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New Results on Finite-Time Stability of Switched Linear Systems with Average Dwell Time ⋆

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Abstract: This paper is concerned with the finite-time stability (FTS) analysis for a class of switched linear systems with average dwell time (ADT), a more practical concept of extended finite time stability (EFTS) is proposed as the first attempt. A parameter-dependent description approach at the instants of each subsystem switched in and off is invoked to obtain the EFTS criterion. It has been shown the obtained criterion is less conservative than the existing results. The switched systems with only unstable subsystems is firstly addressed with ADT and mode-dependent ADT, the obtained results of ADT are extended to the ones with both stable and unstable subsystems. A numerical example is given to verify the theoretical findings.

Keywords: Switched Systems, Extended Finite-Time Stability, Average Dwell Time, Stable and Unstable Subsystems.

1. INTRODUCTION

Switched systems, which provide a unified, systematic framework for modeling practical systems with switching features, have been made significant contributions in theoretical research (Blanchini et al (2012), Persis et al (2003), Zattoni et al (2013), Zhao et al (2012a)) and practices such as power electronics, Tse et al (2002), flight control systems, Pellanda et al (2002) and other fields, e.g., Geromel et al (2008), Hespanha et al (2004). Most existing literature focused on the performance of switched systems in the infinite-time interval, see for example, Zhang et al (2010), Zhao et al (2012b). However, there are some cases where large values of the state are infeasible, such as the terminal guidance of missile and vehicle active suspension system, see Amato et al (2012). For this purpose, the finite-time stability (FTS) is addressed. Specifically, a system is said to be FTS, if the state (norm) with given criteria can hold below a fixed upper bound over the specified interval.

On another research front, switching signal, which is used to distinguish the switched systems from the common time-varying systems, has played an important role for the system performance. Currently, plenty of existing studies can be roughly categorized into two classes: (a) time-dependent switching laws such as average dwell time (ADT) (Zhang et al (2012)) and dwell time(DT)(Morse (1996)); (b) state-dependent switching approach, e.g., Deaecto et al (2010). Obvious contributions in the switched systems domain have shown that ADT switching scheme is more practical and flexible than others, see Vu et al (2007). Then, an mode-dependent ADT approach is proposed, which can reduce the conservativeness of the ADT criterion by adequately exploring the mode-dependent information in Zhao et al (2012b). Furthermore, if more available informations about each subsystem are known, the conservativeness are further reduced, such as the total activation time ratio between different kinds of subsystems, which will be addressed in this paper.

Additionally, many remarkable achievements on the issues of the switched systems with both stable and unstable subsystems have been made. To mention a few, the exponentially stability of switched systems containing both stable and unstable subsystems was studied firstly in infinite-time interval in Zhai et al (2001). By limiting the ratio of the total activation time between stable and unstable subsystems, the stability conditions have been derived with ADT. An extended version is also discussed for asynchronous switched systems with ADT in Zhang et al (2010). The switched systems involving with stabilizable and unstablizable subsystems have been investigated on the tracking control issues in Li et al (2009). To our knowledge, however, few literature have been available on this subject in the finite-time interval. Most published results about FTS of the switched systems just simply considered each subsystem being unstable, see Du et al (2010), which motivates us for this study.

Greatly different from previous works, the approach proposed in this note is inspired by state-dependent dwell time description approach in Persis et al (2003), mode-dependent ADT technique in Zhao et al (2012b) and the method for dealing with the stability of switched systems with stable and unstable subsystems in Zhai et al (2001). Twofold contributions will be addressed in this paper. Firstly, the mode-dependent ADT approach will be developed to reduce the conservativeness over the existing FTS criteria. Then by the extended finite-time stability

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(EFTS) for switched systems, a more practical case of switched system involving with both stable and unstable subsystems will also be considered. The remainder of this paper is organised as follows. In Section 2, the general FTS and EFTS concepts are firstly defined. The EFTS conditions of different switched systems based on improved switching schemes are derived with a novel parameter-dependent running time description approach in Section 3 and a numerical example is given in Section 4 to show the effectiveness of the obtained theoretical results. Conclusions of this paper are given in the last section.

Notations: The notations used throughout this paper are fairly standard, and can be found in the relevant literature of switched systems. We omit them here due to the space limit.

2. PROBLEM STATEMENTS AND PRELIMINARIES

Consider a class of time-invariant continuous-time switched linear systems given by

$$\dot{x}(t) = A_{\sigma(t)}x(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $\sigma(t) : [0,\infty] \to S \triangleq \{1,2,\cdots,r,r+1,\cdots,l\}$ is a piecewise constant function which is deterministic and right continuous, called a switching signal, where l is the number of subsystems. The switching sequence can be described as $\sigma : \{(t_0,\sigma(t_0)),(t_1,\sigma(t_1)),\cdots,(t_k,\sigma(t_k)),\cdots\}$, when $t \in [t_k,t_{k+1})$, we say the $\sigma(t_k)$ subsystem is active. When $A_{\sigma(t)}$ is Hurwitz stable for $\sigma(t) = i$, we say $i \in \Phi \triangleq \{1,2,\cdots,r\}$, where Φ is the set of indexes for all stable subsystems. Otherwise, $i \in \tilde{\Phi} \triangleq \{r+1,\cdots,l\}$, thus we have $S = \Phi \cup \tilde{\Phi}$.

Definition 1. (Finite-time stability, FTS, Du et al (2010)). Given a positive definite matrix R, three positive constants c_1, c_2, T_f with $c_1 < c_2$, and a switching signal σ. System (1) is said to be FTS with respect to (c_1,c_2,R,T_f,σ), if $\forall t \in [0,T_f]$

$$x^T(0)Rx(0) \le c_1 \Longrightarrow x^T(t)Rx(t) \le c_2.$$

In this note, we will consider the case that the stable and unstable subsystems are both existed in the switched system, and the total activation time ratio between unstable subsystems and stable subsystems is known as *a priori*. Besides, the evaluation criterion of the system is assumed to be mode-dependent, which means that the $x^T(t)Rx(t)$ in Definition 1 will be analyzed as $x^T(t)R_{\sigma(t)}x(t)$. Then, an extended finite-time stability (EFTS) concept for switched systems containing both stable and unstable subsystems is presented as follows.

Definition 2. (Extended Finite-time stability, EFTS). Given four positive constants c_1, c_2, a, T_f, a set of positive definite matrix $\Omega \triangleq \{R_i \mid i \in S\}$ and a switching signal $\sigma_a(t)$ by the activation time ratio a between unstable and stable subsystems, system (1) is said to be EFTS with respect to $(c_1,c_2,R,\Omega,T_f,\sigma_a)$, if $\forall t \in [0,T_f]$

$$x^T(0)Rx(0) \le c_1 \Longrightarrow x^T(t)R_{\sigma_a(t)}x(t) \le c_2$$

Note that, when $a \to \infty$, the admissible switching signal permits all the activate subsystems are unstable ones in the whole interval of $[0,T_f]$, which is actually the case considered in the existing literature, e.g., Du et al (2010). Thus, the switching signal σ_a will be restituted by σ throughout this paper.

3. MAIN RESULTS

In this section, a general switched linear system with only unstable subsystems will be firstly considered by multiple Lyapunov-like function (MLF) subject to ADT switching, where a parameter-dependent running time description approach will be proposed to obtain the FTS criteria.

Lemma 1. Consider the continuous-time switched system $\dot{x}(t) = f_{\sigma(t)}(t)$ and let ρ_1,ρ_2, $\beta_i > 0$, $\mu_i > 1$, $\forall\sigma(t) = i \in S$ be given constants. Suppose there exist Lyapunov functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}, \sigma(t) \in S$, such that, $\forall\sigma(t) = i \in S$,

$$\rho_1\|x(t)\|^2 \le V_i(x(t)) \le \rho_2\|x(t)\|^2 \tag{2}$$

$$\dot{V}_i(x(t)) \le \beta_i V_i(x(t)) \tag{3}$$

$$\frac{c_1}{c_2}e^{\beta_M T_f} \le \frac{\rho_1\lambda_1\mu_m}{\rho_2\lambda_2\lambda_3} \tag{4}$$

where $\lambda_1 \triangleq \inf_{i\in S}\{\lambda_{\min}(R_i)\}$, $\lambda_2 \triangleq \sup_{i\in S}\{\lambda_{\max}(R_i)\}$, $\lambda_3 \triangleq \frac{\lambda_2}{\lambda_{\min}(R)}$, $\mu_M \triangleq \sup_{i\in S}\{\mu_i\}$, $\mu_m \triangleq \inf_{i\in S}\{\mu_i\}$ and $\beta_M \triangleq \sup_{i\in S}\{\beta_i\}$,

$$\forall\left(\sigma(t_k) = i, \sigma\left(t_k^-\right) = j\right) \in S \times S, i \ne j$$

$$V_i(x(t_k)) \le \mu_j V_j\left(x\left(t_k^-\right)\right). \tag{5}$$

If the switching signal σ with ADT satisfies

$$\tau_a \ge \tau_a^* = \frac{T_f \ln\mu_M}{\ln(\rho_1\lambda_1\mu_m c_2) - \ln(\rho_2\lambda_3\lambda_2 c_1) - \beta_M T_f} \tag{6}$$

the system holds EFTS with respect to $(c_1,c_2,R,\Omega,T_f,\sigma)$.

Proof. By integrating (3) for $t \in [t_k,t_{k+1}), \forall t \in [0,T_f]$, we have

$$V_{\sigma(t_k)}(x(t)) \le e^{\beta_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}(t_k) \tag{7}$$

Let $t_0,t_1,t_2,\cdots,t_k$ be the switching instants in the interval $[0,t]$, $t \in [0,T_f]$, where $t_0 = 0$. Then the running time between two successive switching instants could be denoted as $\tilde{\tau}_{\sigma(t_{k-1})} = t_k - t_{k-1}$. A parameter-dependent approach is proposed to describe the running time as follows. For arbitrary $\tilde{\tau}_{\sigma(t_{k-1})}$, there exist $q_{\phi(t_{k-1})}$, by which $\tilde{\tau}_{\sigma(t_{k-1})}$ could be rewritten as

$$\tilde{\tau}_{\sigma(t_{k-1})} = t_k - t_{k-1} = \frac{\ln q_{\phi(t_{k-1})} - \ln\mu_{\sigma(t_{k-1})}}{\beta_{\sigma(t_{k-1})}} \tag{8}$$

where $\phi(t_{k-1})$ is a special switching signal, which determines not only the value of $q_{\phi(t)}$ at the switching interval of $t \in [t_{k-1},t_k)$, but also the according value range $q_{\phi(t_{k-1})} \in \left[\mu_{\sigma(t_{k-1})},\infty\right)$.

For arbitrarily given switching sequence, it follows from (5), (7) and (8) that

$$\begin{aligned}
V_{\sigma(t_k)}(x(t)) &\le e^{\beta_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}(x(t_k)) \\
&\le \frac{q_{\phi(t_k)}}{\mu_{\sigma(t_k)}}\mu_{\sigma(t_{k-1})}V_{\sigma(t_{k-1})}\left(x\left(t_k^-\right)\right) \\
&\le \frac{q_{\phi(t_k)}}{\mu_m}\mu_{\sigma(t_{k-1})}e^{\beta_{\sigma(t_{k-1})}(t_k-t_{k-1})} \\
&\quad \times V_{\sigma(t_{k-1})}(x(t_{k-1})) \le \cdots
\end{aligned}$$

$$\begin{aligned} &\leq \frac{q_{\phi(t_k)}}{\mu_m} q_{\phi(t_{k-1})} \cdots q_{\phi(t_1)} q_{\phi(t_0)} V_{\sigma(t_0)}\left(x\left(t_0\right)\right) \\ &= \frac{\Psi_{[t_0,t]} V_{\sigma(t_0)}\left(x\left(t_0\right)\right)}{\mu_m} \end{aligned} \tag{9}$$

where $\Psi_{[t_0,t]} \triangleq q_{\phi(t_k)} q_{\phi(t_{k-1})} q_{\phi(t_{k-2})} \cdots q_{\phi(t_1)} q_{\phi(t_0)}$ stands for the product of all activation time description parameters from $t_0 \to t$.

Thus, for $\tau_{[t_0,t]} \in [0, T_f]$, the following expression can be obtained

$$\begin{aligned} T_f \geq \tau_{[t_0,t]} &= \tilde{\tau}_{\sigma(t_0)} + \tilde{\tau}_{\sigma(t_1)} + \cdots + \tilde{\tau}_{\sigma(t_k)} \\ &= \frac{\ln q_{\phi(t_0)} - \ln \mu_{\sigma(t_0)}}{\beta_{\sigma(t_0)}} + \frac{\ln q_{\phi(t_k)} - \ln \mu_{\sigma(t_k)}}{\beta_{\sigma(t_k)}} \\ &\geq \frac{\ln\left(q_{\phi(t_k)} \cdots q_{\phi(t_1)} q_{\phi(t_0)}\right) - N_{[t_0,t]} \ln \mu_M}{\beta_M} \\ &= \frac{\ln \Psi_{[t_0,t]} - N_{[t_0,t]} \ln \mu_M}{\beta_M} \end{aligned} \tag{10}$$

In addition, based on (2), it yields that

$$V_{\sigma(t_k)}\left(x\left(t\right)\right) \geq \rho_1 \|x\left(t\right)\|^2 \geq \frac{\rho_1}{\lambda_2} x^T\left(t\right) R_{\sigma(t_k)} x\left(t\right) \tag{11}$$

Similarly,

$$\begin{aligned} V_{\sigma(t_0)}\left(x\left(t_0\right)\right) &\leq \rho_2 \|x\left(t_0\right)\|^2 \leq \frac{\rho_2}{\lambda_1} x^T\left(t_0\right) R_{\sigma(t_0)} x\left(t_0\right) \\ &\leq \frac{\rho_2}{\lambda_1} q_{\phi(t_0^-)} x^T\left(t_0\right) R x\left(t_0\right) \leq \frac{\rho_2 \lambda_3 c_1}{\lambda_1} \end{aligned} \tag{12}$$

where $q_{\phi(t_0^-)}$ represents the gain from $x^T(0) R x(0)$ to $x^T(t_0) R_{\sigma(t_0)} x(t_0)$ and there is no dwell time.

In view of (9)-(12), one has that

$$\begin{aligned} &x^T\left(t\right) R_{\sigma(t)} x\left(t\right) \\ &\leq \frac{V_{\sigma(t_k)}\left(x\left(t\right)\right) \lambda_2}{\rho_1} \leq \frac{\Psi_{[t_0,t]} V_{\sigma(t_0)}\left(x\left(t_0\right)\right) \lambda_2}{\rho_1 \mu_m} \\ &\leq \frac{\Psi_{[t_0,t]} \rho_2 \lambda_3 \lambda_2 c_1}{\rho_1 \lambda_1 \mu_m} \leq e^{\beta_M T_f + N_{[t_0,t]} \ln \mu_M} \frac{\rho_2 \lambda_3 \lambda_2 c_1}{\rho_1 \lambda_1 \mu_m} \end{aligned} \tag{13}$$

Based on (4) and $N_{[t_0,t]} \leq N^* = \frac{T_f}{\tau_a^*}$ in (6), we have

$$e^{\beta_M T_f + N_{[t_0,t]} \ln \mu_M} \leq \frac{\rho_1 \lambda_1 \mu_m c_2}{\rho_2 \lambda_3 \lambda_2 c_1} \tag{14}$$

Then, it follows from (13) and (14) that,

$$x^T\left(t\right) R_{\sigma(t)} x\left(t\right) \leq e^{\beta_M T_f + N_{[t_0,t]} \ln \mu_M} \frac{\rho_2 \lambda_3 \lambda_2 c_1}{\rho_1 \lambda_1 \mu_m} \leq c_2$$

Thus, we can conclude the system holds ETFS with respect to $(c_1, c_2, R, \Omega, T_f, \sigma)$. □

Remark 1. Construct the Lyapunov functions $V_{\sigma(t)}\left(x\left(t\right)\right) = x^T\left(t\right) \tilde{P}_{\sigma(t)} x\left(t\right)$ for switched linear systems (1) and suppose that $P_{\sigma(t)} \triangleq R^{-1/2} \tilde{P}_{\sigma(t)} R^{-1/2}$, $\forall t \in [0, T_f]$. Then ρ_1 and ρ_2 in (2) could be respectively obtained as $\rho_1 \triangleq \inf_{i \in S}\left\{\lambda_{\min}\left(P_i\right)\right\}$ and $\rho_2 \triangleq \sup_{i \in S}\left\{\lambda_{\max}\left(P_i\right)\right\}$. If we further let $R_i \equiv R$, $\mu_M = \mu_m \equiv \mu_i$ and $\beta_M \equiv \beta_i$, $\forall i \in S$, then a similar result could be found in Du et al (2010). By rewriting the results in Lemma 1 and Du et al (2010), the following results could be obtained

$$\begin{cases} \tau \geq \tau_a^* = \dfrac{T_f \ln \mu_M}{\ln\left(\rho_1 \mu_M c_2\right) - \ln\left(\rho_2 c_1\right) - \beta_M T_f} & \text{(15a)} \\ \tau \geq \bar{\tau}_a^* = \dfrac{T_f \ln \mu_M}{\ln\left(\rho_1 c_2\right) - \ln\left(\rho_2 c_1\right) - \beta_M T_f} & \text{(15b)} \end{cases}$$

where τ_a^* and $\bar{\tau}_a^*$ are the admissible switching signal of Lemma 1 and Du et al (2010), respectively. By the given definition of μ_M that $\mu_M > 1$, the conservativeness of ADT is reduced in this paper. The same results will also be derived when $R_{\sigma(t)}$ is mode-dependent, which will be demonstrated in the numerical simulation later.

Remark 2. A class of switched nonlinear systems is addressed for the EFTS analysis in Lemma 1, which is more general than the existing literature only considered switched linear systems. Besides, the proposed parameter-dependent running time description approach in (8) can reduce the conservativeness of the ADT criterion by Remark 1. Actually, the parameter μ_i in the above derivation is only used for describing the maximum ratio between the abrupt values of Lyapunov-like function owing to the switch of two different subsystems at the switching instant. The improvement of the developed technique in this note is attributed to the parameter-dependent description approach adequately considering the relationship between the running time and the parameter μ_i at each subsystem, then by using (8) in (9), an redundant μ_i can be cancelled. However, in the existing results, such as Du et al (2010), the running number of μ_i is always considered to be equivalent to the number of all the subsystems, which introduces the conservativeness of the switching signal with ADT approach.

3.1 EFTS of Switched Systems with Unstable Subsystems

In this section, the EFTS of switched systems consisting of only unstable subsystems will be addressed with mode-dependent ADT switching approach.

Theorem 1. Consider the continuous-time switched system $\dot{x}\left(t\right) = f_{\sigma(t)}\left(t\right)$ and let ρ_1, ρ_2, $\beta_i > 0$, $\mu_i > 1$, $\forall \sigma\left(t\right) = i \in S$ be the given constants. Suppose there exist Lyapunov functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}, \sigma\left(t\right) \in S$, such that, $\forall \sigma\left(t\right) = i \in S$,

$$\begin{gathered} \rho_1 \|x\left(t\right)\|^2 \leq V_i(x\left(t\right)) \leq \rho_2 \|x\left(t\right)\|^2 \\ \dot{V}_i(x\left(t\right)) \leq \beta_i V_i(x\left(t\right)) \\ \frac{c_1}{c_2} e^{\beta_M T_f} \leq \frac{\rho_1 \lambda_1 \mu_m}{\rho_2 \lambda_2 \lambda_3} \end{gathered}$$

where λ_1, λ_2, λ_3, μ_M, μ_m and β_M are denoted as in Lemma 1.

$$\begin{gathered} \forall\left(\sigma\left(t_k\right) = i, \sigma\left(t_k^-\right) = j\right) \in S \times S, i \neq j \\ V_i\left(x\left(t_k\right)\right) \leq \mu_j V_j\left(x\left(t_k^-\right)\right). \end{gathered}$$

If the mode-dependent ADT of i-th subsystem, $\forall i \in S$ satisfies

$$\tau_{ai} \geq \tau_{ai}^* = \frac{T_f \ln \mu_i}{\ln\left(\rho_1 \lambda_1 \mu_m c_2\right) - \ln\left(\rho_2 \lambda_3 \lambda_2 c_1\right) - \beta_i T_f} \tag{16}$$

the system holds EFTS with respect to $(c_1, c_2, R, \Omega, T_f, \sigma)$.

Proof. By performing a similar derivation process in Lemma 1, we have,

$$V_{\sigma(t_k)}\left(x\left(t\right)\right) \leq \frac{q_{\phi(t_k)}}{\mu_{\sigma(t_k)}} q_{\phi(t_{k-1})} \cdots q_{\phi(t_1)} q_{\phi(t_0)} V_{\sigma(t_0)}\left(x\left(t_0\right)\right)$$

$$= \prod_{i=1}^{l} \prod_{j=1}^{N_i} \left(q_{\phi(t)=(i,j)}\right) \frac{V_{\sigma(t_0)}\left(x\left(t_0\right)\right)}{\mu_{\sigma(t_k)}}$$
$$\leq \prod_{i=1}^{l} \Psi^i_{[t_0,t]} \frac{V_{\sigma(t_0)}\left(x\left(t_0\right)\right)}{\mu_m} \quad (17)$$

where $q_{\phi(t)=(i,j)}$ denotes the value of q for the j-th running of the i-th subsystem, N_i is the switching number in the interval $[t_0, t]$; $\Psi^i_{[t_0,t]} = \prod_{j=1}^{N_i}\left(q_{\phi(t)=i,j}\right)$ stands for the product of all the $q_{\phi(t)=(i,j)}$ of i-th subsystem from $t_0 \rightarrow t$.

Let T_i be the total running time of i-th subsystem in the interval $[t_0, t]$, which could also be expressed by the proposed parameter-dependent running time in (8) that

$$T_i = \frac{\ln q_{\phi(t)=(i,1)} - \ln\mu_i}{\beta_i} + \cdots + \frac{\ln q_{\phi(t)=(i,N_i)} - \ln\mu_i}{\beta_i}$$
$$= \left(\ln\left(\prod_{j=1}^{N_i} \ln q_{\phi(t)=(i,j)}\right) - N_i \ln\mu_i\right) / \beta_i$$
$$= \left(\ln\Psi^i_{[t_0,t]} - N_i \ln\mu_i\right) / \beta_i \quad (18)$$

By the mode-dependent ADT in (16), (17) and (18), it implies that

$$\prod_{i=1}^{l} \Psi^i_{[t_0,t]} = \prod_{i=1}^{l} \exp\left(\beta_i T_i + N_i \ln\mu_i\right)$$
$$= \prod_{i=1}^{l} \exp\left(\beta_i T_i + \frac{T_i \ln\left(\rho_1\lambda_1\mu_m c_2\right)}{T_f} - \frac{T_i \ln\left(\rho_2\lambda_3\lambda_2 c_1\right) + \beta_i T_i T_f}{T_f}\right)$$
$$= \exp\left(\sum_{i=1}^{l}\left(\frac{T_i}{T_f}\ln\frac{\rho_1\lambda_1\mu_m c_2}{\rho_2\lambda_3\lambda_2 c_1}\right)\right)$$
$$= \frac{\rho_1\lambda_1\mu_m c_2}{\rho_2\lambda_3\lambda_2 c_1} \quad (19)$$

Based on similar lines of (11) and (12), in follows from (17) and (19) that

$$x^T(t) R_{\sigma(t)} x(t) \leq \frac{\prod_{i=1}^{l} \Psi^i_{[t_0,t]} V_{\sigma(t_0)}\left(x\left(t_0\right)\right)\lambda_2}{\rho_1\mu_m}$$
$$\leq \frac{\rho_1\lambda_1\mu_m c_2}{\rho_2\lambda_3\lambda_2 c_1}\frac{\rho_2\lambda_3 c_1}{\lambda_1}\frac{\lambda_2}{\rho_1\mu_m} = c_2$$

Then we can conclude the system holds EFTS with respect to $(c_1, c_2, R, \Omega, T_f, \sigma)$. □

Remark 3. The mode-dependent ADT is an improved criterion which can be easily certified by comparing (6) with (16). It is obviously that $\tau_i^* \leq \tau_a^*$ because of $\beta_i \leq \beta_M$ and $\mu_i \leq \mu_M$, $\forall i \in S$, which means that the system could tolerate faster switching. Therefore, it can be asserted that the conservativeness of the presented criteria in Theorem 1 is reduced than the general ADT criteria in Lemma 1 and Du et al (2010) owing to the mode-dependent features.

3.2 EFTS of Switched Systems with Both Stable and Unstable Subsystems

In this section, the EFTS of the system (1) with both stable and unstable subsystems will be analyzed, a set of admissible switching signal with ADT will be derived under a given ratio a of the total running time between the unstable and stable subsystems.

Theorem 2. Consider the continuous-time switched system $\dot{x}(t) = f_{\sigma(t)}(t)$ and let ρ_1, ρ_2, $\mu_i > 1$, $\forall\sigma(t) = i \in S$, $\alpha_i > 0$, $\forall i \in \Phi$ and $\beta_i > 0$, $\forall i \in \tilde{\Phi}$ be the given constants. Suppose there exist Lyapunov functions $V_{\sigma(t)} : \mathbb{R}^n \rightarrow \mathbb{R}, \sigma(t) \in S$, such that,$\forall\sigma(t) = i \in S$,

$$\rho_1\|x(t)\|^2 \leq V_i(x(t)) \leq \rho_2\|x(t)\|^2, \quad \forall i \in S$$
$$\dot{V}_i(x(t)) \leq \begin{cases} -\alpha_i V_i(x(t)), & \forall i \in \Phi \\ \beta_i V_i(x(t)), & \forall i \in \tilde{\Phi} \end{cases} \quad (20)$$
$$\exp\left(\frac{T_f(a\beta_M - \alpha_m)}{1+a}\right) \leq \frac{\lambda_1\rho_1\mu_m c_2}{\rho_2\lambda_2\lambda_3 c_1}$$

where $\mu_m \triangleq \inf_{i\in S}\{\mu_i\}$, $\alpha_m \triangleq \inf_{i\in\Phi}\{\alpha_i\}$ and $\beta_M \triangleq \sup_{i\in\tilde{\Phi}}\{\beta_j\}$, μ_M, λ_1, λ_2 and λ_3 are denoted as in Lemma 1.

$$\forall\left(\sigma(t_k) = i, \sigma\left(t_k^-\right) = j\right) \in S \times S, i \neq j$$
$$V_i\left(x\left(t_k\right)\right) \leq \mu_j V_j\left(x\left(t_k^-\right)\right).$$

If the switching signal with ADT satisfies $\frac{T_{\uparrow[t_0,t]}}{T_{\downarrow[t_0,t]}} \leq a$ and

$$\tau_a \geq \tau_a^* = \frac{(1+a)T_f \ln\mu_M}{(1+a)\left(\varphi_1 - \varphi_2\right) - (a\beta_M - \alpha_m)T_f} \quad (21)$$

where $\varphi_1 = \ln\left(\rho_1\lambda_1\mu_m c_2\right)$ and $\varphi_2 = \ln\left(\rho_2\lambda_3\lambda_2 c_1\right)$, then the system holds EFTS with respect to $(c_1, c_2, R, \Omega, T_f, \sigma_a)$

Proof. By integrating (20) for $t \in [t_k, t_{k+1})$, it yields that

$$\begin{cases} V_{\sigma(t_k)}\left(x(t)\right) \leq e^{-\alpha_{\sigma(t_k)}(t-t_k)} V_{\sigma(t_k)}\left(x\left(t_k\right)\right), & \sigma(t_k) \in \Phi \\ V_{\sigma(t_k)}\left(x(t)\right) \leq e^{\beta_{\sigma(t_k)}(t-t_k)} V_{\sigma(t_k)}\left(x\left(t_k\right)\right), & \sigma(t_k) \in \tilde{\Phi} \end{cases}$$

Similar to the approach of (8), the running time of switched systems with both stable and unstable subsystems are described respectively as

$$\begin{cases} \tau_{\sigma(t_{k-1})} = t_k - t_{k-1} = \dfrac{\ln\mu_{\sigma(t_{k-1})} - \ln p_{\phi(t_{k-1})}}{\alpha_{\sigma(t_{k-1})}} \\ \tilde{\tau}_{\sigma(t_{k-1})} = t_k - t_{k-1} = \dfrac{\ln q_{\phi(t_{k-1})} - \ln\mu_{\sigma(t_{k-1})}}{\beta_{\sigma(t_{k-1})}} \end{cases} \quad (22)$$

where $\tau_{\sigma(t_{k-1})}$ and $\tilde{\tau}_{\sigma(t_{k-1})}$ represent the running time of the interval $t \in [t_{k-1}, t_k)$ respectively for $\sigma(t_{k-1}) \in \Phi$ and $\sigma(t_{k-1}) \in \tilde{\Phi}$. $\phi(t_{k-1})$ is a special switching signal which determines the value rang of $p_{\phi(t)}$ and $q_{\phi(t)}$, which can be specifically expressed as $p_{\phi(t_{k-1})} \in (0, \mu_i]$ and $q_{\phi(t_{k-1})} \in [\mu_i, \infty)$ respectively for $\sigma(t_{k-1}) = i \in \Phi$ and $\sigma(t_{k-1}) = i \in \tilde{\Phi}$.

Based on aforementioned (22), and arbitrarily assigning a feasible switching sequence $\sigma(t_k) \in \Phi$, $\sigma(t_{k-1}) \in \tilde{\Phi} \cdots$, $\sigma(t_0) \in \Phi$, then we have

$$V_{\sigma(t_k)}\left(x(t)\right) \leq \frac{p_{\phi(t_k)}}{\mu_m} q_{\phi(t_{k-1})} \cdots p_{\phi(t_0)} V_{\sigma(t_0)}\left(x\left(t_0\right)\right)$$
$$= \Psi_{[t_0,t]}(t)\,\tilde{\Psi}_{(t_0,t]}(t)\,V_{\sigma(t_0)}\left(x\left(t_0\right)\right)/\mu_m \quad (23)$$

where $\Psi_{[t_0,t]}(t) \triangleq \prod_{\sigma(t)\in\Phi} p_{\phi(t)}$, $\tilde{\Psi}_{[t_0,t]}(t) \triangleq \prod_{\sigma(t)\in\tilde{\Phi}} q_{\phi(t)}$. Other feasible switching sequences will admit the same results.

Based on the given switching sequence in (23), it follows from (22) that

$$\begin{aligned}
T_f \geq \tau_{[t_0,t]} &= \tau_{\sigma(t_0)} + \cdots + \tilde{\tau}_{\sigma(t_{k-1})} + \tau_{\sigma(t_k)} \\
&= \frac{\ln\mu_{\sigma(t_0)} - \ln p_{\phi(t_0)}}{\alpha_{\sigma(t_0)}} + \cdots + \frac{\ln q_{\phi(t_{k-1})} - \ln\mu_{\sigma(t_{k-1})}}{\beta_{\sigma(t_{k-1})}} \\
&\quad + \frac{\ln\mu_{\sigma(t_k)} - \ln p_{\phi(t_k)}}{\alpha_{\sigma(t_k)}} \\
&= \sum_{i=1}^{r} \frac{N_i \ln\mu_i - \sum_{j=1}^{N_i}\left(\ln p_{\phi(t)=(i,j)}\right)}{\alpha} \\
&\quad + \sum_{i=r+1}^{l} \frac{\sum_{j=1}^{N_i}\left(\ln q_{\phi(t)=(i,j)}\right) - N_i \ln\mu_i}{\beta_i} \\
&= T_{\downharpoonleft[t_0,t]} + T_{\upharpoonleft[t_0,t]}
\end{aligned} \tag{24}$$

where N_i denotes the switching number of i-th subsystem for $[t_0, t]$. $T_{\downharpoonleft[t_0,t]}$ and $T_{\upharpoonleft[t_0,t]}$ are the total running time of all the stable and unstable subsystems during $[t_0, t]$ respectively.

Let $N_{\downharpoonleft[t_0,t]}$ and $N_{\upharpoonleft[t_0,t]}$ be the total switching numbers of stable and unstable subsystems in $[t_0, t]$ respectively, then we have $N_{\downharpoonleft[t_0,t]} = \sum_{i=1}^{r} N_i$ and $N_{\upharpoonleft[t_0,t]} = \sum_{i=r+1}^{l} N_i$. By (24), the following derivation could be obtained

$$\begin{aligned}
T_{\downharpoonleft[t_0,t]} &= \sum_{i=1}^{r} \frac{N_i \ln\mu_i - \sum_{j=1}^{N_i}\left(\ln p_{\phi(t)=(i,j)}\right)}{\alpha_i} \\
&\leq \frac{N_{\downharpoonleft[t_0,t]} \ln\mu_M - \sum_{i=1}^{r}\sum_{j=1}^{N_i}\left(\ln p_{\phi(t)=(i,j)}\right)}{\alpha_m}
\end{aligned}$$

Consequently,

$$\Psi_{[t_0,t]}(t) \leq \exp(N_{\downharpoonleft[t_0,t]} \ln\mu_M - \alpha_m T_{\downharpoonleft[t_0,t]})$$

Similarly,

$$\tilde{\Psi}_{[t_0,t]}(t) \leq \exp(N_{\upharpoonleft[t_0,t]} \ln\mu_M + \alpha_M T_{\upharpoonleft[t_0,t]})$$

Thus, we have

$$\begin{aligned}
&\Psi_{[t_0,t]}(t)\,\tilde{\Psi}_{[t_0,t]}(t) \\
&\leq \exp\left(N_{[t_0,t]} \ln\mu_M + \alpha_M T_{\upharpoonleft[t_0,t]} - \alpha_m T_{\downharpoonleft[t_0,t]}\right)
\end{aligned} \tag{25}$$

Analogous to the techniques of (11) and (12), it follows from (23) and (25) that

$$\begin{aligned}
x^T(t) R_{\sigma(t)} x(t) &\leq \frac{\Psi_{[t_0,t]}(t)\,\tilde{\Psi}_{[t_0,t]}(t)\lambda_3\lambda_2 c_1}{\lambda_1\mu_m} \\
&\leq \exp\Bigg(N_{[t_0,t]} \ln\mu_M + \alpha_M T_{\upharpoonleft[t_0,t]} \\
&\qquad -\alpha_m T_{\downharpoonleft[t_0,t]}\Bigg)\frac{\rho_2\lambda_3\lambda_2 c_1}{\rho_1\lambda_1\mu_m}
\end{aligned} \tag{26}$$

Based on $T_{[t_0,t]} \triangleq T_{\downharpoonleft[t_0,t]} + T_{\upharpoonleft[t_0,t]}$ and the known $\frac{T_{\upharpoonleft[t_0,t]}}{T_{\downharpoonleft[t_0,t]}} \leq a$, we have

$$\begin{aligned}
&N_{[t_0,t]} \ln\mu_M + \alpha_M T_{\upharpoonleft[t_0,t]} - \alpha_m T_{\downharpoonleft[t_0,t]} \\
&\leq N_{[t_0,t]} \ln\mu_M + \frac{a}{1+a}\alpha_M T_{[t_0,t]} - \frac{1}{1+a}\alpha_m T_{[t_0,t]} \\
&= N_{[t_0,t]} \ln\mu_M + \frac{T_{[t_0,t]}\left(a\alpha_M - \alpha_m\right)}{1+a}.
\end{aligned} \tag{27}$$

Table 1. Admissible ADT of different criteria

Switching schemes	ADT	Switching Ratios
Du et al (2010)	$\bar{\tau}_a^* = 0.8725$	*none*
Lemma 1	$\tau_a^* = 0.8025$	*none*
Theorem 1	$\begin{cases}\tau_{a1}^* = 0.6038 \\ \tau_{a2}^* = 0.8025\end{cases}$	*none*
Theorem 2	$\tau_a^* = 0.6785$	$a = 2.6$
Remark 4	$\begin{cases}\tau_0^* = 0.4840 \\ \tau_\infty^* = 0.8025\end{cases}$	$\begin{cases}a \equiv 0 \\ a \to \infty\end{cases}$

Combining (26) and (27), by $N_{[t_0,t]} \leq N^* = \frac{T_f}{\tau_a^*}$ in (21), it implies that,

$$x^T(t) R_{\sigma(t)} x(t) \leq c_2$$

then we conclude that the system holds EFTS with respect to $(c_1, c_2, R, \Omega, T_f, \sigma_a)$. □

Remark 4. For the switching signal in (21), if the activation time ratio $a \to \infty$, there are no constraints of the running time between the stable and unstable subsystems, the admissible criteria of switching signal with ADT can be obtained as

$$\tau_a \geq \tau_\infty^* = \frac{T_f \ln\mu_M}{\ln(\rho_1\lambda_1\mu_m c_2) - \ln(\rho_2\lambda_3\lambda_2 c_1) - \beta_M T_f}$$

Likewise, if $a \equiv 0$, which means that only stable subsystems are permitted to run, the admissible ADT can be given as

$$\tau_a \geq \tau_0^* = \frac{T_f \ln\mu_M}{\ln(\rho_1\lambda_1\mu_m c_2) - \ln(\rho_2\lambda_3\lambda_2 c_1) + \alpha_m T_f}$$

Comparing with the two cases of $a \to \infty$ and $a \equiv 0$, it is obvious that the introduction of stable subsystems will reduce the conservativeness of the switching signal with ADT. The advantage will also be further illustrated in the following example.

4. NUMERICAL EXAMPLE

In this section, a numerical example will be presented to demonstrate the validity of the results obtained above.

Consider the continuous-time switched linear system with four subsystems, parts referred to Du et al (2010)

$$A_1 = \begin{bmatrix} 0 & -1.0 \\ 2.0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & -2.0 \\ 1.0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.22 & 0.01 \\ -0.2 & -0.04 \end{bmatrix},\ A_4 = \begin{bmatrix} -0.12 & -0.16 \\ -0.15 & -0.22 \end{bmatrix}$$

where $c_1 = 1$, $c_2 = 20$, $T_f = 10$, $R_1 = \mathrm{diag}\{1.02, 1.01\}$, $R_2 = \mathrm{diag}\{1.03, 1.02\}$, $R_3 = \mathrm{diag}\{1.01, 1.02\}$, $R_4 = \mathrm{diag}\{1.03, 1.00\}$, $R = I$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1.05$ and $\beta_1 = \beta_2 = \alpha_3 = 0.01, \alpha_4 = 0.03$. Subsystems 1 and 2 are unstable while 3 and 4 are Hurwitz stable. Table I lists the admissible ADT for different switching criteria.

Obviously, by the employing of proposed parameter-dependent running time description approach, Lemma 1 can obtain less conservative criteria of switching signal than that in Du et al (2010). Additionally, owing to the existence of stable subsystems, the admissible ADT is

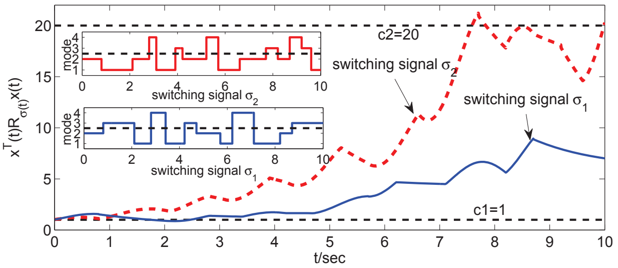

Fig. 1. The response of switched system with both stable and unstable subsystems under switching signal σ_1 and σ_2

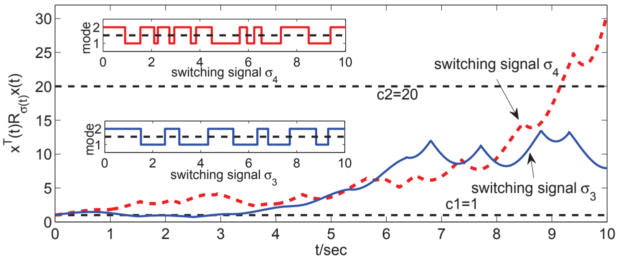

Fig. 2. The response of switched system with unstable subsystems under switching signal σ_3 and σ_4

relaxed to $\tau_a^* = 0.6785$ with the given ratio of switching time $a = 2.6$ by Theorem 2, and the feasible criteria of EFTS for switched subsystems with both stable and unstable subsystems are less conservative than Lemma 1 with only unstable subsystems. Furthermore, if the ADT exceed the admissible value range, $x^T(t)R_{\sigma(t)}x(t)$ may exceed the upper bound. Fig.2 shows the trajectories of $x^T(t)R_{\sigma(t)}x(t)$ containing stable and unstable subsystems with feasible switching signals σ_1 and infeasible switching signals σ_2.

Besides, by the developed with mode-dependent ADT switching signal in Theorem 1, the admissible criteria of each mode can be obtained as $\tau_{a1}^* = 0.6038$ and $\tau_{a2}^* = 0.8025$, which can adequately demonstrate the advantage of mode-dependent approach than Lemma 1 and Du et al (2010). Actually, EFTS of switched systems consisting of only unstable subsystems is just one special case of Theorem 2 when $a \to \infty$ by Remark 4. The simulation results for the switched systems with only unstable subsystems for feasible switching signal σ_3 and infeasible switching signal σ_4 are presented in Fig.3, respectively.

5. CONCLUSION

The so-called EFTS problems for the switched systems with only unstable subsystems and both stable and unstable subsystems have been investigated respectively. The existence criteria of EFTS with ADT and mode-dependent ADT in nonlinear setting are given by a parameter-dependent running time description approach. The obtained results are also extended to the switched systems with both stable and unstable subsystems by a given ratio of switching time between unstable and stable subsystems. The developed techniques provide a less conservative switching signal. The simulation results also demonstrate the advantages of the improved techniques.

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