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Probabilistic constrained stochastic model predictive control for Markovian jump linear systems with additive disturbance ⋆

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Abstract: This paper is concerned with stochastic model predictive control for Markovian jump linear systems with additive disturbance, where the systems are subject to soft constraints on the system state and the disturbance sequence is finitely supported with joint cumulative distribution function given. By resorting to the maximal disturbance invariant set of the system, a model predictive control law is given based on a dynamic controller which is with guaranteed recursive feasibility and ensures the probabilistic constraints on the states. By optimizing the volume of the disturbance invariant set, the dynamic controller is given. The closed loop system under this control law is proven to be stable in the mean square sense. Finally, a numerical example is given to illustrate the developed results.

1. INTRODUCTION

Persistent efforts in stochastic control theory during the past decades make it increasingly mature. However, traditional stochastic control usually does not take constraints into account. Since in real applications, especially in industrial process control, constraints are very common due to performance requirements and limitations on process equipment. When dealing with constraints, model predictive control is usually a good choice because of its special receding horizon implementation mechanism. So stochastic model predictive control (SMPC) has been attracting more and more attention, such as Li et al. [2002], Cannon et al. [2009a,b,c], Kouvaritakis et al. [2010], Cannon et al. [2011], Hokayem et al. [2012], Cannon et al. [2012].

Probabilistic constraints are a common feature in SMPC. For example, stress cycles experienced by the blades in wind turbines are subject to a probabilistic constraint in order to achieve a specified fatigue life (see Cannon et al. [2009c]). An MPC controller is developed for SISO systems with chance constraints on the outputs in Li et al. [2002]. By designing a probability-invariant set, soft constraints can be guaranteed for systems with stochastic multiplicative uncertainty in Cannon et al. [2009b]. SMPC for systems with both multiplicative and additive stochastic uncertainty has been developed using tubes in Cannon et al. [2011]. Another type of soft constraint, say, the number of violations of constraints in a given time horizon is less than a specified integer, is considered in Cannon et al. [2009a,c]. Use of probabilistic distributions of disturbance has been explicitly made to deal with probabilistic constraints on system states in Kouvaritakis et al. [2010], in this way the conservatism can be reduced significantly. For the case in which the system state can not be measured, SMPC with output feedback has been investigated in Hokayem et al. [2012], Cannon et al. [2012].

It is noted that the above results all deal with systems with continuous state. To the best knowledge of the authors, little literature is concerned with probabilistic constrained SMPC for hybrid systems, especially, for Markovian jump linear systems (MJLS). Extensive applications of MJLS in economic systems, aircraft control systems, networked control systems, etc. (see do Valle Costa et al. [2005], Zou et al. [2010] and references therein) make the study of probabilistic constrained SMPC for MJLS a very attractive topic. Furthermore by dealing with constraints as soft constraints rather than hard constraints, control performance can be improved significantly, and the domain of attraction can be enlarged considerably. Motivated by these mentioned above, SMPC for MJLS with finitely supported additive disturbance and soft constraints on system states is studied in this paper.

The controller design is based on a dynamic controller and the maximal disturbance invariant set (MDIS) for the resulting augmented system. A recursively feasible MPC controller guaranteeing the soft constraints on system states is given by steering one step ahead augmented system state into the MDIS. And the dynamic controller dynamics are designed by optimizing the volume of the MDIS. The closed loop system under this controller is proved to be mean square stable.

The paper is organized as follows. The problem formulation is given in Section 2. The main results including the

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MPC control law design, optimization of the dynamic controller, and the MPC algorithm are presented in Section 3. In Section 4, a numerical example is given to illustrate the developed results. Finally, some conclusions are drawn in Section 5.

2. PROBLEM FORMULATION

Consider an MJLS modeled by:

$$x_{k+1} = A_{r_k} x_k + B_{r_k} u_k + G w_k, \tag{1}$$

where $x_k \in R^{n_x}$, $u_k \in R^{n_u}$, $w_k \in R^{n_w}$ are system state, control input, and uncertain stochastic disturbance, respectively; $r_k, k = 0, 1, \dots$ is a discrete time Markov chain taking values in a finite integer set $M = \{1, 2, \dots, m\}$ with transition probability matrix $T = \{\rho_{ij}\}_{i,j\in M}$, where $\rho_{ij} = \Pr\{r_{k+1} = j | r_k = i\}$; And A_{r_k}, B_{r_k}, G are appropriate matrices of conformable dimensions; The disturbance sequence $w_k, k = 0, 1, \dots$ is assumed i.i.d and independent of r_k, we also assume that w_k lies in an ellipsoid:

$$w_k \in W = \{w \in R^{n_w} : w^T \Lambda^{-1} w \le 1\}, \tag{2}$$

with $\mathbb{E}[w_k] = 0$, $\mathbb{E}[w_k w_k^T] = \Xi$ and joint cumulative distribution function given as:

$$\Pr\{w_{k,i} \le \alpha_i,\ i = 1, \dots, n_w\} = F(\alpha_1, \alpha_2, \dots, \alpha_{n_w}). \tag{3}$$

As pointed out by Kouvaritakis et al. [2010], finite support assumption of random disturbance of (2) is more general in real applications compared with the one of infinite support.

System (1) is assumed to be subject to probabilistic constraints on system states as follows:

$$\Pr\{|g^T x_k| \le h\} \ge p,\ k \ge 1, \tag{4}$$

where $g \in R^{n_x}$ and $h \in R$ are used to account for constraints on system states; and $0 < p < 1$ is a specified probability in which constraints are satisfied. Without loss of generality, only one-dimensional constraints on linear combination of only system state are considered. However, the case with more constraints and even for the constraints on both system states and control inputs can be dealt with by easy extension.

Our objective is to design an MPC controller to stabilize the system (1) subject to (2)-(4) in the mean square sense, while at each time k minimizing the following quadratic cost:

$$J_k = \sum_{n=0}^{\infty} \mathbb{E}_k[x_{k+n|k}^T Q_{r_{k+n|k}} x_{k+n|k} + u_{k+n|k}^T R_{r_{k+n|k}} u_{k+n|k}], \tag{5}$$

where $Q_i \ge 0$, $R_i > 0$, $i \in M$ are appropriate weighting matrices; and $x_{k+n|k}$, $u_{k+n|k}$, $r_{k+n|k}$ are predicted future states, inputs, and modes at time $k+n$ predicted at time k, respectively; by $\mathbb{E}_k[\cdot]$ we denote the expected value of a corresponding random variable conditional on the information available at time k.

To ensure (5) is well defined, we also assume that the disturbance sequence $w_k, k = 0, 1, \dots$ is with bounded energy, i.e., $\sum_{k=0}^{\infty} \mathbb{E}[w_k^T w_k] = \varpi < \infty$. If it is not the case, long run average cost should be considered (cf.do Valle Costa et al. [2005], Kouvaritakis et al. [2010]).

At time instant k, once the optimisation (5) is solved, only the first control move, that is, u_k is implemented. Then the optimization problem will be reformulated and solved based on the predicted information and new measurements at the next time instant $k + 1$. This procedure repeats as time evolves.

The problem formulation stated here refers to that in Kouvaritakis et al. [2010] for the linear time invariant system. However, due to the existence of Markovian jump of the modes, the prediction of the future system states becomes more complicated. And also it becomes more difficult to achieve a guarantee of the recursive feasibility of the resulting MPC algorithm. So results in Kouvaritakis et al. [2010] can not be applied directly. To overcome these problems, set-valued predictions will be adopted and we will resort to the MDIS to guarantee the recursive feasibility of the MPC algorithm.

3. MAIN RESULTS

3.1 Controller structure

Since the MDIS plays an important role in our results, a brief description of it is given before presenting the controller design.

For an MJLS with the following dynamics:

$$x_{k+1} = A_{r_k} x_k + B_{r_k} w_k, \tag{6}$$

$$y_k = C x_k, \tag{7}$$

the maximal disturbance invariant set $O_\infty\,(A, B, C, M, W, Y)$ or simply O_∞ is defined as:

$$O_\infty = \left\{x_0 : y_k \in Y, \forall k \in Z^+, \forall r_k \in M, \forall w_k \in W\right\},$$

where $C \in R^{n_y \times n_x}$ is the output matrix; Z^+ is the set of nonnegative integers; and Y is a compact set containing the origin;

$$y_0 = C x_0,$$
$$y_k = C A_{r_{k-1}} \cdots A_{r_0} x_0 + \sum_{l=0}^{k-1} C A_{r_{k-1}} \cdots A_{r_{l+1}} B_{r_l} w_l, k \ge 1.$$

From the above definition, we can conclude that

$$x_0 \in O_\infty \Rightarrow x_1 = A_{r_0} x_0 + B_{r_0} w_0 \in O_\infty,$$
$$\forall r_0 \in M, \forall w_0 \in W.$$

That is, O_∞ is disturbance invariant.

The above definition follows from Kolmanovsky and Gilbert [1998]. For more details about maximal invariant sets, readers may refer to Gilbert and Tan [1991], Kolmanovsky and Gilbert [1998], Pluymers et al. [2005].

Now we are ready to present the predictive controller design. For system (1)-(5), referring to Kouvaritakis et al. [2000], Cannon and Kouvaritakis [2005], we adopt the following predictive controller:

$$u_{k+n|k} = K_{r_{k+n|k}} x_{k+n|k} + c_{k+n|k}, n \ge 0, \tag{8}$$

where $K_i, i \in M$ are off-line designed to be optimal in some sense, which also guarantee the asymptotical stability of the closed loop system $x_{k+1} = \Phi_{r_k} x_k$ irrespective of the mode transition, where $\Phi_i = A_i + B_i K_i, i \in M$; and

$c_{k+n|k}, n \geq 0$ are perturbations on the optimal control $K_{r_{k+n|k}} x_{k+n|k}, n \geq 0$ generated by the dynamic controller:

$$x^c_{k+n+1|k} = A^c_{r_{k+n|k}} x^c_{k+n|k}, \tag{9}$$

$$c_{k+n|k} = C^c x^c_{k+n|k}, \tag{10}$$

where $x^c_{k+n|k} \in R^{n_x}, n \geq 0$ is the dynamic controller state, with its initial state x^c_k as the optimization variable of the MPC algorithm; $A^c_i, i \in M, C^c$ will be designed to maximize the volume of the MDIS.

It is noted that the control law with the perturbations given by a dynamic controller is usually with a larger initial feasible region than that with finite number of perturbations (see Kouvaritakis et al. [2000], Cannon and Kouvaritakis [2005] for more details).

By combining (1) and (8)-(10), the whole autonomous system under control law (8) can be described by

$$X_{k+n+1|k} = \bar{A}_{r_{k+n|k}} X_{k+n|k} + \bar{G} w_{k+n|k}, \tag{11}$$

with $X_k = \begin{bmatrix} x_k \\ x^c_k \end{bmatrix}$, $\bar{A}_i = \begin{bmatrix} \Phi_i & B_i C^c \\ 0 & A^c_i \end{bmatrix}, i \in M$, $\bar{G} = \begin{bmatrix} G \\ 0 \end{bmatrix}$.

And the constraints (4) and (5) can be rewritten in terms of the autonomous state X_k, respectively as

$$\Pr\{|\bar{g}^T X_{k+n|k}| \leq h\} \geq p, \; n \geq 1, \tag{12}$$

$$J_k = \sum_{n=0}^{\infty} \mathbb{E}_k [X_{k+n|k} \mathbb{Q}_{r_{k+n|k}} X_{k+n|k}], \tag{13}$$

where $\bar{g} = \begin{bmatrix} g^T & 0 \end{bmatrix}^T$, $\mathbb{Q}_i = \begin{bmatrix} K_i^T R_i K_i + Q_i & K_i^T R_i C^c \\ C^{cT} R_i K_i & C^{cT} R_i C^c \end{bmatrix}, i \in M$.

In the sequel, MPC controller design will be based on the autonomous system description (11)-(13) due to its equivalence to the original one (1)-(5).

To guarantee the satisfaction of the constraints (12), we give the following lemma.

Lemma 1. For system (11), constraints (12) are satisfied if we can find an initial state x^c_k of the dynamic controller such that

$$\bar{A}_{r_k} X_k \in O_\infty \left(\bar{A}, \bar{A}\bar{G}, \bar{g}^T, M, W, \bar{Y}\right), \tag{14}$$

where $\bar{Y} = [-h + \bar{\gamma}, h - \bar{\gamma}]$, and $\bar{\gamma}$ is the smallest real number satisfying

$$\Pr\left\{|\bar{g}^T \bar{G} w_k| = |g^T G w_k| \leq \bar{\gamma}\right\} = p. \tag{15}$$

Proof. Set $\bar{X}_0 = \bar{A}_{r_k} X_k$, from (14) we have

$$-h + \bar{\gamma} \leq \bar{g}^T \bar{A}_{r_{k-1}} \cdots \bar{A}_{r_0} \bar{X}_0 + \sum_{l=0}^{k-1} \bar{g}^T \bar{A}_{r_{k-1}} \cdots \bar{A}_{r_{l+1}} \left(\bar{A}_{r_l} \bar{G}\right) w_l \leq h - \bar{\gamma}, k \geq 0.$$

Since the above inequalities hold for all realizations of the mode sequence $r_k, k = 0, 1, \ldots$, and $w_k, k \geq 0$ have the identical distributions, the above inequalities can be rewritten equivalently as:

$$-h + \bar{\gamma} \leq \bar{g}^T \bar{A}_{r_{k+n-1|k}} \cdots \bar{A}_{r_{k+1|k}} \bar{X}_0 + \sum_{l=1}^{n-1} \bar{g}^T \bar{A}_{r_{k+n-1|k}} \cdots \bar{A}_{r_{k+l+1|k}} \left(\bar{A}_{r_{k+l|k}} \bar{G}\right) w_{k+l-1|k} \leq h - \bar{\gamma}, n \geq 1$$

By combining the above inequalities with (15) and substituting $\bar{X}_0 = \bar{A}_{r_k} X_k$ into them, we get that

$$-h \leq \bar{g}^T X_{k+n|k} = \bar{g}^T \bar{A}_{r_{k+n-1|k}} \cdots \bar{A}_{r_{k+1|k}} \bar{A}_{r_k} X_k + \sum_{l=0}^{n-1} \bar{g}^T \bar{A}_{r_{k+n-1|k}} \cdots \bar{A}_{r_{k+l+1|k}} \bar{G} w_{k+l|k} \leq h, n \geq 1,$$

is satisfied with at least probability p, which completes the proof. □

3.2 The dynamic controller design

By restricting the one step ahead augmented state into the corresponding MDIS, the probabilistic constraints (12) can be guaranteed. This necessitates an algorithm to obtain the MDIS. For linear time invariant systems, Kolmanovsky and Gilbert [1998] have developed efficient algorithms to solve it. However, for MJLS no algorithm exists. Theoretically, such algorithms can be obtained by extending the results in Kolmanovsky and Gilbert [1998]. However, due to the mode uncertainty, the number of inequalities involved will grow exponentially as the increase of the horizon, namely, "curse of dimensionality ", which makes it intractable especially for large dimensional systems.

An efficient way to solve this problem is to approximate the MDIS by an ellipsoidal set. So in this section, we will give an ellipsoidal approximation of $O_\infty \left(\bar{A}, \bar{A}\bar{G}, \bar{g}^T, M, W, \bar{Y}\right)$.

Before doing it, a lemma from Kolmanovsky and Gilbert [1998] is needed.

Lemma 2. (Kolmanovsky and Gilbert [1998]) For system $x_{k+1} = Ax_k + Bw_k$ with $w_k \in \{w : w^T \Lambda^{-1} w \leq 1\}$, the set $\{x : x^T P x \leq 1\}, P = P^T > 0$ is disturbance invariant, if there exists a positive number $0 < \xi < 1$ such that

$$P^{-1} - \xi^{-1} A P^{-1} A^T - (1 - \xi)^{-1} B \Lambda B^T > 0. \tag{16}$$

Now we are ready to give the following lemma.

Lemma 3. The ellipsoidal set $O = \{X : X^T P X \leq 1\}$ is an inner approximation of $O_\infty \left(\bar{A}, \bar{A}\bar{G}, \bar{g}^T, M, W, \bar{Y}\right)$, that is, for the following system

$$X_{k+1} = \bar{A}_{r_k} X_k + \bar{A}_{r_k} \bar{G} w_k \tag{17}$$

$$Y_k = \bar{g}^T X_k \tag{18}$$

$X_k \in O \Rightarrow X_{k+1} \in O$ and $\bar{g}^T X \in [-h + \bar{\gamma} \quad h - \bar{\gamma}], \forall X \in O$, if there exists a positive number $0 < \xi < 1$ and an appropriate matrix $P = P^T > 0$ satisfying

$$\begin{bmatrix} P & P\bar{A}_i & P\bar{A}_i \bar{G} \\ * & \xi P & 0 \\ * & * & (1 - \xi)\Lambda^{-1} \end{bmatrix} > 0, i \in M, \tag{19}$$

$$\begin{bmatrix} (h - \bar{\gamma})^2 & \bar{g}^T \\ * & P \end{bmatrix} > 0, \tag{20}$$

where $*$ induces a symmetric structure in a matrix.

Proof. By applying the Schur complement, (19) is satisfied iff

$$P - \xi^{-1} P \bar{A}_i P^{-1} (P\bar{A}_i)^T - (1 - \xi)^{-1} P \bar{A}_i \bar{G} \Lambda (P \bar{A}_i \bar{G})^T > 0$$

for all $i \in M$. Then pre- and post-multiplying both sides of the above inequalities by P^{-1}, we obtain

$$P^{-1} - \xi \bar{A}_{r_k} P^{-1} \bar{A} {r_k}^T - (1 - \xi)^{-1} \bar{A}_{r_k} \bar{G} \Lambda (\bar{A}_{r_k} \bar{G})^T > 0,$$

which guarantees the disturbance invariance of the set O. To prove $\bar{g}^T X \in [-h+\bar{\gamma} \quad h-\bar{\gamma}], \forall X \in O$, using the Schur complement, (20) can be rewritten equivalently as

$$\bar{g}^T P^{-1} \bar{g} \leq (h-\bar{\gamma})^2.$$

Then we have

$$\begin{aligned} |\bar{g}^T X| &= |\bar{g}^T P^{-1/2} P^{1/2} X| \leq \|\bar{g}^T P^{-1/2}\| \|P^{1/2} X\| \\ &\leq (\bar{g}^T P^{-1} \bar{g})^{1/2} \leq h - \bar{\gamma}, \ \forall X \in O. \end{aligned}$$

completing the proof. □

Notice that the existence of a matrix P satisfying (19)-(20) guarantees the asymptotical stability of the system (17) without disturbance. Also should be pointed out is that the above approximation is made for given $\bar{A}_i, i \in M$ and $\bar{G}$. From design point of view, we want to design $\bar{A}_i, i \in M$ to make the MDIS as large as possible. To this end, following Cannon and Kouvaritakis [2005] we parameterize P and P^{-1} as follows

$$P = \begin{bmatrix} \mathbb{X}^{-1} & \mathbb{X}^{-1} U \\ U^T \mathbb{X}^{-1} & \hat{U} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \mathbb{Y} & V \\ V^T & \hat{V} \end{bmatrix}, \tag{21}$$

where $\mathbb{X} \in R^{n_x \times n_x}, \mathbb{Y} \in R^{n_x \times n_x}, U \in R^{n_x \times n_x}, \hat{U} \in R^{n_x \times n_x}, V \in R^{n_x \times n_x}, \hat{V} \in R^{n_x \times n_x}$. To guarantee $PP^{-1} = I$, it must have $UV^T = \mathbb{X} - \mathbb{Y}$. Similar to Geromel et al. [2009], we can always find $\mathbb{X}, \mathbb{Y}, U, \hat{U}, V, \hat{V}$ satisfying $PP^{-1} = I$, if $\begin{bmatrix} \mathbb{Y} & \mathbb{X} \\ \mathbb{X} & \mathbb{X} \end{bmatrix} > 0$.

To design $\bar{A}_i, i \in M$, more precisely $A_i^c, C^c, i \in M$, we give the following lemma.

Lemma 4. With (21), for $0 < \xi < 1$ the satisfaction of (19)-(20) is equivalent to the existence of matrices $\mathbb{Y}, \mathbb{X}, \Upsilon, \Gamma_i, i \in M$ of appropriate dimensions satisfying the following LMIs

$$\begin{bmatrix} \mathbb{Y} & \mathbb{X} & \Phi_i \mathbb{Y} + B_i \Upsilon & \Phi_i \mathbb{X} & \Phi_i G \\ \mathbb{X} & \mathbb{X} & \Phi_i \mathbb{Y} + B_i \Upsilon + \Gamma_i & \Phi_i \mathbb{X} & \Phi_i G \\ * & * & \xi \mathbb{Y} & \xi \mathbb{X} & 0 \\ * & * & \xi \mathbb{X} & \xi \mathbb{X} & 0 \\ * & * & * & * & (1-\xi)\Lambda^{-1} \end{bmatrix} > 0, \tag{22}$$

$$\begin{bmatrix} (h-\bar{\gamma})^2 & g^T \mathbb{Y} & g^T \mathbb{X} \\ * & \mathbb{Y} & \mathbb{X} \\ * & \mathbb{X} & \mathbb{X} \end{bmatrix} > 0, \tag{23}$$

where $\Upsilon = C^c V^T, \Gamma_i = U A_i^c V^T, i \in M$.

Proof. Pre- and post-multiplying (19), respectively by $diag\{\Pi^T \ \Pi^T \ I\}$ and $diag\{\Pi \ \Pi \ I\}$ with $\Pi = \begin{bmatrix} \mathbb{Y} & \mathbb{X} \\ V^T & 0 \end{bmatrix}$ we obtain (22); Similarly, pre- and post-multiplying (20), respectively by $diag\{1 \ \Pi^T\}$ and $diag\{1 \ \Pi\}$, we can obtain (23), which prove the equivalence of (19)-(20) to (22)-(23). □

For design purpose, we want the inner ellipsoidal approximation O is as large as possible. Furthermore as pointed out by Kouvaritakis et al. [2000], Cannon and Kouvaritakis [2005], for $O = \{X : X^T P X \leq 1\}$ with P given by (21), its projection onto the x_k space is $\{x : x^T \mathbb{Y}^{-1} x \leq 1\}$. So our dynamic controller can be designed to maximize the volume of this set, which is proportional to its determinant. The optimization problem is stated as follows:

$$\begin{aligned} \min_{\mathbb{X}, \mathbb{Y}, \Upsilon, \Gamma_i} \quad & -logdet \ \mathbb{Y} \\ s.t. \quad & (22)-(23) \end{aligned} \tag{24}$$

Once the optimization is solved, we can choose any non-singular U, and then solve $UV^T = \mathbb{X} - \mathbb{Y}$ for V. Finally we can obtain $A_i^c = U^{-1} \Gamma_i V^{-T}$ and $C^c = \Upsilon V^{-T}$. The condition (14) should also be modified as

$$\bar{A}_{r_k} X_k \in O. \tag{25}$$

Notice that although the set O is not the MDIS, it is still disturbance invariant, then the constraints (12) can still be guaranteed.

3.3 MPC algorithm

After the design of the MPC control law and the dynamic controller, we move on to the MPC algorithm in this subsection. We will first compute the cost function (13) of system (11).

Lemma 5. The cost function (13) of system (11) can be given by

$$J_k = X_k^T \Theta_{r_k} X_k + \varpi_k, \tag{26}$$

where $\Theta_i, i \in M$ solve the coupled Lyapunov equations:

$$\Theta_i - \bar{A}_i^T (\sum_{j \in M} \rho_{ij} \Theta_j) \bar{A}_i = \mathbb{Q}_i, i \in M; \tag{27}$$

$\varpi_k = \sum_{n=0}^{\infty} \sum_{i \in M} \pi_i(k+n|k) tr[\bar{G} \Xi \bar{G}^T (\sum_{j \in M} \rho_{ij} \Theta_j)]$, $\pi_i(k+n|k) = \Pr\{r_{k+n|k} = i | r_k\}, n \geq 0;$

Proof. At prediction time $k+n|k, n \geq 0$, by (27) we have

$$\begin{aligned} &\mathbb{E}[X_{k+n+1|k}^T \Theta_{r_{k+n+1|k}} X_{k+n+1|k} | r_{k+n|k}] \\ &-X_{k+n|k}^T \Theta_{r_{k+n|k}} X_{k+n|k} \\ =& \mathbb{E}_{k+n|k}[(\bar{A}_{r_{k+n|k}} X_{k+n|k} + \bar{G} w_{k+n|k})^T \Theta_{r_{k+n+1|k}} \\ &\times (\bar{A}_{r_{k+n|k}} X_{k+n|k} + \bar{G} w_{k+n|k})] - X_{k+n|k}^T \Theta_{r_{k+n|k}} X_{k+n|k} \\ =& X_{k+n|k}^T \bar{A}_{r_{k+n|k}}^T (\sum_{j \in M} \rho_{r_{k+n|k} j} \Theta_j) \bar{A}_{r_{k+n|k}} X_{k+n|k} \\ &-X_{k+n|k}^T \Theta_{r_{k+n|k}} X_{k+n|k} \\ &+\mathbb{E}[w_{k+n|k}^T \bar{G}^T (\sum_{j \in M} \rho_{r_{k+n|k} j} \Theta_j) \bar{G} w_{k+n|k}] \\ =& -X_{k+n|k}^T \mathbb{Q}_{r_{k+n|k}} X_{k+n|k} + tr[\bar{G} \Xi \bar{G}^T (\sum_{j \in M} \rho_{r_{k+n|k} j} \Theta_j)]. \end{aligned}$$

Then taking the expected values on both sides of the above inequality based on the information available at time instant k, and summing them up from $n = 0$ to ∞ and we obtain

$$\begin{aligned} &\sum_{i=0}^{\infty} \mathbb{E}_k [X_{k+n|k}^T \mathbb{Q}_{r_{k+n|k}} X_{k+n|k}] \\ =& X_k^T \Theta_{r_k} X_k - \mathbb{E}_k [X_{k+\infty|k}^T \Theta_{r_{k+\infty|k}} X_{k+\infty|k}] + \varpi_k \\ =& X_k^T \Theta_{r_k} X_k + \varpi_k \end{aligned}$$

The last equality follows from the finite energy assumption of the disturbance sequence and the fact that $\mathbb{E}_k [X_{k+\infty|k}^T \Theta_{r_{k+\infty|k}} X_{k+\infty|k}] = 0$ due to the asymptotical stability furthermore the mean square stability of the disturbance free system. This completes the proof. □

Let $\lambda = \max_{i\in M} \lambda(G^T(\sum_{j\in M} \rho_{ij}\Theta_j)G)$, from the above arguments we have

$$\begin{aligned}&\varpi_k\\ &= \sum_{n=0}^{\infty}\sum_{i\in M}\pi_i(k+n|k)tr[\bar{G}\Xi\bar{G}^T(\sum_{j\in M}\rho_{ij}\Theta_j)]\\ &= \sum_{n=0}^{\infty}\sum_{i\in M}\pi_i(k+n|k)\mathbb{E}[w_{k+n|k}^T\bar{G}^T(\sum_{j\in M}\rho_{ij}\Theta_j)\bar{G}w_{k+n|k}]\\ &\leq \lambda\sum_{n=0}^{\infty}\mathbb{E}[w_{k+n|k}^T w_{k+n|k}] \leq \lambda\varpi < \infty,\end{aligned}$$

which implies J_k in (26) is well defined. See do Valle Costa et al. [2005] for more details.

Now we are ready to give the MPC algorithm in terms of the optimization at each time instant k.

MPC Algorithm At each time instant k, solve the following optimization for x_k^c:

$$\begin{aligned}&\min_{x_k^c}\ X_k^T\Theta_{r_k}X_k\\ &s.t.\ \ X_k^T\bar{A}_{r_k}^T P\bar{A}_{r_k}X_k \leq 1,\end{aligned} \tag{28}$$

where $X_k = \left[\, x_k^T\ (x_k^c)^T \,\right]^T$. Then implement the control move $u_k = K_{r_k}x_k + C^c x_k^c$.

It is noted that, since the second term of the cost function (26) has nothing to do with the optimization variable x_k^c, it is omitted in the objective function of the optimization. And also the minimization of (28) is a quadratically constrained quadratic programming problem. To solve it, we will translate it into a semi-definite program, for which efficient solvers exist.

By applying Schur complement, optimization (28) can be equivalently translated into the following optimization.

$$\min_{x_k^c,\eta}\ \eta \tag{29}$$

$$s.t.\ \begin{bmatrix}\eta & *\\ X_k & \Theta_{r_k}^{-1}\end{bmatrix} > 0 \tag{30}$$

$$\begin{bmatrix}1 & *\\ \bar{A}_{r_k}X_k & P^{-1}\end{bmatrix} > 0 \tag{31}$$

As pointed out by Kouvaritakis et al. [2000, 2002], similar quadratically constrained quadratic programming problems can be translated into a line search problem, for which efficient methods such as Newton-Raphson method can be used. So in this way, the computation of our MPC algorithm can be further reduced.

Regarding the closed loop stabilizability and recursive feasibility of the MPC algorithm, we give the following theorem.

Theorem 6. Consider the system (11) with partial initial state x_k, if the optimization problem (29)-(31) is feasible at time instant k, then it is feasible at all subsequent time instants $k+i, i \geq 0$, and the receding horizon implementation of the MPC algorithm guarantees the mean square stability of the closed loop system.

Proof. For recursive stability, assume that the optimal solution at time instant k is $\{\eta_k^*,\ (x_k^c)^*\}$, which guarantees $X_k^T\Theta_{r_k}X_k \leq \eta_k^*$ and $\bar{A}_{r_k}X_k \in O$. Then construct a solution at time instant $k+1$ as $\{\eta_{k+1} = X_{k+1}^T\Theta_{r_{k+1}}X_{k+1}, x_{k+1}^c = A_{r_k}^c(x_k^c)^*\}$. It is clear that this choice of η_{k+1} guarantees (30). Since the set O is disturbance invariant, then for all realizations of the mode r_{k+1} and disturbance w_k, $\bar{A}_{r_{k+1}}\bar{A}_{r_k}X_k + \bar{A}_{r_{k+1}}\bar{G}w_k \in O$, that is $\bar{A}_{r_{k+1}}\begin{bmatrix}\Phi_{r_k}x_k + B_{r_k}C^c + Gw_k\\ A_{r_k}^c(x_k^c)^*\end{bmatrix} = \bar{A}_{r_{k+1}}X_{k+1} \in O$, implying the satisfaction of (31). So we can conclude that $\{\eta_{k+1} = X_{k+1}^T\Theta_{r_{k+1}}X_{k+1}, x_{k+1}^c = A_{r_k}^c(x_k^c)^*\}$ is indeed a feasible solution at time $k+1$. By induction, the optimization problem (29)-(31) is feasible at all time instants $k+n, n \geq 0$.

For mean square stability, from the proof of Lemma 5 and the feasibility of $\{X_{k+1}^T\Theta_{r_{k+1}}X_{k+1}, A_{r_k}^c(x_k^c)^*\}$, we have

$$\begin{aligned}\eta_k^* + \varpi_k &\geq X_k^T\Theta_{r_k}X_k + \varpi_k\\ &= \mathbb{E}_k[X_{k+1}^T\Theta_{r_{k+1}}X_{k+1} + \varpi_{k+1}|r_k] + X_k^T\mathbb{Q}_{r_k}X_k\\ &= \mathbb{E}_k[\eta_{k+1} + \varpi_{k+1}|r_k = i] + X_k^T\mathbb{Q}_{r_k}X_k.\end{aligned}$$

Moreover, by optimality principle η_{k+1} will be further optimized at time instant $k+1$. Then $\{\eta_k^*+\varpi_k, k \geq 0\}$ is a supermartingale, implying $\lim_{i\to\infty}\mathbb{E}_k[X_{k+n}^T\mathbb{Q}_{r_{k+n}}X_{k+n}] = 0$. Since $\mathbb{Q}_i > 0$, we have $\lim_{i\to\infty}\mathbb{E}_k\left[X_{k+n}^T X_{k+n}\right] = 0$, which in turn guarantees the mean square stability of the closed loop system. □

Finally notice that by adopting the control law $u_{k+n|k} = K_{r_{k+n|k}}x_{k+n|k}$ with $K_i, i \in M$ optimized online, the corresponding MPC controller can also guarantee the recursive feasibility and mean square stability of the closed loop system. However, since more LMIs are involved in order to solve for the feedback control gains $K_i, i \in M$, which makes the online computation burden prohibitive.

4. A NUMERICAL EXAMPLE

In this section, a numerical example is given to illustrate the efficiency of the developed results. Consider an MJLS with two modes $M = \{1, 2\}$, the system data are given by

$$A_1 = \begin{bmatrix}0 & 1\\ -2.5 & 3.2\end{bmatrix}, A_2 = \begin{bmatrix}0 & 1\\ 5.3 & -5.2\end{bmatrix}, B_1 = B_2 = \begin{bmatrix}0\\ 1\end{bmatrix},$$
$$G = \begin{bmatrix}0\\ 0.5\end{bmatrix}, T = \begin{bmatrix}0.75 & 0.25\\ 0.2 & 0.8\end{bmatrix}, Q_1 = 0.5I, Q_2 = I,$$
$$R_1 = 0.5, R_2 = 1;$$

And $w_k, 0 \leq k \leq 10$ are mutually independent and uniformly distributed in the interval $[-0.6, 0.6]$; $w_k = 0, k > 10$.

Consider the following probabilistic constraints on system states

$$\Pr\{|x_k^1 + 2x_k^2| \leq 4\} \geq 0.8,\ \ k \geq 1.$$

Then $\bar{\gamma}$ satisfying (15) can be chose as 0.48.

By solving optimization problem (24), we can obtain an ellipsoidal approximation of $O_\infty\left(\bar{A}, \bar{A}\bar{G}, \bar{g}^T, M, W, \bar{Y}\right)$ as shown in Fig. 1. Also is shown in this figure is the initial feasible region of the MPC algorithm.

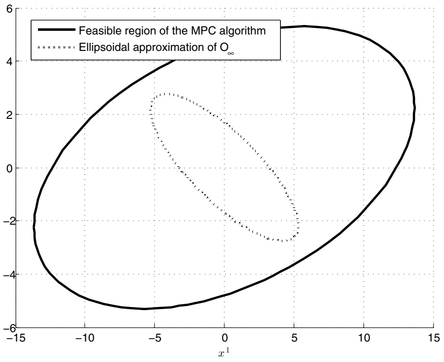

Fig. 1. Ellipsoidal approximation of O_∞ and the feasible region of the MPC algorithm

For initial conditions $x_0 = [13.2 \quad 3.38]^T$ and $r_0 = 1$, 1000 simulations have been performed. The trajectories $x_k^1 + 2x_k^2$ of the closed loop system under the controller (29)-(31) are shown in Fig. 2, where the two solid parallel lines are constraints $|x_k^1 + 2x_k^2| = 4$. It can be seen at time instant $k = 2$, the constrains are not always satisfied. There are 9.6% trajectories violate the constraints. However, at each time instant the MPC controller remains feasible. It is also obvious that the closed loop system under the designed MPC controller is guaranteed to be stable.

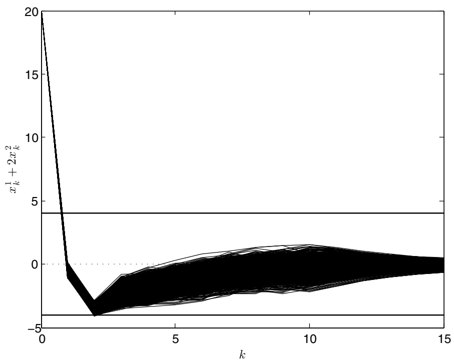

Fig. 2. Trajectories of the closed loop system

5. CONCLUSION

In this paper, probabilistic constrained stochastic model predictive control for MJLS with additive bounded l_2 disturbance is investigated. By steering the one step ahead prediction state into an disturbance invariant set, a model predictive control law is given. The dynamic controller dynamics are obtained by optimizing the volume of the MDIS. Finally recursive feasibility and mean square stabilizability of the developed MPC controller is proven. Simulation results verify the efficiency the presented results.

It should be pointed out that since an ellipsoidal approximation of the MDIS is used in the MPC algorithm, then our design is in some sense conservative. Efficient algorithms to compute the real MDIS form our future research topic.

REFERENCES

Mark Cannon and Basil Kouvaritakis. Optimizing prediction dynamics for robust MPC. *Automatic Control, IEEE Transactions on*, 50(11):1892–1897, 2005.

Mark Cannon, Basil Kouvaritakis, and Desmond Ng. Probabilistic tubes in linear stochastic model predictive control. *Systems & Control Letters*, 58(10):747–753, 2009a.

Mark Cannon, Basil Kouvaritakis, and Xingjian Wu. Model predictive control for systems with stochastic multiplicative uncertainty and probabilistic constraints. *Automatica*, 45(1):167–172, 2009b.

Mark Cannon, Basil Kouvaritakis, and Xingjian Wu. Probabilistic constrained MPC for multiplicative and additive stochastic uncertainty. *Automatic Control, IEEE Transactions on*, 54(7):1626–1632, 2009c.

Mark Cannon, Basil Kouvaritakis, Saša V Raković, and Qifeng Cheng. Stochastic tubes in model predictive control with probabilistic constraints. *Automatic Control, IEEE Transactions on*, 56(1):194–200, 2011.

Mark Cannon, Qifeng Cheng, Basil Kouvaritakis, and Saša V Raković. Stochastic tube MPC with state estimation. *Automatica*, 48(3):536–541, 2012.

Oswaldo Luiz do Valle Costa, M Marcelo Dutra Fragoso, and Ricardo Paulino Marques. *Discrete time Markov jump linear systems*. Springer, 2005.

José C Geromel, Alim PC Gonçalves, and André R Fioravanti. Dynamic output feedback control of discrete-time Markov jump linear systems through linear matrix inequalities. *SIAM Journal on Control and Optimization*, 48(2):573–593, 2009.

Elmer G Gilbert and K Tin Tan. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *Automatic Control, IEEE Transactions on*, 36(9):1008–1020, 1991.

Peter Hokayem, Eugenio Cinquemani, Debasish Chatterjee, Federico Ramponi, and John Lygeros. Stochastic receding horizon control with output feedback and bounded controls. *Automatica*, 48(1):77–88, 2012.

Ilya Kolmanovsky and Elmer G Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4(4):317–367, 1998.

Basil Kouvaritakis, JA Rossiter, and Jan Schuurmans. Efficient robust predictive control. *Automatic Control, IEEE Transactions on*, 45(8):1545–1549, 2000.

Basil Kouvaritakis, Mark Cannon, and J Anthony Rossiter. Who needs QP for linear MPC anyway? *Automatica*, 38(5):879–884, 2002.

Basil Kouvaritakis, Mark Cannon, Saša V Raković, and Qifeng Cheng. Explicit use of probabilistic distributions in linear predictive control. *Automatica*, 46(10):1719–1724, 2010.

Pu Li, Moritz Wendt, and Günter Wozny. A probabilistically constrained model predictive controller. *Automatica*, 38(7):1171–1176, 2002.

B. Pluymers, J.A. Rossiter, J.A.K. Suykens, and B. De Moor. The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty. In *American Control Conference, 2005. Proceedings of the 2005*, pages 804–809. IEEE, 2005.

Y Zou, T Chen, and S Li. Network-based predictive control of multirate systems. *IET control theory & applications*, 4(7):1145–1156, 2010.