i} | A_{m,i}x + a_{m,i} \in \mathcal{R}_{n,j}\}$$

Note that due to $\bigcup_{i \in \mathcal{I}_m} \mathcal{R}_{m,i} = \mathbb{R}^{n_x}, \forall m \in \mathcal{M}$, it holds $\bigcup_{j \in \mathcal{I}_n} \mathcal{R}_{m,i}^{n,j} = \mathcal{R}_{m,i}, \forall m \times n \in \mathcal{M} \times \mathcal{M}$.

To present the main objectives of this paper more precisely, the definition of stochastic stability of system (1) is also required as below.

Definition 2. System (1) is said to be stochastically stable if, for $u_{m,i}(k) \equiv 0, k \geq 0$ and any initial condition $x_0 \in \mathbb{R}^{n_x}, r_0 \in \mathcal{M}$, the following holds

$$E\left\{\sum_{k=0}^{\infty} \|x(k)\|^2 |_{x_0, r_0}\right\} < \infty \quad (2)$$

Therefore, the objectives in this paper are to derive the stochastic stability criterion for system (1) and to design a state-feedback stabilizing controller such that the resulting closed-loop system is stochastically stable. The mode-dependent and region-dependent affine controller is considered here with the form:

$$u_{m,i}(k) = K_{m,i}x(k) + g_{m,i} \quad (3)$$

where $K_{m,i}, g_{m,i}$ are the controller gains to be determined and $g_{m,i}$ is considered to be $0, \forall i \in \mathcal{I}_m^0, \forall m \in \mathcal{M}$.

Before proceeding further, to facilitate later treatments of affine term in developing stability and stabilization criteria, we shall introduce an ellipsoidal outer approximation of polyhedral region $\mathcal{R}_{m,i}$ such that $\mathcal{R}_{m,i} \subseteq \mathcal{E}_{m,i}$, with $\mathcal{E}_{m,i} \triangleq \{x | \|E_{m,i}x + e_{m,i}\| \leq 1\}$. Then, let the vertices of $\mathcal{R}_{m,i}$ be denoted by $v \in \mathcal{V}_{m,i}$, it holds that $v \in \mathcal{E}_{m,i}$. The ellipsoid can be optimized as below in a sense of covering $\mathcal{R}_{m,i}$ most tightly, and the parameters $(E_{m,i}, e_{m,i})$ can be also given correspondingly. More details can be referred to Rungtweesuk and Wongsaisuwan [2012], Vandenberghe et al. [1998]:

$$\begin{aligned} &\min_{E_{m,i}, e_{m,i}} \det\left[E_{m,i}^{-1}\right] \\ &s.t. \begin{bmatrix} I & E_{m,i}v + e_{m,i} \\ * & 1 \end{bmatrix} \geq 0, v \in \mathcal{V}_{m,i} \\ &E_{m,i} = E_{m,i}^T > 0 \end{aligned} \quad (4)$$

The above optimization problem is convex and can be solved efficiently by YALMIP toolbox, Lofberg [2004].

3. MAIN RESULTS

3.1 Stability Analysis

It has been recognized that the dynamics of system (1) are known to be $\Xi_{m,i}$ when the system state belongs to region $\mathcal{R}_{m,i}$ at the current time k. However, there is no clue in which region the system will operate at the next step $k+1$ due to the mode transitions governed by the Markov chain. However, the whole set of AASPs, denoted by $\Omega_{m,i}$, can be determined such that the stability criterion can be established less conservatively compared with the case by using the set of complete adjacent switching paths $\bar{\Omega}_{m,i} \triangleq \left\{\omega | \mathcal{R}_{m,i}^{\omega}\right\}$, with $\Omega_{m,i} \subseteq \bar{\Omega}_{m,i}$.

Towards this end, consider $\mathcal{R}_{n,j_n}, \forall j_n \in \mathcal{I}_n$ for $n = 1, 2, \ldots, M$ as the target regions and compute the intersection of the one-step controllable regions to $\mathcal{R}_{n,j_n}$ from $\mathcal{R}_{m,i}$:

$$\mathcal{R}_{m,i}^{\omega} \triangleq \mathcal{R}_{m,i}^{1,j_1} \cap \mathcal{R}_{m,i}^{2,j_2} \cap \ldots \cap \mathcal{R}_{m,i}^{M,j_M} \quad (5)$$

where $\omega \triangleq ((1, j_1); (2, j_2); \ldots; (M, j_M))$ and the intersection $\mathcal{R}^{\omega}_{m,i}$ is a subpartition of region $\mathcal{R}_{m,i}$.

Then the set of all AASPs associated with region $\mathcal{R}_{m,i}$ can be determined as:

$$\Omega_{m,i} \triangleq \left\{\omega | \mathcal{R}^{\omega}_{m,i} \neq \varnothing\right\}, \forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$$

Remark 1. For the ω such that $\mathcal{R}^{\omega}_{m,i} \neq \varnothing$, it contains the indices of regions $\mathcal{R}_{1,j_1}, \mathcal{R}_{2,j_2}, \ldots, \mathcal{R}_{M,j_M}$ into which the system state can move within one-step from the current subregion $\mathcal{R}^{\omega}_{m,i}$, with the corresponding transition probabilities $\pi_{m1}, \pi_{m2}, \ldots, \pi_{mM}$. Similar to the partition of $\mathcal{I}_m$, $\Omega_{m,i}$ can be partitioned as $\Omega_{m,i} = \Omega^0_{m,i} \cup \Omega^1_{m,i}, \forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$, where $\Omega^0_{m,i} \triangleq \left\{\omega \in \Omega_{m,i} | 0 \in \mathcal{R}^{\omega}_{m,i}\right\}$. Note that $\omega \in \Omega^0_{m,i}$ implies $i \in \mathcal{I}^0_m, \forall m \in \mathcal{M}$, i.e., $a_{m,i} = 0, \forall \omega \in \Omega^0_{m,i}, \forall m \in \mathcal{M}$.

With the above definition of $\mathcal{R}^{\omega}_{m,i}$, the following fact holds true.

Proposition 1. Given any region $\mathcal{R}_{m,i}, \forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$, it can be exactly filled with subregions $\mathcal{R}^{\omega}_{m,i}, \forall \omega \in \Omega_{m,i}$, namely, $\forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$,

$$\bigcup_{\omega \in \Omega_{m,i}} \mathcal{R}^{\omega}_{m,i} = \mathcal{R}_{m,i} \tag{6}$$

$$\mathcal{R}^{\omega}_{m,i} \cap \mathcal{R}^{\omega'}_{m,i} = \varnothing, \forall \omega \neq \omega' \in \Omega_{m,i} \times \Omega_{m,i} \tag{7}$$

Proof. Part (I) (for(6)). Due to the fact that $\bar{\Omega}_{m,i} = \Omega_{m,i} \cup \left\{\omega | \mathcal{R}^{\omega}_{m,i} = \varnothing\right\}$, and $\bigcup_{\omega \in \Omega_{m,i}} \mathcal{R}^{\omega}_{m,i} = \bigcup_{\omega \in \Omega_{m,i}} \mathcal{R}^{\omega}_{m,i} \cup \varnothing$, it follows that

$$\begin{aligned}
&\bigcup_{\omega \in \Omega_{m,i}} \mathcal{R}^{\omega}_{m,i} \\
&= \bigcup_{\omega \in \bar{\Omega}_{m,i}} \mathcal{R}^{\omega}_{m,i} \\
&= \bigcup_{j_1 \in \mathcal{I}_1} \cdots \bigcup_{j_M \in \mathcal{I}_M} (\mathcal{R}^{1,j_1}_{m,i} \cap \ldots \cap \mathcal{R}^{M,j_M}_{m,i}) \\
&= \bigcup_{j_1 \in \mathcal{I}_1} \cdots \bigcup_{j_{M-1} \in \mathcal{I}_{M-1}} (\mathcal{R}^{1,j_1}_{m,i} \cap \ldots \cap \mathcal{R}^{M-1,j_{M-1}}_{m,i}) \\
&\quad \cap (\bigcup_{j_M \in \mathcal{I}_M} \mathcal{R}^{M,j_M}_{m,i}) \\
&= \bigcup_{j_1 \in \mathcal{I}_1} \cdots \bigcup_{j_{M-2} \in \mathcal{I}_{M-2}} (\mathcal{R}^{1,j_1}_{m,i} \cap \ldots \cap \mathcal{R}^{M-1,j_{M-1}}_{m,i}) \\
&\quad \cap (\bigcup_{j_{M-1} \in \mathcal{I}_{M-1}} \mathcal{R}^{M-1,j_{M-1}}_{m,i}) \cap (\bigcup_{j_M \in \mathcal{I}_M} \mathcal{R}^{M,j_M}_{m,i}) \\
&\quad \vdots \\
&= \bigcap_{n \in \mathcal{M}} (\bigcup_{j_n \in \mathcal{I}_n} \mathcal{R}^{n,j_n}_{m,i}) \\
&= \bigcap_{n \in \mathcal{M}} \mathcal{R}_{m,i} \\
&= \mathcal{R}_{m,i}
\end{aligned}$$

Part (II) (for(7)). The proof can be done by contradiction. Suppose there exists system state $x^* \in \mathcal{R}^{\omega}_{m,i} \cap \mathcal{R}^{\omega'}_{m,i}$ for some $\omega \neq \omega' \in \Omega_{m,i} \times \Omega_{m,i}$. By the definition of $\mathcal{R}^{\omega}_{m,i}$, it follows that $x^* \in \left(\mathcal{R}^{1,j_1}_{m,i} \cap \mathcal{R}^{2,j_2}_{m,i} \cap \ldots \cap \mathcal{R}^{M,j_M}_{m,i}\right) \cap \left(\mathcal{R}^{1,j'_1}_{m,i} \cap \mathcal{R}^{2,j'_2}_{m,i} \cap \ldots \cap \mathcal{R}^{M,j'_M}_{m,i}\right)$, which also implies that $x^* \in \mathcal{R}^{n,j_n}_{m,i} \cap \mathcal{R}^{n,j'_n}_{m,i}, \forall j_n \neq j'_n \in \mathcal{I}_m \times \mathcal{I}_m$. Recall the definition of $\mathcal{R}^{n,j}_{m,i} \triangleq \{x \in \mathcal{R}_{m,i} | A_{m,i}x + a_{m,i} \in \mathcal{R}_{n,j}\}$, it yields that

$$\begin{cases} A_{m,i}x^* + a_{m,i} \in \mathcal{R}_{n,j_n} \\ A_{m,i}x^* + a_{m,i} \in \mathcal{R}_{n,j'_n} \end{cases}, \forall j_n \neq j'_n \in \mathcal{I}_m \times \mathcal{I}_m$$

which contradicts $\mathcal{R}_{m,i} \cap \mathcal{R}_{m,j} = \varnothing, \forall i \neq j \in \mathcal{I}_m \times \mathcal{I}_m, \forall m \in \mathcal{M}$. Thus it is unlikely that $x^* \in \mathcal{R}^{\omega}_{m,i} \cap \mathcal{R}^{\omega'}_{m,i}, \forall \omega \neq \omega' \in \Omega_{m,i} \times \Omega_{m,i}$, i.e., $\mathcal{R}^{\omega}_{m,i} \cap \mathcal{R}^{\omega'}_{m,i} = \varnothing$. $\square$

Algorithm 1. Determination of the AASPs set:

Input: $A_{m,i}, B_{m,i}, a_{m,i}$ and $\mathcal{R}_{m,i}, \forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$.

Step 1: Initialization. Set $k = 0$, and the AASPs set $\Omega^{(k)}_{m,i} = \varnothing$.

Step 2: Construct the set $\mathcal{J}^n_{m,i} \triangleq \{j_n \in \mathcal{I}_n | \mathcal{R}^{n,j_n}_{m,i} \neq \varnothing\}, \forall m \in \mathcal{M}$, and let $\mathbb{N}[\mathcal{J}^n_{m,i}]$ denote the number of elements in $\mathcal{J}^n_{m,i}$.

Step 3: Select a new set of $j_1, j_2, \ldots, j_M$ from $\mathcal{J}^1_{m,i}, \mathcal{J}^2_{m,i}, \ldots, \mathcal{J}^M_{m,i}$, and calculate $\mathcal{R}^{\omega}_{m,i} = \mathcal{R}^{1,j_1}_{m,i} \cap \mathcal{R}^{2,j_2}_{m,i} \cap \ldots \cap \mathcal{R}^{M,j_M}_{m,i}$. Set $k = k + 1$.

Step 4: Check if $k = \prod_{n \in \mathcal{M}} \mathbb{N}[\mathcal{J}^n_{m,i}]$, if yes, **exit** and **output** $\Omega_{m,i}$; Otherwise, check if $\mathcal{R}^{\omega}_{m,i} \neq \varnothing$, if yes, set $\Omega^{(k)}_{m,i} = \Omega^{(k-1)}_{m,i} \cup \omega$ and goto **Step 3**; Otherwise, also goto **Step 3**.

Now, the following theorem presents a stability criterion for the underlying systems. Unless otherwise stated, the following derivations all hold for $\forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$.

Theorem 1. The unforced system (1) with $u_{m,i}(k) \equiv 0$ is stochastically stable if there exist a set of positive definite matrices $P_{m,i}$ and scalar parameters $\gamma^{\omega}_{m,i} < 0, \forall \omega \in \Omega^1_{m,i}$ such that the following LMIs hold:

$$\begin{bmatrix} -\mathcal{P} & \mathcal{L}^A_{m,i} \\ * & -P_{m,i} \end{bmatrix} < 0, \forall \omega \in \Omega^0_{m,i} \tag{8}$$

$$\begin{bmatrix} -\mathcal{P} & \mathcal{L}_{m,i} \\ * & \Lambda_{m,i} \end{bmatrix} < 0, \forall \omega \in \Omega^1_{m,i} \tag{9}$$

where $\mathcal{P} \triangleq diag\left\{P_{1,j_1}, \ldots, P_{M,j_M}\right\}$, $\mathcal{L}_{m,i} \triangleq \left[\, \mathcal{L}^A_{m,i} \; \mathcal{L}^a_{m,i} \,\right]$ and

$$\Lambda_{m,i} \triangleq \begin{bmatrix} -P_{m,i} + \Lambda^E_{m,i} & \gamma^{\omega}_{m,i} \left(E^{\omega}_{m,i}\right)^T e^{\omega}_{m,i} \\ * & \Lambda^e_{m,i} \end{bmatrix}$$

with $\Lambda^E_{m,i} \triangleq \gamma^{\omega}_{m,i} \left(E^{\omega}_{m,i}\right)^T E^{\omega}_{m,i}, \Lambda^e_{m,i} \triangleq (\gamma^{\omega}_{m,i} \left(e^{\omega}_{m,i}\right)^T e^{\omega}_{m,i} - 1)$ and

$$\mathcal{L}^A_{m,i} \triangleq \left[\sqrt{\pi_{m1}} A^T_{m,i} P^T_{1,j_1}, \ldots, \sqrt{\pi_{mM}} A^T_{m,i} P^T_{M,j_M}\right]^T$$
$$\mathcal{L}^a_{m,i} \triangleq \left[\sqrt{\pi_{m1}} a^T_{m,i} P^T_{1,j_1}, \ldots, \sqrt{\pi_{mM}} a^T_{m,i} P^T_{M,j_M}\right]^T$$

Proof. Without loss of generality, we just show that (9) holds since (8) can be regarded as a special case of (9), when $a_{m,i} = 0$ and the $\mathcal{S}$-procedure given in Section II is not needed.

Consider the Lyapunov function associated with the local model $\Xi_{m,i}$:

$$V_{m,i}(x(k)) = x^T(k) P_{m,i} x(k)$$

where $P_{m,i}$ is a positive definite matrix to be determined.

Based on Proposition 1, the AASPs provided by $\Omega_{m,i}$ can be utilized in deriving the mathematical expectation of Lyapunov function: $\forall \omega \in \Omega_{m,i}$, (bearing in mind $\omega = ((1, j_1); (2, j_2); \ldots; (M, j_M))$)

$$\begin{aligned}
&\Delta V_{m,i}(x_k) \\
&\triangleq \mathbb{E}\left[V_{n,j}(x_{k+1})\right]|_{x_k\in\mathcal{R}_{m,i}} - V_{m,i}(x_k) \\
&= \mathbb{E}\left[x_{k+1}^T P_{n,j} x_{k+1}\right]|_{x_k\in\mathcal{R}_{m,i}} - x_k^T P_{m,i} x_k \\
&= \sum_{n\in\mathcal{M}} \pi_{mn}\left(A_{m,i}x_k + a_{m,i}\right)^T P_{n,j}(*) - x_k^T P_{m,i} x_k
\end{aligned} \quad (10)$$

If $\Delta V_{m,i}(x_k) < 0$, following a similar vein in the proof of Lemma 1 in Zhang [2009], it can be shown that $E\left\{\sum_{k=0}^{\infty}\|x(k)\|^2|_{x_0,r_0}\right\} < \infty$, which implies that the unforced system (1) with $u_{m,i}(k) \equiv 0$ is stable in stochastic sense. Define $\bar{x} \triangleq \left[\, x^T \; 1 \,\right]^T$, and $\Delta V_{m,i}(x_k)$ can be rewritten as: $\forall\omega \in \Omega_{m,i}$,

$$\sum_{n\in\mathcal{M}} \pi_{mn}\bar{x}^T \begin{bmatrix} A_{m,i}^T P_{n,j} A_{m,i} - P_{m,i} & A_{m,i}^T P_{n,j} a_{m,i} \\ * & a_{m,i}^T P_{n,j} a_{m,i} \end{bmatrix} \bar{x} \quad (11)$$

Note that the term $\sum_{n\in\mathcal{M}} \pi_{mn} a_{m,i}^T P_{n,j} a_{m,i}$ in the left-hand-side of (11) is great than 0, which implies that it is impossible for $\Delta V_{m,i}(x_k) < 0$ to admit a feasible solution. To overcome this difficulty, a $\mathcal{S}$-procedure of using the region information is employed via the afore-introduced ellipsoidal outer approximation $\mathcal{E}_{m,i}$. For further relaxation, the subregion $\mathcal{R}_{m,i}^{\omega}$ of $\mathcal{R}_{m,i}$ and corresponding ellipsoidal outer approximation $\mathcal{E}_{m,i}^{\omega} = \left\{x \mid \left\|E_{m,i}^{\omega}x + e_{m,i}^{\omega}\right\| \le 1\right\}$ can be chosen instead. Ellipsoid $\mathcal{E}_{m,i}^{\omega}$ can be also constructed as: $\forall\omega \in \Omega_{m,i}$,

$$\bar{x}^T \begin{bmatrix} \left(E_{m,i}^{\omega}\right)^T E_{m,i}^{\omega} & \left(E_{m,i}^{\omega}\right)^T e_{m,i}^{\omega} \\ * & \left(e_{m,i}^{\omega}\right)^T e_{m,i}^{\omega} - 1 \end{bmatrix} \bar{x} \le 0$$

Then, the following inequality implies (11) with $\gamma_{m,i}^{\omega} < 0, \forall\omega \in \Omega_{m,i}$

$$\bar{x}^T \left\{ \begin{array}{c} \sum_{n\in\mathcal{M}} \pi_{mn} \begin{bmatrix} A_{m,i}^T P_{n,j} A_{m,i} & A_{m,i}^T P_{n,j} a_{m,i} \\ * & a_{m,i}^T P_{n,j} a_{m,i} \end{bmatrix} \\ +\Lambda_{m,i} \end{array} \right\} \bar{x} < 0$$

further,

$$\sum_{n\in\mathcal{M}} \pi_{mn} \begin{bmatrix} A_{m,i}^T \\ a_{m,i}^T \end{bmatrix} P_{n,j}(*) + \Lambda_{m,i} < 0$$

By Schur complement, (9) guarantees that the above inequality holds and accordingly $\Delta V_{m,i}(x(k)) < 0$, namely, the stochastic stability of system (1) is ensured and this completes the proof. □

Remark 2. Note that in Theorem 1, if $\bar{\Omega}_{m,i}$ is used instead of $\Omega_{m,i}$, the numbers of resulting LMIs will be increased. Therefore, the presented criterion is less conservative compared with the case of using $\bar{\Omega}_{m,i}$. In addition, for the cases that the Markov transition probabilities matrix $\Pi = I$, that is, each subsystem can only switch to itself, the considered system reduces to a series of conventional PWA systems and the stability criterion established in Theorem 1 is still valid. In addition, if the indexes set $\mathcal{M}$ only contains one element, i.e., $\mathcal{M} \triangleq \{1\}$, the considered system reduces further to a single conventional PWA system. Then, the corresponding stability conditions can reduce to the ones presented in the following corollary.

Corollary 1. The PWA system reduced from the unforced system (1) is stable, if there exist a set of positive definite matrices P_i and scalar parameters $\gamma_i < 0, \forall i \in \mathcal{I}^1$ such that $A_i^T P_j A_i - P_i < 0, \forall i \in \mathcal{I}^0$ and $diag\left\{A_i^T P_j A_i, 0\right\} + \Lambda_i < 0, \forall i \in \mathcal{I}^1$, where Λ_i is denoted in Theorem 1 (with the index m being removed out correspondingly).

Remark 3. In addition, for the cases that the Markov transition probabilities are completely unknown, the concerned system can be viewed as arbitrarily switched PWA systems. Accordingly, (10) can be reformed as $\Delta V_{m,i}(x_k) = \sum_{n\in\mathcal{M}} \pi_{mn}[(A_{m,i}x_k + a_{m,i})^T P_{n,j}(*) - x_k^T P_{m,i} x_k], \forall\omega \in \Omega_{m,i}$ and $\Delta V_{m,i}(x(k)) < 0$ still can be guaranteed by $(A_{m,i}x_k + a_{m,i})^T P_{n,j}(*) - x_k^T P_{m,i} x_k < 0, \forall\omega \in \Omega_{m,i}$ despite that the transition probabilities π_{mn} are not known *a priori*. It is straightforward that without precise knowledge on the switching of subsystems, the stability condition will be more conservative which is presented in the following corollary. More details on the relation between arbitrarily switched systems and Markov jump systems with completely unknown transition probabilities can be referred to Zhang and Boukas [2009].

Corollary 2. The arbitrarily switched PWA systems extended from the unforced system (1) is stable, if LMIs (8)-(9) admit a set of positive definite matrices $P_{m,i}$ and scalar parameters $\gamma_{m,i}^{\omega} < 0, \forall\omega \in \Omega_{m,i}^1, \forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$, where $\mathcal{L}_{m,i}^A$ and $\mathcal{L}_{m,i}^a$ in (8)-(9) are replaced by $\mathcal{L}_{m,i}^A \triangleq [A_{m,i}^T P_{1,j_1}^T, \ldots, A_{m,i}^T P_{M,j_M}^T]^T, \mathcal{L}_{m,i}^a \triangleq [a_{m,i}^T P_{1,j_1}^T, \ldots, a_{m,i}^T P_{M,j_M}^T]^T$.

3.2 Stabilization

In this subsection, based on the above stability criterion, the stabilization problem of system (1) with control input $u_{m,i}$ will be addressed. The following theorem presents sufficient conditions for the existence of a simultaneous mode-dependent and region-dependent affine stabilizing controller with form (3). Note that the complete adjacent switching paths set $\bar{\Omega}_{m,i}$ will be used for the controller design as below, since the controller gains are obtained *a posteriori* such that the AASPs set $\Omega_{m,i}$ determined via open-loop system will not be applicable any more. Likewise, unless otherwise stated, the following derivations all hold for $\forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$.

Theorem 2. Consider the system (1), if there exist a set of positive definite matrices $Q_{m,i}, U_{m,i}$, vectors $g_{m,i}, \forall i \in \mathcal{I}_m^1, \forall m \in \mathcal{M}$ and scalars $\gamma_{m,i} < 0, \forall\omega \in \bar{\Omega}_{m,i}^1, \forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$ such that

$$\begin{bmatrix} -\mathcal{Q} & \bar{\mathcal{L}}_{m,i}^A \\ * & -Q_{m,i} \end{bmatrix} < 0, \forall\omega \in \bar{\Omega}_{m,i}^0 \quad (12)$$

$$\begin{bmatrix} -\mathcal{Q} & \bar{\mathcal{L}}_{m,i}^A & \bar{\mathcal{L}}_{m,i}^a \\ * & -Q_{m,i} & \gamma_{m,i} Q_{m,i} E_{m,i}^T e_{m,i} \\ * & * & \gamma_{m,i} e_{m,i}^T e_{m,i} - 1 \end{bmatrix} < 0, \forall\omega \in \bar{\Omega}_{m,i}^1 \quad (13)$$

where $\mathcal{Q} \triangleq diag\left\{Q_{1,j_1}, \ldots, Q_{M,j_M}\right\}$ and

$$\bar{\mathcal{L}}_{m,i}^A \triangleq \left[\sqrt{\pi_{m1}}\tilde{A}_{m,i}^T, \ldots, \sqrt{\pi_{mM}}\tilde{A}_{m,i}^T\right]^T$$
$$\bar{\mathcal{L}}_{m,i}^a \triangleq \left[\sqrt{\pi_{m1}}\tilde{a}_{m,i}^T, \ldots, \sqrt{\pi_{mM}}\tilde{a}_{m,i}^T\right]^T$$

with $\tilde{A}_{m,i} \triangleq A_{m,i}Q_{m,i} + B_{m,i}U_{m,i}, \tilde{a}_{m,i} \triangleq a_{m,i} + B_{m,i}g_{m,i}$

then a mode-dependent and region-dependent affine controller of the form (3) can be obtained to guarantee the stochastic stability of the resulting closed-loop system. Moreover, if the LMIs (12)-(13) have a feasible solution, the admissible controller gain is given by $g_{m,i}$ and

$$K_{m,i} = U_{m,i} Q_{m,i}^{-1} \tag{14}$$

Proof. Consider system (1) with control input $u_{m,i}$, replace $A_{m,i}$ and $a_{m,i}$ of (8)-(9) by $A_{m,i} + B_{m,i}K_{m,i}, a_{m,i} + B_{m,i}g_{m,i}$, respectively, and set $Q_{m,i} \triangleq P_{m,i}^{-1}, U_{m,i} \triangleq K_{m,i}Q_{m,i}$. Due to the fact that the system matrix $A_{m,i}$ and affine term $a_{m,i}$ of the resulting closed-loop system are not known *a priori* while designing the controller, the AASPs set $\Omega_{m,i}$ can not be determined as before by Algorithm 1. Then, the LMIs (8)-(9) have to be satisfied for the afore-introduced complete adjacent switching pathes set $\bar{\Omega}_{m,i}$ and the $\mathcal{E}_{m,i}$ structured $\mathcal{S}$-procedure is employed instead. Besides, considering the negative of element $\gamma_{m,i}E_{m,i}^T E_{m,i}$, it gives rise to the following inequality (15) that implies (9).

$$\begin{bmatrix} -\mathcal{P} & \mathcal{L}_{m,i}^A & \mathcal{L}_{m,i}^a \\ * & -P_{m,i} & \gamma_{m,i}E_{m,i}^T e_{m,i} \\ * & * & \gamma_{m,i}e_{m,i}^T e_{m,i} - 1 \end{bmatrix} < 0, \forall \omega \in \bar{\Omega}_{m,i}^1 \tag{15}$$

Performing a congruence transformation to (13) by $\mathcal{Q}_{m,i} \triangleq diag\{\mathcal{P}, P_{m,i}, 1\}$ yields (15). In addition, similar to the proof in Theorem 1, for $\omega \in \bar{\Omega}_{m,i}^0$, namely, $a_{m,i} = 0, g_{m,i} = 0$, the $\mathcal{S}$-procedure will not be needed, and it is trivial to show that (12) ensures (8). Moreover, if a feasible solution of LMIs (12)-(13) exists, the gains of admissible controller are given by (14). This completes the proof. □

Remark 4. Note that the condition (13) is actually a bilinear matrix inequality (BMI) problem. However, as $\gamma_{m,i}$ is a scalar variable, (13) can be solved as a strict LMI while treating $\gamma_{m,i}$ known *a priori*, and a bisection approach can be used to obtain the optimum of $\gamma_{m,i}$.

Remark 5. In addition, comparable to Remark 2 and Remark 3, the conditions in Theorem 2 can also reduce to the stabilization problems of conventional PWA systems or arbitrarily switched PWA systems.

4. NUMERICAL EXAMPLE

Consider a Markov jump nonlinear system with three modes, where each mode is approximated by a PWA system with respective region partitions shown in Fig. 1. An artificial bound $\|x\|_\infty \le 40$ is given for the use of outer ellipsoidal approximation methodology. The system matrices are given as below:

$$A_{1,1} = \begin{bmatrix} 0.96 & 0.12 \\ 0.12 & 1.08 \end{bmatrix} \quad A_{2,1} = \begin{bmatrix} 1.08 & 0.24 \\ 0.24 & 0.60 \end{bmatrix}$$
$$A_{1,2} = \begin{bmatrix} 0.48 & 0.12 \\ 0.12 & 0.60 \end{bmatrix} \quad A_{2,2} = \begin{bmatrix} 0.60 & 0.24 \\ 0.12 & 1.08 \end{bmatrix}$$
$$A_{1,3} = \begin{bmatrix} 0.96 & 0.24 \\ 0.12 & 0.84 \end{bmatrix} \quad A_{2,3} = \begin{bmatrix} 0.36 & 0.12 \\ 0.24 & 0.60 \end{bmatrix}$$
$$A_{1,4} = \begin{bmatrix} 0.60 & 0.12 \\ 0.48 & 0.72 \end{bmatrix} \quad A_{3,1} = \begin{bmatrix} 1.08 & 0.12 \\ 0.12 & 0.60 \end{bmatrix}$$
$$A_{1,5} = \begin{bmatrix} 0.36 & 0.12 \\ 0.24 & 0.48 \end{bmatrix} \quad A_{3,2} = \begin{bmatrix} 0.72 & 0.12 \\ 0.36 & 0.48 \end{bmatrix}$$
$$A_{3,3} = \begin{bmatrix} 0.84 & 0.12 \\ 0.24 & 0.72 \end{bmatrix} \quad A_{3,4} = \begin{bmatrix} 0.48 & 0.60 \\ 0.24 & 0.84 \end{bmatrix}$$

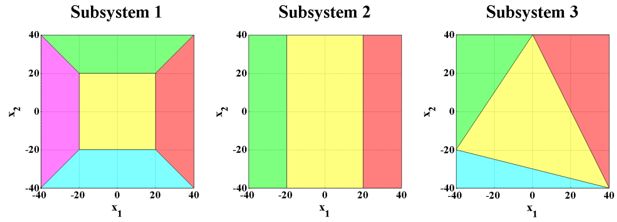


Fig. 1. Region partitions of each subsystem

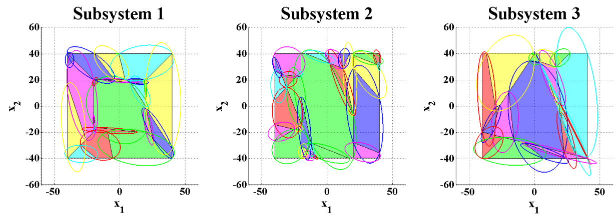


Fig. 2. Subregion with corresponding ellipsoidal outer approximation for the open-loop system

and $B_{m,i} = [\,1\ 1\,]^T, \forall i \in \mathcal{I}_m, \forall m \in \mathcal{M}$. The affine terms are given by:

$$\begin{aligned} a_{1,1} &= [\,0\ 0\,]^T & a_{2,1} &= [\,0\ 0\,]^T \\ a_{1,2} &= [\,15.6\ -3.6\,]^T & a_{2,2} &= [\,-4.8\ -1.6\,]^T \\ a_{1,3} &= [\,1.44\ -0.96\,]^T & a_{2,3} &= [\,2.4\ -8\,]^T \\ a_{1,4} &= [\,-12\ 14.4\,]^T & a_{3,1} &= [\,0\ 0\,]^T \\ a_{1,5} &= [\,-7.6\ 7.6\,]^T & a_{3,2} &= [\,1.6\ 1.6\,]^T \\ a_{3,3} &= [\,2.8\ -5.2\,]^T & a_{3,4} &= [\,0.8\ 0.8\,]^T \end{aligned}$$

The transition probabilities matrix Π is

$$\Pi = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

Based on (5) and by Algorithm 1, the subregion $\mathcal{R}_{m,i}^\omega$ for the open-loop system can be obtained, and the corresponding ellipsoidal outer approximation $\mathcal{E}_{m,i}$ is calculated while solving (4), both of which are shown in Fig. 2. Firstly, by Theorem 1, it can be checked that the given open-loop system is not stable in the stochastic sense. Then, a mode-dependent and region-dependent affine stabilizing controller is designed by Theorem 2. The controller gains are omitted here due to space limit. The variations of Lyapunov function of each closed-loop subsystems with the obtained controller are given in Fig. 3.

Given the initial condition $x_0 = [\,-38\ 35\,]^T, r_0 = 2$, a possible subsystems evolution is given in Fig. 4(b) and the the corresponding state trajectory of the closed-loop system is shown in Fig. 4(a). It can be seen from the curves that the solved mode-dependent and region-dependent affine controller is effective.

5. CONCLUSIONS

The stability and stabilization problems of a class of discrete-time Markov jump nonlinear systems are investigated, where the nonlinearities are approximated by PWA dynamics. The concept of AASPs set and an efficient algorithm to determine it have been addressed, upon which a less conservative stability criterion has been

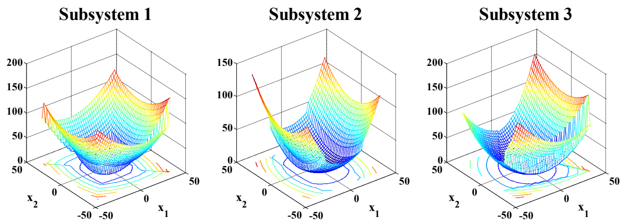
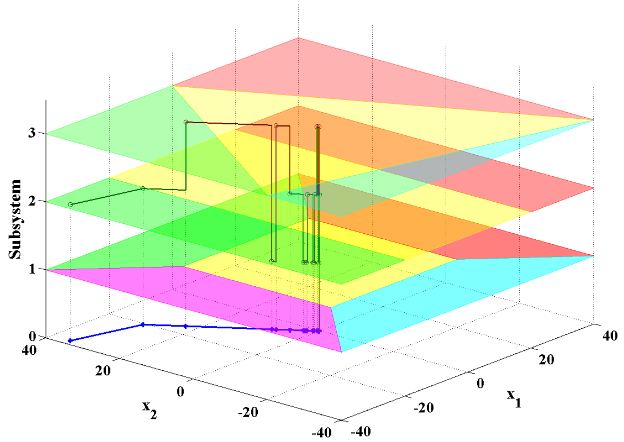
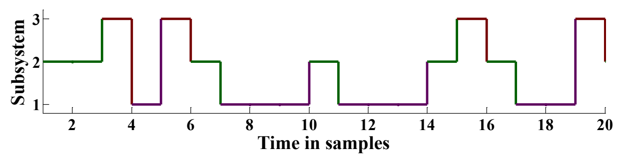


Fig. 3. Lyapunov function of the resulting closed-loop system



(a) State trajectory



(b) Subsystems evolution generated by Π

Fig. 4. State trajectory corresponding to the subsystems evolution generated by Π

established. Furthermore, a mode-dependent and region-dependent affine controller has been designed to guarantee the stochastic stability of the resulting closed-loop system. One of future research is to extend the ideas and methodologies used in the paper to other control problems of Markov jump piecewise-affine systems.

REFERENCES

Boukas, E.K. (2006). *Stochastic switching systems: Analysis and design*. Springer.

Boukas, E.K. and Yang, H. (1996). Optimal control of manufacturing flow and preventive maintenance. *Automatic Control, IEEE Transactions on*, 41(6), 881–885.

Costa, O.L.V., do Val, J.B.R., and Geromel, J.C. (1999). Continuous-time state-feedback H_2-control of Markovian jump linear systems via convex analysis. *Automatica*, 35(2), 259–268.

Gao, Y., Liu, Z., and Chen, H. (2009). Robust H_∞ control for constrained discrete-time piecewise affine systems with time-varying parametric uncertainties. *IET control theory & applications*, 3(8), 1132–1144.

Lazar, M. (2006). *Model predictive control of hybrid systems: Stability and robustness*. Ph.D. thesis, Eindhoven University of Technology.

Lin, Z., Lin, Y., and Zhang, W. (2009). A unified design for state and output feedback h control of nonlinear stochastic markovian jump systems with state and disturbance-dependent noise. *Automatica*, 45(12), 2955–2962.

Lin, Z., Liu, J., Zhang, W., and Niu, Y. (2011). Stabilization of interconnected nonlinear stochastic markovian jump systems via dissipativity approach. *Automatica*, 47(12), 2796–2800.

Lofberg, J. (2004). YALMIP: A toolbox for modeling and optimization in matlab. In *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, 284–289. IEEE.

Rantzer, A. and Johansson, M. (2000). Piecewise linear quadratic optimal control. *Automatic Control, IEEE Transactions on*, 45(4), 629–637.

Rungtweesuk, P. and Wongsaisuwan, M. (2012). State feedback redesign for PWA system with regional pole placement performance using LMI. In *Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology (ECTI-CON), 2012 9th International Conference on*, 1–4. IEEE.

Shi, P., Boukas, E.K., and Agarwal, R.K. (1999). Control of Markovian jump discrete-time systems with norm bounded uncertainty and unknown delay. *Automatic Control, IEEE Transactions on*, 44(11), 2139–2144.

Sontag, E. (1981). Nonlinear regulation: The piecewise linear approach. *Automatic Control, IEEE Transactions on*, 26(2), 346–358.

Vandenberghe, L., Boyd, S., and Wu, S.P. (1998). Determinant maximization with linear matrix inequality constraints. *SIAM journal on matrix analysis and applications*, 19(2), 499–533.

Willsky, A. and Levy, B. (1979). Stochastic stability research for complex power systems. *Lab. Inf. Decision Systems, MIT, Report no. ET-76-C-01-2295*.

Wu, Z.G., Shi, P., Su, H., and Chu, J. (2014). Asynchronous L_2–L_∞ filtering for discrete-time stochastic Markov jump systems with randomly occurred sensor nonlinearities. *Automatica*, 50(1), 180–186.

Xu, S., Chen, T., and Lam, J. (2003). Robust H_∞ filtering for uncertain Markovian jump systems with mode-dependent time delays. *Automatic Control, IEEE Transactions on*, 48(5), 900–907.

Zhang, L., Huang, B., and Lam, J. (2003). H_∞ model reduction of Markovian jump linear systems. *Systems & Control Letters*, 50(2), 103–118.

Zhang, L. (2009). H_∞ estimation for discrete-time piecewise homogeneous Markov jump linear systems. *Automatica*, 45(11), 2570–2576.

Zhang, L. and Boukas, E.K. (2009). Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities. *Automatica*, 45(2), 463–468.

Zhang, L., Gao, H., and Kaynak, O. (2013). Network-induced constraints in networked control systems-a survey. *IEEE Transactions on Industrial Informatics*, 9(1), 403–416.

Zhang, L. and Lam, J. (2010). Necessary and sufficient conditions for analysis and synthesis of Markov jump linear systems with incomplete transition descriptions. *Automatic Control, IEEE Transactions on*, 55(7), 1695–1701.