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Adaptive output feedback second order sliding mode control with unknown bound of perturbation

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Abstract: An adaptive output feedback is presented. The adaptive gain is used to relax the restriction of knowing the perturbation bound. it is proved that with the adaptive gain both, the controller and the observer converge to the origin. A Lyapunov approach is used to proved the results and give restrictions on the fixed gains of the controller and the observer.

Keywords: Sliding mode control, Lyapunov functions, adaptive control, variable structure control.

1. INTRODUCTION

Sliding-mode control is considered as one of the main methods for control and observation under uncertainty conditions (Utkin , 1992), (Utkin , 2009). Classical sliding modes can bring an output σ to zero in finite time and can keep it in zero exactly by using high-frequency control switching, in a very accurate and robust way, but it requires the relative degree of σ to be one with respect to the control variable u. This relative degree restriction can be removed by means of higher-order sliding modes (HOSM). For relative degree $r > 1$, it is possible, using HOSM algorithms, to bring to zero in finite time not only σ but also $r-1$ of its time derivatives $\sigma = \dot{\sigma} = \ldots = \sigma(r-1) = 0$, and they will be kept at zero exactly and in a robust manner (Fridman, Levant , 2002), (Levant, 1993).

Twisting controller (TC) is historically the first 2-SMC algorithm (Levant, 1993), (Levant , 1985) that drives the output and its derivative of the system with relative degree two to the origin in finite time in the presence of the bounded disturbance (even dry friction), whose boundary is known.

In (Utkin, et. al. , 2011) a second order sliding mode controller is presented with adaptive gain. The adaptation process is based on the estimation of the equivalent control, which is obtained by filtering the control signal by a low-pass filter. The disadvantage of this method is that the equivalent control signal is delayed by the effect of the filter. Also the bound of the perturbation derivative should be known.

The problem of state observation for a system whose dynamics may involve poorly known, perhaps even non locally Lipschitz functions and whose output measurement may be corrupted by noise is address in (Sanfelice, Praly , 2011). The proposed method is based on the study of differential inequalities involving quadratic functions of the error system in two coordinate frames plus the gain adaptation law.

The objective and the contribution of this paper are in developing the adaptive-gain twisting control (ATC) algorithm in the presence of the bounded disturbances with the unknown boundaries, using the Lyapunov approach (Moreno, Osorio , 2012), (Polyakiv, Poznyak , 2009), (Shtessel, et. al. , 2010), (Kochalummoottil , 2011). Furthermore only the state x_1 is available, in other to estimate the state x_2 a higher order sliding mode observer is introduce with adaptive gain.

The main results of this paper are (i) the state feedback controller with adaptive gain, (ii) the observer with adaptive gain and (iii) the output feedback controller with adaptive gains resulting from the interconnection. A Lyapunov approach is used to prove the results. The novelty on this work is that the perturbation bound exist but in not known.

2. PROBLEM STATEMENT AND MAIN RESULT

Consider the following SISO dynamical system (Levant , 2007)

$$\dot{\xi} = f(t,\xi) + g(t,\xi)u$$
$$\sigma = h(t,\xi)$$

where $\xi \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is the control variable, and $\sigma \in \mathbb{R}$ is the measured output. The functions f, g, h and the dimension n are unknown. It is assumed that the system has a well defined relative degree 2. Under these conditions, taking the second order total time derivative of the output σ, and defining $x_1 = \sigma, x_2 = \dot{\sigma}$, it is obtained

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= a(t,\xi) + u\end{aligned} \qquad (1)$$

where $a(t,\xi)$ is some unknown scalar function that is bounded, i.e.

$$|a(t,\xi)| \leq a_p$$

In this paper the coefficient of the controller is considered a one. The results presented here can be easily extended to the case where the coefficient is unknown but the upper and lower bounds of it are known.
The control task is to drive the output σ to zero in finite time and to keep $\sigma \equiv 0$, despite the perturbation, using only the measurement of the output σ.
A discontinuous globally bounded controller that drives the output σ to the origin in finite time and keeps $\sigma \equiv 0$ is the Twisting Algorithm (Levant, 1993). In order to implement this controller the states should be available. In the case that is considered in this paper, the only information available is the state x_1. The state x_2 has to be determined by means of an observer with the use of the state x_1. This problem has been solved in (Moreno, 2012), (Levant , 1985) with the knowledge of the perturbation bound a_p.
In this paper it is assumed that the perturbation $a(\xi,t)$ is bounded. The bound is considered as unknown. To cope with this problem the gains of the controller and the observer are adapted. In section 3 a twisting controller is designed with adaptive gain, assuming that both states are available. In section 4 an observer with adaptive gain is designed to estimate the state x_2. Finally in section 5 it is stated that the closed loop system enforce the states to the origin.

3. ADAPTIVE TWISTING

Considering that the states x_1, x_2 are available, the twisting controller drives the output σ to zero in finite time if the gains are selected properly. The restriction on the gains depends on the bound of the perturbation $a(t,\xi)$, that means that the bound a_p should be known. Given that the bound a_p is unknown, an adaptive gain $L(t)$ is introduced so the twisting controller with adaptive gain is

$$u = -L(t)\left(k_1 sign(x_1) + k_2 sign(x_2)\right) \qquad (2)$$

where the adaptive gain dynamics are defined by

$$\dot{L}(t) = \begin{cases} l(t), & if \;\; x = 0 \\ 0, & if \;\; x \neq 0 \end{cases} \qquad (3)$$

were $l(t) > 0$ is a positive function. The closed loop system becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -L(t)\left(k_1 sign(x_1) + k_2 sign(x_2)\right) + a(\xi,t) \end{aligned} \qquad (4)$$

Theorem 1. Consider the system (1) with a bounded perturbation $|a(\xi,t)|$ and the controller (2) with adaptive gain. Assuming that the bound a_p is unknown, if the controller gains are selected such that

$$k_1 > k_2 > 0$$

then the trajectories of the closed loop system (4) will reach the equilibrium point in finite time. Moreover the adaptive gain $L(t)$ will remain bounded for all time.

□

Note that after reaching the origin the states will remain there unless the perturbation grow once the equilibrium point is reached. In that case the adaptive gain $L(t)$ will grow until it reaches a value that is large enough to drive again the trajectories to the equilibrium point

Proof. To prove that the controller (2) drives the output σ to zero in finite time, without knowing the perturbation bound a_p, a Lyapunov function (Moreno, 2012) is used. The following change of variables is introduce

$$\begin{aligned} z_1 &= \left(\frac{\kappa}{L^q}\right)^2 \frac{x_1}{L} \\ z_2 &= \left(\frac{\kappa}{L^q}\right) \frac{x_2}{L} \end{aligned}$$

where $\kappa > 0$ is some positive constant, and $0 < q \in \mathbb{R}$. In the new coordinates the system (1) is given by

$$\begin{aligned} \dot{z}_1 =& -(2q+1)\left(\frac{\dot{L}}{L}\right) z_1 + \left(\frac{\kappa}{L^q}\right) z_2 \\ \dot{z}_2 =& -(q+1)\left(\frac{\dot{L}}{L}\right) z_2 - \left(\frac{\kappa}{L^q}\right)\left[k_1 sign(z_1) + k_2 sign(z_2)\right] \\ &+ \left(\frac{\kappa}{L^q}\right)\frac{a(t,\xi)}{L} \end{aligned}$$

Consider the following Lyapunov function (Moreno, 2012) in the new coordinates

$$V(z) = \left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right)^{3/2} + \pi_2 z_1 z_2 \qquad (5)$$

where π and π_2 are positive constant that are chosen such that the inequality $k_2 > |\pi_1 - k_1| - \pi_2 \frac{2}{3}(2)^{\frac{1}{2}}$ is fulfilled.
It has been proved in (Moreno, 2012) that the following inequalities holds

$$\eta_1 |z_1|^{3/2} + \eta_2 |z_2|^3 \leq V(z) \leq \eta_3 |z_1|^{3/2} + \eta_4 |z_2|^3$$

where

$$\begin{aligned} \eta_1 &\triangleq \pi_1^{\frac{3}{2}} - \frac{2}{3}\pi_2 \left(\nu_b 2^{\frac{1}{2}} \left(\frac{\pi_2}{3}\right)^{\frac{1}{3}} + (1-\nu_b)\left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}} \pi_1\right)^{\frac{2}{3}}, \\ \eta_2 &\triangleq \left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\pi_2 \left(\nu 2^{\frac{1}{2}} \left(\frac{\pi_2}{3}\right)^{\frac{1}{2}} + (1-\nu)\left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}} \pi_1\right)^{-3} \\ \eta_3 &\triangleq \left(\phi_{max} \pi_1^{\frac{3}{2}} + \frac{2}{3}\pi_2 c^{\frac{3}{3}}\right) \\ \eta_4 &\triangleq \left(\phi_{max}\left(\frac{1}{2}\right)^{\frac{3}{2}} + \frac{1}{3}\pi_2 c^{-3}\right) \\ & 0 < \nu_b < 1, \;\; c > 0 \end{aligned}$$

The derivative of the function (5) can be bounded as follows

$$\begin{aligned} \dot{V}(z) \leq& -\left(\frac{\kappa}{L^q}\right)\left[1 - \frac{W_a(z)}{W_0(z)}\frac{a(t,\xi)|}{L}\right] W_0(z) \\ & - (2q+1)\frac{3}{2}\left(\frac{\dot{L}}{L}\right) W_d(z) \end{aligned}$$

where

$$W_0(z) = \frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}(k_2 - |\pi_1 - k_1|)|z_2| - \pi_2 z_2^2 + \pi_2(k_1 - k_2)|z_1|$$

$$W_a(z) = \frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}|z_2| + \pi_2|z_1|$$

$$W_d(z) = \left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left(\pi_1|z_1| + \frac{q+1}{q+\frac{1}{2}}\frac{1}{2}z_2^2\right) + \frac{q+\frac{2}{3}}{q+\frac{1}{2}}\pi_2 z_1 z_2$$

Given that $\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right) \geq \left(\frac{1}{2}\right)^{\frac{1}{2}}|z_2|$, for the function W_0, the following inequality holds

$$W_0 \geq \frac{3}{2}\left(\frac{1}{2}\right)^{\frac{1}{2}}\left(k_2 - |\pi_1 - k_1| - \pi_2\frac{2}{3}(2)^{\frac{1}{2}}\right)|z_2|^2 + + \pi_2(k_1 - k_2)|z_1|$$

Given that the constants $\pi > 0$ and $\pi_2 > 0$ are selected such that $k_2 > |\pi_1 - k_1| - \pi_2\frac{2}{3}(2)^{\frac{1}{2}}$, the function W_0 is positive definite if

$$k_1 > k_2 > 0 \tag{6}$$

The function $W_a(z)$ is also positive definite, and $W_d(z) \to V(z)$ uniformly as $q \to \infty$. This implies that for q sufficiently large $W_d(z)$ is also positive definite.

The function

$$\nu(z) = \frac{W_a(z)}{W_0(z)}$$

is homogeneous of degree zero with weights (2,1), this means that

$$\nu(k^2 z_1, kz_2) = \frac{W_a(k^2 z_1, kz_2)}{W_0(k^2 z_1, kz_2)} = \frac{k^2 W_a(z)}{k^2 W_0(z)} = \nu(z)$$

and so all values of the function are taken on the unit homogeneous ball $B_h = \left\{z | \left(|z_1| + |z_2|^2\right)^{\frac{1}{2}} = 1\right\}$. since $W_0(z)$ is positive definite and $W_a(z)$ takes only finite values on B_h, then there exist a maximum and a minimum value taken by $\nu(z)$ on the unit ball, and therefore on the whole space, i.e.

$$0 < \nu_{min} \leq \nu(z) \leq \nu_{max}$$

In the same manner it can be proved that

$$0 < \alpha_{min} \leq \frac{W_0(z)}{V^{\frac{2}{3}}} \leq \alpha_{max},$$

$$0 < \delta_{min} \leq \frac{W_d(z)}{V(z)} \leq \delta_{max}$$

Thus, the derivative of the Lyapunov function satisfies

$$\dot{V}(z) \leq -\left(\frac{\kappa}{L^q}\right)\left[\alpha_{min} - \nu_{max}\alpha_{max}\frac{|a(t,\xi)|}{L}\right]V^{\frac{2}{3}}(z) - (2q+1)\frac{3}{2}\left(\frac{\dot{L}}{L}\right)W_d(z) \tag{7}$$

$L(t)$ will reach a value which is sufficient large to drive $z(t)$ to zero, therefore the state $x(t)$ eventually go to zero, and L will stop growing remaining positive and bounded for all $t \geq 0$.

△

Remark 2. Theoretically, the adaptive gain will stop growing when the states are exactly zero. In application, this condition can not be achieve because the measurement will always be noisy. In this case, the gain will stop growing when the state reach a neighborhood of the origin, i.e. it should stops growing when $|x| < \epsilon, \epsilon > 0$. Then the states will remain in a neighborhood of the origin ϵ and practical stability will be achieved. The selection of parameter ϵ will depend on the bounds of the noise signals.

4. ADAPTIVE SUPER TWISTING OBSERVER

In this section an adaptive observer for the plant (1) without input is designed (for the closed loop system the input will be considered). In(Moreno, 2012) a super-twisting based observer is presented with constant gain. Here the observer will be design with adaptive gain. Consider the super-twisting based observer

$$\begin{aligned}\dot{\hat{x}}_1 &= -h_1\gamma\phi_1(e_1) + \hat{x}_2\\ \dot{\hat{x}}_2 &= -h_2\gamma^2\phi_2(e_1)\end{aligned} \tag{8}$$

where the no-linearities are given by

$$\phi_1 = \mu_1|e_1|^{1/2}sign(e_1)$$

$$\phi_2 = \frac{\mu_1^2}{2}sign(e_1)$$

and the estimation error is $e_1 = \hat{x}_1 - x_1$.
The gain γ is an adaptive gain, i.e. $\gamma = \gamma(t)$, which dynamics are given by

$$\dot{\gamma} = \begin{cases} s(t), & if \;\; e_1 \not\equiv 0\\ 0, & if \;\; e_1 \equiv 0\end{cases} \tag{9}$$

were $s(t) > 0$ is a positive function.

Theorem 3. Consider the observer (8) with constant gains h_1, h_2 and an adaptive gain $\gamma(t)$ with a bounded perturbation. Then, if $h_1 > 0$ and $h_2 > 0$ the trajectories of the estimation error will reach the origin in finite time. Moreover the adaptive gain $\gamma(t)$ will remain bounded for all time.

□

Note that if the perturbation grows after the origin is reached the adaptive gain $L(t)$ will grow until it reaches a value that is large enough to drive again the trajectories to the origin.

Proof.

For the observation errors ($e_1 = \hat{x}_1 - x_1, e_2 = \hat{x}_2 - x_2$) the following equalities hold

$$\begin{aligned}\dot{e}_1 &= -h_1\gamma\phi_1(e_1) + e_2\\ \dot{e}_2 &= -h_2\gamma^2\phi_2(e_1) - a(\xi,t)\end{aligned} \tag{10}$$

To prove that the adaptive gain observer converge in finite time, the following vector is introduce

$$\epsilon = \begin{bmatrix} \phi_1(e_1) \\ e_2 \end{bmatrix}$$

its derivative is given by

$$\begin{aligned} \dot{\epsilon} &= \phi_1'(e_1) \begin{bmatrix} -h_1\gamma\phi_1(e_1) + e_2 \\ -h_2\gamma^2\phi_1(e_1) - \dfrac{a(\xi,t)}{\phi_1'(e_1)} \end{bmatrix} \\ &= \phi_1'(e_1)\left((A_0 - \Gamma H_0 C_0) + \tilde{\rho}\right) \end{aligned}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \Gamma = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^2 \end{bmatrix}, H_0 = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix},$$

$$C_0 = [1 \;\; 0], \tilde{\rho} = \begin{bmatrix} 0 \\ \dfrac{-2|e_1|^{1/2} a(\xi,t)}{\mu_1} \end{bmatrix}$$

Consider the following lemma

Lemma 4. Consider the positive constant $h_1 > 0$, $h_2 > 0$, the algebraic Lyapunov equation (ALE)

$$A^T P + P A^T = -Q$$

and the symmetric matrix R

$$R = NP + PN, \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Given the matrix $Q = Q^T = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix} > 0$ and the matrix$A = \begin{bmatrix} -h_1 & 1 \\ -h_2 & 0 \end{bmatrix}$that is Hurwitz. If the elements of the matrix Q are chosen such that the following inequalities hold

$$\begin{aligned} q_2 &= 1 \\ 2h_1q_1q_3 + h_1^2 q_1 + q_1^2 + h_2 q_1 + 2h_1h_2q_3 &+ \\ +q_1h_2 + h_2^2 + \frac{7h_1^2h_2}{16} &> 0 \end{aligned}$$

then, the matrices P, R are positive definite.

□

A change of variables is introduce

$$\zeta = \Gamma^{-1}\epsilon = \begin{bmatrix} \dfrac{\epsilon_1}{\gamma} \\ \dfrac{\epsilon_2}{\gamma^2} \end{bmatrix}$$

given that $\Gamma^{-1}A_0\Gamma = \gamma A_0$ and $C_0\Gamma = \gamma C_0$, the derivative of the variable ζ is

$$\begin{aligned} \dot{\zeta} &= \Gamma^{-1}\phi_1'(e_1)\left[(A_0 - \Gamma H_0 C_0)\Gamma\zeta + \tilde{\rho}\right] + \frac{\dot{\gamma}}{\gamma}N\zeta \\ &= \phi_1'(e_1)\left[\gamma(A_0 - H_0C_0) + \Gamma^{-1}\tilde{\rho}\right] - \frac{\dot{\gamma}}{\gamma}N\zeta \end{aligned}$$

where N is given in lemma 4

The following Lyapunov function is defined

$$V = \zeta^T P \zeta \tag{11}$$

where P is the solution of the algebraic Lyapunov function

$$(A_0 - H_0C_0)^T P + P(A_0 - H_0C_0) = -Q$$

choosing the gains h_1, h_2, such that the matrix $A_0 - H_0C_0$ is Hurwitz, and an arbitrary symmetric positive definite matrix $Q = Q^T > 0$ the solution P is unique and symmetric positive definite.

Deriving the equation (11) it is obtained

$$\begin{aligned} \dot{V} = \phi_1'(\gamma\zeta^T\left[(A_0 - H_0C_0)^T P + P(A_0 - H_0C_0)\right]\zeta &+ \\ +2\zeta^T P\Gamma^{-1}\tilde{\rho}) - \frac{\dot{\gamma}}{\gamma}\zeta^T R\zeta \end{aligned}$$

note that

$$||\Gamma^{-1}\tilde{\rho}||^2 \le 4a_p^2||\zeta||^2 \tag{12}$$

According to lemma 4, the matrix Q can be chosen such that the matrices P and R are positive definite. Thus the following inequalities hold for the positive symmetric matrix R

$$0 < \lambda_{min}(R)||\zeta||^2 \le \zeta^T R\zeta \le \lambda_{max}(R)||\zeta||^2$$

and $Q = Q^T > 0$ holds with

$$0 < \lambda_{min}(Q)||\zeta||^2 \le \zeta^T Q\zeta \le \lambda_{max}(Q)||\zeta||^2$$

then, considering (12)

$$\begin{aligned} \dot{V} \le (-\phi_1'(e_1)(\gamma\lambda_{min}(Q) - 4a_p\lambda_{max}(P)) \\ - \frac{\dot{\gamma}}{\gamma}|\lambda_{max}(R)|\Big)\, ||\zeta||^2 \end{aligned}$$

.

It is clear that after a finite time the following inequality holds

$$\gamma\lambda_{min}(Q) - 2k\lambda_{max}(P) > 0$$

Given the inequality

$$\lambda_{min}(P)||\zeta||^2 \le \zeta^T P\zeta \le \lambda_{max}(P)||\zeta||^2 \tag{13}$$

then

$$\begin{aligned} \dot{V} \le -\left(\lambda_{min}(Q) - \frac{2k\lambda_{max}(P)}{\gamma}\right)\frac{\mu_1^2}{2\lambda_{max}^{1/2}(P)}V^{1/2} &+ \\ - \frac{\dot{\gamma}}{\gamma\lambda_{min}(P)}|\lambda_{max}(R)|V \end{aligned}$$

It is observed that that there exist a time t_o where the gain $\gamma(t)$ is large enough to compensate the perturbation term and make the derivative of the Lyapunov function negative. Therefore the Lyapunov function will converge to zero in finite time. As a consequence the gain $\gamma(t)$ will stop growing in finite time, thus, it will remain bounded.

△

Remark 5. The adaptive gain will stop growing when the error is exactly zero. In application this condition can not be achieve, therefore, in this case, the gain will stop growing when the error reach a neighborhood of the origin, i.e. it stops growing when $|e_1| < \delta, \delta > 0$, then the estimation errors will remain bounded and practical stability will be achieved. Information of the noise signal should be available in order to select the parameters of the algorithm including the boundδ.

5. MAIN RESULT: THE CLOSED LOOP STABILITY

In this section the main result is presented.

Theorem 6. Consider the system (1)where $a(t,\xi)$ is some unknown scalar function that is bounded by an unknown constant a_p. The output feedback controller, with adaptive gain defined by (3),

$$u = -L(t)\left(k_1 sign(x_1) + k_2 sign(\hat{x}_2)\right) \tag{14}$$

where $\hat{x}_2$ is provided by the observer with adaptive gain (9)

$$\begin{aligned}\dot{\hat{x}}_1 &= -h_1\gamma\phi_1(e_1) + \hat{x}_2\\ \dot{\hat{x}}_2 &= -h_2\gamma^2\phi_2(e_1) + u\end{aligned}$$

where the no-linearities are given by

$$\begin{aligned}\phi_1 &= \mu_1|e_1|^{1/2} sign(e_1)\\ \phi_2 &= \frac{\mu_1^2}{2} sign(e_1)\end{aligned}$$

drives the trajectories of the closed loop system to zero in finite time. If the perturbation grows after the origin is reached the states will diverge from the origin and the adaptive gains will grow until the trajectories of the system return to zero in finite time.

□

The proof of theorem 6 is given in the appendix.

Remark 7. The states and the observation errors can be deviated from the origin after they reach it. In this case the adaptation gains will grow until the states and observation errors return to the origin. This fact is observed in the simulation.

6. SIMULATION

Consider the following system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -10 + 5sen(4\pi t)(1+p)\\ y &= x_1\end{aligned} \tag{15}$$

where

$$p = \begin{cases}0, & if\ \ t < 10\\ 65, & if\ \ t \geq 10\end{cases}$$

The controller (14) and the observer (15) were implemented in system (15).

On figures (1, 2) it can be observed that the estimation error for the observer converges to zero in finite time. The

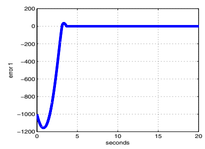

Fig. 1. Error e_1

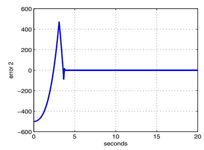

Fig. 2. Error e_2

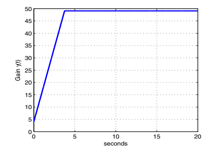

Fig. 3. Gain γ

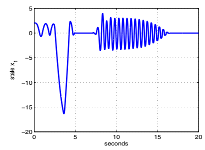

Fig. 4. State x_1

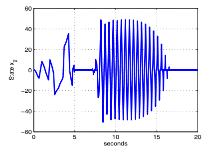

Fig. 5. State x_2

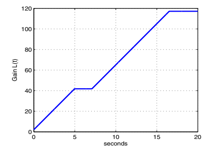

Fig. 6. Gain L

states also converges to the origin in finite time. The gains γ and L stop growing and as result the control signal stops growing.

When the perturbation grows, it can be observed that the sates deviate from the origin and the adaptive gain $L(t)$ stars growing until the states return to zero. As a result the control signal also grows to cope with the new amplitude of the perturbation.

7. CONCLUSIONS

An adaptive output feedback second order sliding mode controller were introduce. It has been proven that both, the controller and the observer, converge to zero in finite time with an adaptive gain despite that the perturbation bound is not known. The adaptive gains grows until the observer errors and the states converge to zero. As seen in the simulation, if the perturbation magnitude grows after the observation error and the states converge to zero, they will diverge and the gains will grow again, until they are large enough to compensate the perturbation. Therefore it can not be guaranteed that the sliding mode will be preserved after it is reached. The Lyapunov analysis was essential to prove that both adaptive techniques will respond to any bounded perturbation.

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8. APPENDIX

A.1 proof of theorem 6

Defining the estimation errors as $e_1 = \hat{x}_1 - x_1$ and $e_2 = \hat{x}_2 - x_2$, the closed loop dynamics can be written as

$$C : \begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = -L(t)\left(k_1 sign(x_1) + k_2 sign(x_2)\right) \\ & \quad +a(t,\xi) + \chi(t,\xi,x_2,e_2) \end{cases}$$

$$O : \begin{cases} \dot{e}_1 = -h_1\gamma(t)\phi_1(e_1) + e_2 \\ \dot{e}_2 = -h_2\gamma^2(t)\phi_2(e_1) - a(\xi,t) \end{cases}$$

where the perturbation terms are given by

$$\chi(t,\xi,x_2,e_2) \triangleq k_2[sign(x_2(t)) - sign(x_2(t)+e_2(t))].$$

The perturbation term vanishes when $e_2 = 0$. This perturbation is uniformly bounded

$$|\chi(t,\xi,x_2,e_2)| \leq 2k_2$$

In section 4, it is proved that the estimation error of system O converges to zero in finite time, i.e. there exists a time $T(e_0) > 0$ such that for all $t \geq T(e_0)$, it follows that $e_1(t) = e_2(t) = 0$. Note that the trajectories of system C cannot escape to infinity in finite time. This is due the fact that $f(x)$ satisfies the growth condition

$$||f(x)|| \leq k_s||x||, \quad \forall ||x|| \geq c_s$$

for some $k_s > 0$, $c_s > 0$, and where $f(x)$ is the right hand side of system C. Moreover, the perturbation signal of system C is also bounded, and since there exist a time where the gain $L(t)$ is large enough to compensate any bounded perturbation, the state $x(t)$ converges to the origin in finite time. $\triangle$

A.2 Proof of lemma 4

Given the matrix

$$P = \begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix} > 0$$

The matrix Q can be written as

$$Q = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix} = \begin{bmatrix} 2\left(h_1p_1 + h_2p_2\right) & h_1p_3 + h_2p_2 - p_1 \\ h_1p_3 + h_2p_2 - p_1 & -2p_3 \end{bmatrix}$$

then the elements of P can be written in terms of the elements of Q

$$p_1 = \frac{q_1 + h_2q_2}{2h_1}$$
$$p_2 = \frac{2h_1q_3 + h_1^2q_2 + q_1 + h_2q_2}{2h_1h_2}$$
$$p_3 = -\frac{q_2}{2}$$

The matrix R can be written as

$$R = -\begin{bmatrix} 2p_1 & 3p_3 \\ 3p_3 & 4p_2 \end{bmatrix}$$

then R is positive definite if the following inequalities hold

$$p_1 > 0$$
$$8p_1p_2 - 9p_3^2 > 0 \tag{16}$$

the condition (16) is equivalent to

$$q_1 + h_2q_2 > 0 \tag{17}$$

$$8\left(\frac{q_1 + h_2q_2}{2h_1}\right)\left(\frac{2h_1q_3 + h_1^2q_2 + q_1 + h_2q_2}{2h_1h_2}\right) - 9\left(\frac{q_2}{2}\right)^2 > 0$$

without loss of generality q_2 can be chosen as $q_2 = 1$, then the condition (17) holds. Condition (16) holds if

$$2h_1q_1q_3 + h_1^2q_1 + q_1^2 + h_2q_1 + 2h_1h_2q_3 + h_1^2h_2 + q_1h_2 + h_2^2 - \frac{9h_1^2h_2}{16} > 0$$

that is always possible since the parameters q_1, q_3 can be chosen large enough. $\triangle$