

Control Contraction Metrics and Universal Stabilizability

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Abstract: In this paper we introduce the concept of universal stabilizability: the condition that every solution of a nonlinear system can be globally stabilized. We give sufficient conditions in terms of the existence of a control contraction metric, which can be found by solving a pointwise linear matrix inequality. Extensions to approximate optimal control are straightforward. The conditions we give are necessary and sufficient for linear systems and certain classes of nonlinear systems, and have interesting connections to the theory of control Lyapunov functions.

1. INTRODUCTION

Constructive control design for nonlinear systems remains a challenging problem, even in the case of full state feedback [Slotine and Li 1991], [Isidori 1995], [Kokotović and Arcak 2001].

The classical Lyapunov stability theory leads to necessary and sufficient conditions in terms of existence of control Lyapunov functions, however these may be difficult to find. A full state feedback linearization allows linear control tools to be applied; however finding such a transform can be non-trivial even when it can be shown that one exists [Isidori 1995].

Constructive design tools tend to be more limited in their scope of application. Backstepping and related methods provide a systematic and constructive procedure [Krstic et al. 1995], but are generally limited to systems of a particular “triangular” structure. For certain mechanical and electrical systems, energy and passivity methods can be used [van der Schaft 1999].

Recently, there has been increased interest in methods for systems analysis and control design based on convex optimization, for example linear matrix inequalities, integral quadratic constraints, and sum-of-squares programming [Boyd et al. 1994], [Megretski and Rantzer 1997], [Parrilo 2000]. For nonlinear control design, the density functions of [Rantzer 2001], [Prajna et al. 2004] and related techniques of occupation measures [Lasserre et al. 2008] and control Lyapunov measures [Vaidya et al. 2010] explicitly address convexity of criteria. Another approach is to piece together stabilized trajectories, with regions of stability verified via sum-of-squares programming [Tedrake et al. 2010].

Constructing a control-Lyapunov or density function requires prior knowledge of the solution to be stabilized. However, in many applications with complex system archi-

tectures, the role of feedback control is to track a setpoint or trajectory that changes in real-time. For nonlinear systems the problem of stabilizing changing trajectories can be quite different to stabilizing a single *a priori* known trajectory.

An alternative to explicit design is model predictive control, based on real-time numerical optimization. For many years this method was limited to relatively slow linear processes, but rapid advances in computer technology and optimization algorithms mean it is emerging as a feasible tool for nonlinear problems (see, e.g. Diehl et al. [2009]). Despite some clear benefits, it generally remains difficult to predict or analyse performance of nonlinear MPC schemes by any method other than exhaustive simulations.

In this paper, we introduce the concept of *universal stabilizability*, i.e. the property that *every* solution of a system is globally stabilizable. Using the tools of contraction analysis [Lohmiller and Slotine 1998b], we give convex sufficient conditions for universal stabilizability, and a constructive control design procedure. Useful properties such as exponential stability and approximate linear quadratic optimality are natural extensions, and for the case of linear systems, our conditions reduce to well-known necessary and sufficient conditions.

Historically, basic convergence results on contracting systems can be traced back to the results of Lewis [1949] in terms of Finsler metrics, and results of Hartman [1961] and Demidovich [1962]. Extensions to analysis of limit cycles have recently been developed by the authors [Manchester and Slotine 2013] building upon the early work of Borg [1960] and Hartman and Olech [1962]. Connections with Finsler structures and Lyapunov theory have recently been explored in detail by Forni and Sepulchre [2012].

The conditions we give are state-dependent linear matrix inequalities. The conditions are formally similar to those studied for a class of nonlinear H^∞ problems in Lu and Doyle [1995] and to state-dependent Riccati equations

* This work was supported in part by the Australian Research Council.

(see, e.g., Cloutier [1997]), however the class of systems and the explicit control construction we consider are quite different. An interesting feature of our method is that it breaks the control-design problem into two stages: computation of a metric, which can be performed in advance of knowledge of the solution to be stabilized, and an on-line path integration to compute the control signal.

Our methods build upon the control design suggested in Lohmiller and Slotine [1998a]. In that paper, it was argued that requiring a nonlinear system to be feedback transformable to a stable linear system – as in feedback linearization – was too strong, since it requires an involutivity condition to be added to the rather natural controllability condition [Isidori 1995]. If instead one required only that the system be feedback transformable to a “nice” stable nonlinear system (a contracting system), then this can be achieved using the controllability condition alone.

Due to space restrictions, we have omitted all proofs from this version of the paper, but a complete version can be found on arXiv.org

2. UNIVERSAL STABILIZABILITY

For most of this paper, we will consider a nonlinear time-dependant control-affine system

$$\dot{x}(t) = f(x(t), t) + B(t)u(t) \quad (1)$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ are state and control, respectively, at time $t \in \mathbb{R}^+ := [0, \infty)$. The function $f : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is assumed to be smooth, and $B : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times m}$ is a time-dependent matrix.

Contraction analysis is the study of (1) by way of the associated system of differential dynamics:

$$\dot{\delta}_x(t) = A(x, t)\delta_x(t) + B(t)\delta_u(t) \quad (2)$$

where $A(x, t) = \frac{\partial}{\partial x} f(x, t)$ is the Jacobian matrix.

In Section 8 we will consider more general systems $\dot{x} = f(x, u, t)$ but for the moment we note that many systems not naturally appearing in the form (1) can be put in that form, either exactly or approximately, by change of variables or introducing new states.

A solution of (1) is a pair of vector signals $(x(t), u(t))$ satisfying (1) over the interval \mathbb{R}^+ . For simplicity, in this paper we will assume that (1) is such that solutions exist and are unique. As such, we will also use the notation $\phi(t, x_0, u)$ to denote the solution of x at time $t > 0$ of (1) starting from initial condition $x(0) = x_0$ and under the application of the control input $u(\tau), \tau \in [0, t]$.

A static state-feedback controller is a function $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$. The main objective is to design such a function so that the behaviour of the closed-loop system

$$\dot{x}(t) = f(x(t), t) + B(t)k(x(t), t) \quad (3)$$

is in some sense desirable, e.g. globally stable or optimal in some sense. Solutions of the closed-loop system with a given state-feedback controller k will be denoted $\phi_k(t, x_0)$.

As is standard, a solution (x^*, u^*) defined on $[0, \infty)$ is said to be globally asymptotically stabilized by a feedback controller $u = k(x, t)$ if both of the following hold:

- (1) For any α there exists an ϵ such that $|x_0 - x^*(0)| < \epsilon$ implies $|\phi_k(t, x_0) - x^*(t)| < \alpha$,

- (2) For any initial condition $x_0 \in \mathbb{R}^n$, the closed loop solution satisfies $|\phi_k(t, x_0) - x^*(t)| \rightarrow 0$.

Global exponential stabilization refers to the stronger condition that there exists a K and λ such that

$$|\phi_k(t, x_0) - x^*(t)| \leq Ke^{-\lambda t}|x_0 - x^*(0)|$$

for all $x(0)$.

In this paper, we will study the following property:

Definition 1. A system of the form (1) is said to be *universally stabilizable by state feedback* if for any solution (x^*, u^*) defined on $t \in [0, \infty)$ there exists a state feedback controller $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that (x^*, u^*) is globally stabilized by $u = k(x, t)$.

Analogously, we also consider the notion of *universally exponentially stabilizable with rate λ* .

Note that universal stabilizability is a stronger condition than global stabilizability of a particular solution.

3. CONTROL CONTRACTION METRICS

The following definition is central to this paper:

Definition 2. A function $V(x, \delta_x, t) = \delta'_x M(x, t)\delta_x$, with $c_1 I \leq M(x, t) \leq c_2 I$ for some $c_2 \geq c_1 > 0$, is said to be a *control contraction metric* for the system (1) if $\frac{\partial V}{\partial x} B(t) = 0$ and

$$\frac{\partial V}{\partial \delta_x} B(t) = 0 \implies \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t) + \frac{\partial V}{\partial \delta_x} A(x, t)\delta_x < 0 \quad (4)$$

for all x, t .

We will also make use of a Riemannian distance function between any two points at a given time $d(x_1, x_2, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined like so: let $\Gamma(x_1, x_2)$ denote the set of all smooth paths connecting x_1 and x_2 , where each $\gamma \in \Gamma(x_1, x_2)$ is parametrised by $s \in [0, 1]$, i.e. $\gamma(s) : [0, 1] \rightarrow \mathbb{R}^n$. The path length of γ is then defined as

$$L(\gamma, t) := \int_0^1 D\left(\gamma(s), \frac{\partial}{\partial s} \gamma(s), t\right) ds$$

where D is a Finsler function [Bao et al. 2000]. In this paper, we will primarily work with functions of the form $D(x, \delta_x, t) = \sqrt{V(x, \delta_x, t)}$. The distance between two points is then defined as

$$d(x_1, x_2, t) = \min_{\gamma \in \Gamma(x_1, x_2)} L(\gamma, t)$$

The existence of a minimizing path, which we denote $\gamma_{x_1}^{x_2}(t, s)$, is implied by the Hopf-Rinow Theorem.

In this conference paper we restrict ourselves to simple choices of V and D . Other choices are possible, including non-quadratic metrics and matrix measures. See, e.g., Lohmiller and Slotine [1998a] and Forni and Sepulchre [2012] for detailed discussion of non-quadratic metrics in contraction analysis.

The main utility of a control contraction metric is that it allows one to construct stabilizing controls by computing path integrals of a suitable δ_u .

Theorem 1. If a control contraction metric exists for a system of the form (1), then the system is universally stabilizable by static state feedback.

4. CONVEX CONDITION AND CONSTRUCTION OF EXPLICIT CONTROL

The second main result of this paper is the following, where we use the notation S_+^n to denote the convex cone of symmetric positive semidefinite matrices:

Theorem 2. Consider the system (1) with differential dynamics (2). If there exists a matrix function $W(x, t) : \mathbb{R}^n \times \mathbb{R} \rightarrow S_+^n$, a function $\rho(x, t) \geq 0$, and constants $\alpha_2 \geq \alpha_1 > 0$ such that

$$\alpha_1 I \leq W \leq \alpha_2 I, \quad (5)$$

$$-\dot{W} + WA' + AW - \rho BB' < 0, \quad (6)$$

for all x, t , then the system (1) is universally stabilizable by static state feedback, and $\delta_x W(x, t)^{-1} \delta_x$ is a control contraction metric.

In the above condition, $\dot{W}(x, t)$ is a matrix with the i, j element given by $\frac{\partial W_{i,j}}{\partial t} + \frac{\partial W_{i,j}}{\partial x}(f(x, t) + B(t)u)$.

Let $\gamma_t(s)$ be a locally minimizing geodesic (with respect to the metric $M = W^{-1}$) connecting $x^*(t)$ to $x(t)$, parameterized by $s \in [0, 1]$, then the control signal:

$$u(t) = u^*(t) - \frac{1}{2} \int_0^1 \rho(\gamma_t(s), t) B(t)' M(\gamma_t(s), t) \frac{\partial \gamma_t}{\partial s} ds$$

is stabilizing to $x^*(t)$.

Remark 1. We note that the conditions are pointwise linear matrix inequalities in $W(x, t)$ and $\rho(x, t)$, and are thus amenable to solution via convex optimization algorithms. We will address this further in Section 6.

For systems of the form (1), the matrix \dot{W} has elements $\dot{W}_{i,j} = \frac{\partial W_{i,j}}{\partial t} + \frac{\partial W_{i,j}}{\partial x}(f(x, t) + B(t)u)$, which are affine in the control signal u . Therefore for inequality (6) to hold for all u , it is necessary that $\frac{\partial W_{i,j}}{\partial x} B(t) = 0$, i.e. $W(x, t)$ must be constant along the subspace S_u of state space in which the control signal directly acts.

4.1 Conditions for Exponential Stabilization

A stronger result is the following:

Theorem 3. Consider the system (1) with differential dynamics (2). If there exists a matrix function $W(x, t) \in S_+^n$, $\rho(x, t) \geq 0$ and $\alpha_2 \geq \alpha_1 > 0$ such that

$$\alpha_1 I \leq W \leq \alpha_2 I, \quad (7)$$

$$-\dot{W} + WA' + AW - \rho BB' \leq -2\lambda W(x), \quad (8)$$

for all x, u, t , then the system is universally exponentially stabilizable with rate λ by state feedback.

5. GUARANTEED QUADRATIC COST CONTROL

Explicit solutions of nonlinear optimal control problems are generally extremely challenging to find. A general approach is to formulate a nonlinear partial differential equation, the Hamilton Jacobi Bellman equation, and attempt to find a so-called viscosity solution [Crandall et al. 1992]. However, in most cases of practical interest this computational problem is intractable.

A less demanding approach, frequently used in robust control, is to find a *guaranteed cost control* [Petersen et al.

2000]. In this framework, one tries to find a controller that guarantees a conservative upper bound on the cost, which is usually obtained via dissipation inequalities. Then one can approach approximate optimal control by minimizing this upper bound. In this section we show that our method can be extended to guaranteed cost control of nonlinear systems, by making use of *differential* dissipation inequalities [Manchester and Slotine 2013].

Theorem 4. Given $Q(t) \geq 0$ and $R(t) > 0$ and the differential dynamics (2), suppose there exists a $W(x, t)$ such that

$$\begin{bmatrix} (\dot{W} - WA' - AW + BR^{-1}B') & WL' \\ LW & I \end{bmatrix} \geq 0 \quad (9)$$

for all x, u, t , with $Q = L'L$. Now, for any feasible trajectory $(x^*(t), u^*(t))$, consider the LQ cost function

$$J = \int_0^\infty \begin{bmatrix} (x(t) - x^*(t)) \\ (u(t) - u^*(t)) \end{bmatrix}' \begin{bmatrix} Q(t) & 0 \\ 0 & R(t) \end{bmatrix} \begin{bmatrix} (x(t) - x^*(t)) \\ (u(t) - u^*(t)) \end{bmatrix} dt.$$

then there exists a state-feedback controller such that

$$J(x(0)) \leq \int_0^1 \delta_x' W(x)^{-1} \delta_x ds$$

with $x = \gamma(t, s)$ and $\delta_x = \frac{\partial \gamma(t, s)}{\partial s}$ over the path $\gamma_{x^*(0)}^{x(0)}(t, s)$.

As approximate optimal control, one could consider maximizing, e.g. the smallest eigenvalue of $W(x)$ or the trace of $W^{-1}(x)$, both of which are concave in $W(x)$, and correspond to “worst case” or “expected” values a particular differential cost.

It is well known that for linear systems, stabilizability implies solvability of the LQR problem if $Q \geq 0, R > 0$ Hespanha [2009]. In fact, for nonlinear systems a similar claim can be made in terms of guaranteed cost controllability.

Theorem 5. Suppose a system of the form (1) is universally stabilizable, as verified by the conditions of Theorem 2, then there exists a solution to the quadratic guaranteed cost control problem in Theorem 4.

6. COMPUTATIONAL APPROACHES

The methods we have described naturally break the control problem down into two stages: firstly, a pointwise LMI must be solved for $W(x)$, giving the metric $\delta_x' M(x) \delta_x$, guaranteeing a particular form of stabilizability; secondly, the on-line computation involves computing a path integral along a line or geodesic. A full discussion is not possible due to space restrictions, but in this section we briefly describe some applicable techniques.

The main calculation is to find a matrix function $W(x, t)$ and a scalar function $\rho(x, t)$ satisfying (5) and (6) for all x, u, t , or the similar conditions in Sections 4.1 and 5. These conditions are convex, but the decision variables W and ρ are infinite dimensional and there are infinitely many LMI constraints (5) and (6), due to the dependence on x and t . To be practically computable, we must reduce these to a finite-dimensional optimization problem.

If $f(x, t) = f(x)$ is a vector of polynomials in x , one can search for polynomial $W(x)$ and $\rho(x)$ satisfying the conditions of the theorem using sum-of-squares programming [Parrilo 2003]. Alternatively, over a compact set in state

space one could specify a finite set of basis functions for W and ρ , and check the inequalities (5) and (6) at a finite grid of points. This again leads to a finite-dimensional set of linear matrix inequalities. A similar method was investigated by Johansen [2000] for computation of Lyapunov functions.

In the simplest case that a $M(t)$ and $\rho(t)$ can be found independent of x and satisfying the conditions, then the shortest path is simply a straight line in state space, and the feedback control is simply the linear gain $u(t) = u^*(t) - \frac{1}{2}\rho(t)B(t)'M(t)(x(t) - x^*(t))$.

When a state-varying metric is found, on-line computation of the geodesic is required. Algorithms for computing geodesics have been developed in several fields, including computational physics, computer graphics, and robot motion planning Mémoli and Sapiro [2001], Boykov and Kolmogorov [2003], LaValle [2006]. The approach described in this subsection may also be applicable to the controller construction in [Lohmiller and Slotine 1998a].

6.1 Open-loop Control via Path Images

An alternative method giving an open-loop control signal that drives $x(t) \rightarrow x^*(t)$ was suggested in [Lohmiller and Slotine 1998a]. This method also uses path integrals of a differential feedback gain $\delta_u = K\delta_x$, although K was constructed by a different method related to the controllability conditions of Isidori [1995]. In particular, it was suggested to compute a path from $x^*(0)$ to $x(0)$, and integrate $\delta_u = K\delta_x$ along this path for $u(0)$. For times $t > 0$, the forward image of each point on the path under the flow of the system is simulated, generating a family of paths $\gamma(s, t)$, and controls $u(s, t)$.

This method can also be applied for the systems we study with $K = -\frac{1}{2}\rho B'M$. One advantage is that it does not require recomputing a geodesic at each time to compute the control signal. A disadvantage is that it does not give a feedback law, and could be non-robust. E.g., if the true system behaviour does not match the simulated forward image of $x(0)$ then there are no guarantees of stability.

One can also imagine a hybrid of this approach and the previous one, in which forward images of the initial path are computed, but with some corrective term based on gradient descent to converge towards geodesics. In this case, the resulting feedback system would be a dynamic state feedback with an infinite-dimensional controller state.

6.2 Two-stage Computation of Explicit Feedback

If $W(x, t)$ has been found using the above-described convex optimization procedure, then one can fix $M(x, t) = W(x, t)^{-1}$ which implies the existence of matrix $K(x)$ such that

$$\delta'_x \left(\dot{M}(x, t) + 2M(x, t)[A(x, t) + B(t)K(x, t)] \right) \delta_x < 0 \quad (10)$$

for all δ_x , i.e., the existence of a differential state feedback $\delta_u = K(x)\delta_x$. However, in order to compute an explicit control law $u = k(x, t)$, this differential relation must be integrable.

Let $K_i(x)$ refer to the i^{th} row of the matrix $K(x)$, then the additional convex constraint

$$\frac{\partial}{\partial x} K_i(x) = \left(\frac{\partial}{\partial x} K_i(x) \right)', \quad i = 1, 2, \dots, m \quad (11)$$

ensures the existence of a function $k(x)$ satisfying $\frac{\partial k}{\partial x} = K(x)$ by the Poincaré lemma. For example, consider a linearly parameterized class of controllers where the i^{th} control element is given by

$$u_i = k_i(x) := \sum_{j=1}^p \kappa_{i,j} \phi_j(x)$$

where $\phi_j(x)$ are smooth basis functions (e.g. a monomial basis for polynomial feedback). In this case,

$$K_i(x) := \sum_{j=1}^p \kappa_{i,j} \frac{\partial}{\partial x} \phi_j(x)$$

and (11) imposes a linear constraint on the coefficients, as long as the set of basis functions is closed under differentiation with respect to x . This can also be considered as a curl condition on a vector field.

7. DISCUSSION

7.1 Necessity for the Case of Linear Systems

In general, the conditions we provide will be sufficient but not necessary. An interesting question is whether there is a class of nonlinear systems for which they are necessary, i.e. they completely describe stabilizability. As a basic requirement, one would expect this class to include all stabilizable linear time-invariant (LTI) systems:

$$\dot{x} = Ax + Bu$$

and indeed this is the case. Our condition for LTI systems with a matrix $W > 0$ independent of x is:

$$AW + WA' - BB' < 0$$

where ρ , also constant, has been absorbed into W . This is well-known Lyapunov condition for stabilizability of linear systems – see, e.g., [Hespanha 2009, Sec 14.5].

The condition we give on computing a guaranteed cost control similarly reduces to the Riccati inequality:

$$AW + WA' - BR^{-1}B' + WQW \leq 0$$

which is known to have a solution if and only if the algebraic Riccati equation:

$$AW + WA' - BR^{-1}B' + WQW = 0$$

has a solution, given by the solution of the Riccati inequality with maximal trace. By multiplying the latter on either side by $M = W^{-1}$, this is the exactly the Riccati inequality associated with the linear quadratic regulator problem. In linear control design, changes of variables such as that from M to W are frequently used to construct linear matrix inequality conditions [Boyd et al. 1994].

7.2 Connection to Control Lyapunov Functions

The distance function $d(x^*(t), x(t), t)$ is a control Lyapunov function for a solution x^* , thus the conditions we give imply the existence of a control Lyapunov function for every solution of the system.

Control Lyapunov functions for systems of the form (1) can be characterized by the condition:

$$\frac{\partial}{\partial x} V(x, t)B(t) = 0 \Rightarrow \frac{\partial}{\partial x} V(x, t)f(x, t) + \frac{\partial}{\partial t} V(x, t) < 0$$

for all x and t . In general, it is computationally challenging to search for a $V(x, t)$ satisfying such a condition implication. If we instead consider a similar condition for the differential dynamics (2) and a function $\delta'_x M(x, t)\delta_x$:

$$MB\delta_x = 0 \Rightarrow \delta'_x(\dot{M} + A'M + MA)\delta < 0$$

Then by Finsler's theorem [Uhlig 1979], or a version of the S-Procedure losslessness theorem [Yakubovich 1971], this is equivalent to the existence of a non-negative function $\rho(x, t)$ for which the following is true:

$$\dot{M} + A'M + MA - \rho MBB'M < 0$$

which, by pre and post-multiplication by W is our condition (6). Therefore our condition guarantees both the existence of a control Lyapunov function in the traditional sense, and a "differential" control Lyapunov function – i.e. a control contraction metric.

7.3 Necessity for Hierarchical Compositions

In this section we show an example class of systems for which the above analysis can be applied: compositions of a stabilizable linear system with a contracting system.

Consider the following hierarchical composition of systems:

$$\dot{z} = \bar{A}z + \bar{B}u, \quad \dot{y} = f_1(y, z),$$

for which f_1 is partially contracting with respect to y , and the pair \bar{A}, \bar{B} is stabilizable. That is, there exists $M_1(y) \geq \alpha I$ and P such that

$$\dot{M}_1(y) + \frac{\partial f_1'}{\partial y} M_1(y) + M_1(y) \frac{\partial f_1}{\partial y} < 0 \quad (12)$$

$$\bar{A}P + P\bar{A}' - \bar{B}\bar{B}' < 0. \quad (13)$$

Let $W_1(y) = M_1^{-1}(y)$, then we also have from (12)

$$\Pi(y) := -\dot{W}_1(y) + W_1(y) \frac{\partial f_1'}{\partial y} + \frac{\partial f_1}{\partial y} W_1(y) < 0 \quad (14)$$

with $W_1(y) \leq \frac{1}{\alpha} I$.

The combined system with state $x := [y', z']'$ is:

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f_1(y, z) \\ \bar{A}z \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} u =: f(x) + Bu$$

and the differential dynamics are (2) with

$$A(y) := \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ 0 & \bar{A} \end{bmatrix}$$

Let us consider a class of metrics parametrised by $\beta > 0$:

$$W(y) = \begin{bmatrix} \beta(y)M_1(y)^{-1} & 0 \\ 0 & P \end{bmatrix}$$

Noting that

$$\dot{W} = \begin{bmatrix} \dot{W}_1 & 0 \\ 0 & 0 \end{bmatrix}, BB' = \begin{bmatrix} 0 & 0 \\ 0 & \bar{B}\bar{B}' \end{bmatrix} \quad (15)$$

we see that (6) reduces to

$$\begin{bmatrix} \beta(y)\Pi(y) & \beta(y)W_y \frac{\partial f}{\partial z} \\ \beta(y) \frac{\partial f'}{\partial z} W_y & \bar{A}'P + P\bar{A} - \bar{B}\bar{B}' \end{bmatrix} < 0. \quad (16)$$

Now, since the top-left block is negative definite by assumption (14), negativity of the entire matrix is equivalent, by the Schur complement, to the following

$$\bar{A}'P + P\bar{A} - \bar{B}\bar{B}' - \beta(y)W_y \frac{\partial f}{\partial z} \Pi(y) \frac{\partial f'}{\partial z} W_y < 0. \quad (17)$$

Now, since $\bar{A}'P + P\bar{A} - \bar{B}\bar{B}' < 0$ by assumption (13), this can clearly be satisfied for sufficiently small $\beta(y) > 0$. Thus we have another class of systems for which the conditions of Theorem 2 are necessary and sufficient.

This situation is clearly simpler than Lyapunov-based control design: if a nonlinear system is Lyapunov stable at the origin, and driven by a stabilizable linear system, it is not so clear that the entire system is stabilizable. This is the fact that led to the development of backstepping [Krstic et al. 1995].

7.4 On Transverse Contraction

The results in this paper also suggest a new interpretation of limit cycle stability. In [Manchester and Slotine 2013] the authors introduced the concept of transverse contraction, a generalisation of prior work by Borg [1960], Hartman and Olech [1962], Leonov et al. [1996], to study the existence of limit cycles in autonomous systems:

$$\dot{x} = f(x).$$

Transverse contraction refers to the existence of a metric function $V(x, \delta_x)$ such that

$$\frac{\partial V(x, \delta)}{\partial x} f(x) + \frac{\partial V(x, \delta)}{\partial \delta} \frac{\partial f(x)}{\partial x} \delta \leq -\lambda V(x, \delta), \quad (18)$$

for all $\delta \neq 0$ such that $\frac{\partial V}{\partial \delta} f(x) = 0$. In [Manchester and Slotine 2013] it was shown that, for a Riemannian metric $\sqrt{\delta' M(x) \delta}$, this is equivalent to a pointwise LMI condition:

$$\dot{W}(x) \geq W(x)A(x)' + A(x)W(x) - \rho(x)Q(x)' + 2\lambda W(x), \quad (19)$$

where $A(x) = \frac{\partial f}{\partial x}$, $Q(x) = f(x)f(x)'$ and $W(x) > 0$. It was shown that this implies stability under time reparametrisation – a.k.a. Zhukovsky stability – which is known to imply existence of a stable limit cycle.

In the context of this paper, that condition is equivalent to stabilizability of the system:

$$\dot{x} = f(x) + f(x)u.$$

The interpretation of this fact is that a control input acting "along" the flow of the system $f(x)$ can speed up or slow down the trajectory, but not change its phase portrait.

8. EXTENSION TO MORE GENERAL NONLINEAR SYSTEMS

In this section we briefly discuss extension to more general nonlinear systems

$$\dot{x}(t) = f(x(t), u(t), t)$$

with differential dynamics

$$\dot{\delta}_x(t) = A(x, u, t)\delta_x(t) + B(x, u, t)\delta_u(t).$$

The significant change from the previous sections is that the A and B matrices now depend on u .

In this case, we can also consider a matrix $W(x, u, t)$ and constant $\rho(x, u, t)$. Then, suppose conditions (5) and

(6) hold for all x, u, t . The path integral equation for the control signal $u(s)$ now depends on $u(s)$, and can be constructed if a solution exists to the differential equation:

$$\frac{d}{ds}u = -\frac{1}{2}\rho(\gamma, u)B(\gamma)'M(\gamma, u)\frac{\partial\gamma}{\partial s}(s)$$

with boundary condition $u(0) = u^*(t)$. If the dependency of these terms on u can be well controlled, then it may be possible to guarantee existence by, e.g., the Bellman-Grönwall lemma. We defer detailed discussion of this issue for a later publication.

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