Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Entanglement Generation in Uncertain Quantum Systems Using Sampling-based Learning Control

Mohamed A. Mabrok * **Daoyi Dong** * **Ian R. Petersen** *
Chunlin Chen **

* *School of Engineering and Information Technology, University of New South Wales at the Australian Defence Force Academy, Canberra ACT 2600, Australia, email: abdallamath@gmail.com, daoyidong@gmail.com, i.r.petersen@gmail.com*
** *Department of Control and System Engineering, Nanjing University, Nanjing 210093, China.*

Abstract: In this paper, we develop a control algorithm to generate entanglement in a quantum system with uncertainties. The system under consideration is an uncertain system of two two-level atoms interacting with each other through a dipole-dipole interaction. The sampling-based learning control (SLC) strategy is employed to find a control law. An SLC strategy contains two steps of training and evaluation. In the training step, we obtain several samples to construct an augmented system by sampling the uncertainties according to a possible distribution of the uncertainty parameters and learn an optimal control law by maximizing the performance index. In the evaluation step, we apply the obtained control law from the training step to additional samples through randomly sampling the uncertainties. Numerical results are presented showing the success of the SLC method in control design for generating entanglement.

Keywords: sampling-based learning control, entanglement generation, quantum control

1. INTRODUCTION

The control of quantum phenomena has received a great deal of attention in the last few years Wiseman and Milburn (2010); D'Alessandro (2007). Many theoretical and experimental aspects of quantum control have been studied (see for example Brif et al. (2010); Dong and Petersen (2010) and the references therein). One significant problem is to investigate the design of control laws for quantum systems with uncertainties since the existence of such uncertainties is unavoidable for most practical quantum systems James et al. (2008);Petersen et al. (2012). In the last several years, some results on quantum robust control have been presented. For instance, in James et al. (2008), an H_∞ controller synthesis problem for a class of quantum linear stochastic systems in the Heisenberg picture has been formulated and solved. In Dong and Petersen (2012), a sliding mode control approach to deal with Hamiltonian uncertainties in two-level quantum systems has been presented. In Petersen et al. (2012), the robust stability of linear quantum systems has been analyzed.

An interesting phenomenon in the quantum domain is `quantum entanglement` Nielsen and Chuang (2000), which is considered as a central topic in current studies of collective effects in many-body quantum systems Ficek and Tana (2002). Generally speaking, quantum entanglement is a physical phenomenon that occurs when pairs (or groups) of particles (subsystems) are generated or interact in ways such that the quantum state of each subsystem must subsequently be described relative to each other. Quantum entanglement has many applications in quantum communication Nielsen and Chuang (2000), quantum teleportation Bouwmeester et al. (1997), coding and quantum cryptography Ekert (1991). With the aid of quantum entanglement, some tasks that are impossible to accomplish by employing classical systems may be achieved. Thus, generation of quantum entanglement is an essential demand in quantum information technology.

Quantum control can play an important role in entanglement generation, especially in the robust preparation of entanglement (e.g., see Xia et al. (2009); Cai et al. (2010); Posazhennikova et al. (2013); Quintana et al. (2013)). In Posazhennikova et al. (2013), it has been shown that the entanglement possibilities can be controlled by changing system parameters. In Xia et al. (2009), coherent control and the creation of entangled states were discussed for a system of two superconducting flux qubits interacting with each other, which relied on the dynamic control of the qubit transition frequencies. In Cai et al. (2010), the radical-pair mechanism was discussed and the question of how quantum control can be used to either enhance or reduce the performance of such a chemical compass was investigated. The results provided a new route to further study the role of radical-pair entanglement via this mechanism.

⋆ This work was supported by the Australian Research Council (DP130101658, FL110100020) and by the Natural Science Foundation of China under Grant Nos.61273327 and 61374092.

When we consider a practical task for the generation of quantum entanglement, it is unavoidable that there exist different uncertainties in the internal Hamiltonian of each subsystem or in the coupling between these subsystems. It is necessary to develop robust methods for entanglement generation when possible uncertainties exist. In this paper, a sampling based-learning control (SLC) method, that was originally presented in Chen et al. (2014) and Dong et al. (2013) for control design of quantum ensembles and quantum systems with uncertainties, is used to design a control law for entanglement generation in an uncertain quantum system. The SLC method includes two steps: 1) The training step: we sample the uncertainties according to possible distributions of uncertainty parameters and construct an augmented system using these samples. Then we develop a gradient flow based learning and optimization algorithm to find the control with desired performance for the augmented system. 2) The testing and evaluation step: we test a number of samples, which are artificially created according to the possible distribution of uncertainty parameters, to evaluate the control performance. The objective is to find a control law using the SLC method to drive two qubits into a maximally entangled state. Numerical results show that the SLC method is an effective control strategy for generating entanglement when Hamiltonian uncertainties are present in the quantum system.

The paper is organized as follows: Section 2 presents the model and problem formulation. Section 3 describes the sampling-based learning control algorithm. Numerical results for applying the SLC algorithm to entanglement generation are given in Section 4. The paper is concluded with final remarks in Section 5.

2. MODEL AND PROBLEM FORMULATION

In this section, we introduce a quantum system consisting of two two-level atoms. A two-level atom can be used as a quantum bit (qubit). Quantum entanglement between two atoms or ions has been demonstrated Eichmann et al. (1993); DeVoe and Brewer (1996); Ficek and Tana (2002) in such systems as two barium ions confined in a spherical Paul trap Eichmann et al. (1993); DeVoe and Brewer (1996); Agarwal (1970). We assume that the two-qubit system under consideration can be approximated as a closed quantum system with the following evolution

$$i|\dot{\psi}(t)\rangle = H(t)\,|\psi(t)\rangle\,. \tag{1}$$

Here, $|\psi(t)\rangle$ is the quantum state of the system which corresponds to a complex vector in the underlying Hilbert space and $H(t)$ is the system Hamiltonian. For the two-qubit system, $|\psi(t)\rangle$ can be represented using the four basis vectors $|g_1, g_2\rangle$, $|e_1, g_2\rangle$, $|g_1, e_2\rangle$ and $|e_1, e_2\rangle$ as follows

$$\begin{aligned}|\psi(t)\rangle = {} & c_1(t)\,|g_1, g_2\rangle + c_2(t)\,|e_1, g_2\rangle \\ & + c_3(t)\,|g_1, e_2\rangle + c_4(t)\,|e_1, e_2\rangle\,,\end{aligned}$$

where $|e_j\rangle$ denotes the excited state of the atom j and $|g_j\rangle$ denotes the ground state of the atom j. Also, the complex coefficients $c_1(t), c_2(t), c_3(t)$ and $c_4(t)$ satisfy the following condition

$$|c_1(t)|^2 + |c_2(t)|^2 + |c_3(t)|^2 + |c_4(t)|^2 = 1.$$

The two two-level system is shown in Fig. 1. Assume that the time dependent system's Hamiltonian is given as follows:

$$H(t) = H_0 + H_{int} + H_u. \tag{2}$$

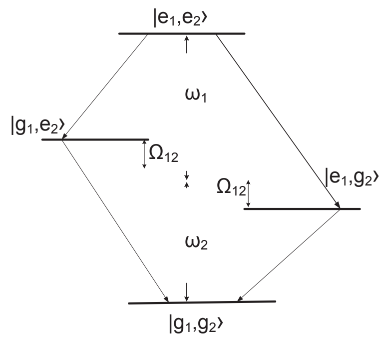

Fig. 1. Schematic energy level diagram of two two-level atoms as an equivalent four-level system.

Here,

$$H_0 = \sum_{i=1}^{2} \omega_i \sigma_z^i, \tag{3}$$

is the Hamiltonian describing the energy of the atoms, ω_i is the atomic transition frequency for atom i and $\sigma_z^i = \frac{1}{2}(|\, e_i\rangle\langle e_i\,| - |\, g_i\rangle\langle g_i\,|)$. The term

$$H_{int} = \sum_{i \neq j}^{2} \Omega_{ij} \sigma_+^i \sigma_-^j,$$

represents the dipole-dipole interaction between the two qubits (atoms) and Ω_{ij} is the dipole-dipole interaction parameter. It is known that most atoms have attractive forces between each other due to dipole moments, when the electrons of an atom leave the positively charged nucleus unshielded. This is called dipole-dipole interaction Jaksch et al. (2000). The last term

$$\begin{aligned}H_u = {} & u_1(\sigma_+^1 + \sigma_-^1) + u_2(\sigma_+^2 + \sigma_-^2) \\ & + u_3 i(\sigma_+^1 - \sigma_-^1) + u_4 i(\sigma_+^2 - \sigma_-^2) \\ & + u_5(\sigma_+^1\sigma_+^2 + \sigma_-^1\sigma_-^2) \\ & + u_6 \sum_{i\neq j}^{2} \sigma_+^i \sigma_-^j + u_7 \sum_{i=1}^{2} \sigma_+^i \sigma_-^i + u_8 \sum_{i=1}^{2} \sigma_-^i \sigma_+^i\end{aligned}$$

represents the possible control Hamiltonian, where $\sigma_+^i = |e_i\rangle\langle g_i|$, and $\sigma_-^i = |g_i\rangle\langle e_i|$. These operators can be represented in matrix form as follows $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

The main goal in this paper is to find the functions $u_i(t)$, which define the control Hamiltonian, to drive the quantum system to a particular target state with a desired level of fidelity even when uncertainties exist. Here, the proposed control Hamiltonian includes different terms. The first four terms $u_1(\sigma_+^1 + \sigma_-^1), u_2(\sigma_+^2 + \sigma_-^2), u_3 i(\sigma_+^1 - \sigma_-^1), u_4 i(\sigma_+^2 - \sigma_-^2)$ are local controls, and the term $u_5(\sigma_+^1\sigma_+^2 + \sigma_-^1\sigma_-^2)$ represents two photon exchange. These terms can be achieved in a quantum system by applying a specific electromagnetic field. The term $u_6 \sum_{i\neq j}^{2} \sigma_+^i \sigma_-^j$ represents the change in the dipole-dipole interaction between the atoms. This term can be controlled by changing the distance between the two atoms, or tuning the frequency of the driving field. The last two terms control the Lamb shift in the atomic resonance frequency of every atom.

In this paper, we are interested in uncertain quantum systems, where the Hamiltonian parameters may not be very well defined. We assume that the Hamiltonian (2) can be written as follows:

$$H(t) = K(\phi)H_0 + \beta(\varphi)H_{int} + f(\theta)H_u, \tag{4}$$

where $K(\phi)$, $\beta(\varphi)$ and $f(\theta)$ represent possible uncertainties in the free Hamiltonian, the interaction Hamiltonian and the control Hamiltonian, respectively. We assume that the parameters ϕ, φ and θ are time-independent and bounded, where $\phi \in [-\chi_1, \chi_1]$ and $\varphi \in [-\chi_2, \chi_2]$ and $\theta \in [-\chi_3, \chi_3]$. The constants $\chi_i \in [0, 1]$ represent the bounds of the uncertainty parameters.

The solution of the Schrödinger equation (1) is given as follows;

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle\,, \tag{5}$$

where $U(t)$ is the propagator operator which satisfies

$$i\dot{U}(t) = H(t)U(t). \tag{6}$$

Also, suppose that $C(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \\ c_4(t) \end{bmatrix}$. Using the Schrödinger equation (1), we have

$$i\dot{C}(t) = H(t)C(t).$$

The Hamiltonian $H(t)$ is written in a matrix form as follows

$$\begin{aligned} H(t) = \\ K(\phi)&\begin{pmatrix} -\nabla & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & \nabla \end{pmatrix} + \beta(\varphi)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_{12} & 0 \\ 0 & \Omega_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ + f(\theta)u_1&\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + f(\theta)u_2\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\ + f(\theta)u_3&\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + f(\theta)u_4\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \\ + f(\theta)u_5&\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + f(\theta)u_6\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ + f(\theta)u_7&\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} + f(\theta)u_8\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $\nabla = \omega_1 + \omega_2$ and $\Lambda = \omega_1 - \omega_2$.

Now, we define the performance function as follows

$$J(u) = |\,\langle\psi(T)|\,\psi_{\text{target}}\rangle|^2, \tag{7}$$

where $|\psi(T)\rangle$ is the state of the system at the end time T of the evolution, and $|\psi_{\text{target}}\rangle$ is the target state. This performance function will be used to measure the fidelity of the system with a given control law. An optimal control law can be found by maximizing $J(u)$.

3. SAMPLING-BASED LEARNING CONTROL ALGORITHM

In this section, we recall the sampling-based learning control (SLC) algorithm that has been presented in Chen et al. (2014), Dong et al. (2013) for control design of inhomogeneous quantum ensembles and quantum systems with uncertainties. The basic idea is to first use some artificial samples to construct an augmented system, then employ a learning algorithm to find an optimal control law for the augmented system, and finally evaluate the control performance using additional samples. A sampled-based learning algorithm contains two steps of "training" and "evaluation".

Training step: In this step, we assume that N samples are obtained through sampling uncertainties according to a particular distribution (e.g., uniform distribution) of the uncertainty parameters ϕ, φ, θ. In the training step, the main task is to find an optimal law $u(t) = u_i^*$ which maximizes the performance index

$$J_N(u) = \frac{1}{N}\sum_{n=1}^{N} |\,\langle\psi_{\phi_n,\varphi_n,\theta_n}(T)|\,\psi_{\text{target}}\rangle|^2. \tag{8}$$

where $|\psi_{\phi_n,\varphi_n,\theta_n}(T)\rangle$ is the final state of the training sample n. For a sample n, the propagator operator $U_{\phi_n,\varphi_n,\theta_n}(t)$ satisfies

$$\frac{d}{dt}U_{\phi_n,\varphi_n,\theta_n}(t) = -iH_{\phi_n,\varphi_n,\theta_n}(t)U_{\phi_n,\varphi_n,\theta_n}(t)$$

where $U_{\phi_n,\varphi_n,\theta_n}(0) = I$. This task to find the optimal control law can be accomplished using a gradient flow based iterative learning algorithm shown in Algorithm 1 (for details, see, e.g., Chen et al. (2014), Dong et al. (2013)).

Evaluation step: We apply the control u^* obtained in the training step to additional samples through randomly sampling the uncertainties. If the fidelity for the tested samples satisfies the required level of fidelity, we accept the designed control law and complete the control design process. Otherwise, we should go back to the training step and generate another optimized control strategy (e.g., restarting the training step with a new initial control strategy or a new set of samples).

4. NUMERICAL RESULTS

In this section, we present numerical simulation results for the proposed two-qubit system presented in Section 2. The main focus is to find an optimal control strategy that drives the system to a particular target state. In particular, we focus in generating entanglement between the two qubits (atoms). Thus, the target state is chosen to be one of the four Bell quantum states (see, e.g., Nielsen and Chuang (2000)). These states are often called maximally entangled states of two qubits. The four Bell states are given as follows:

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|e_1, g_2\rangle \pm |g_1, e_2\rangle) \tag{9}$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle \pm |g_1, g_2\rangle). \tag{10}$$

Algorithm 1 Gradient flow based iterative learning algorithm

Set the index of iterations $k = 0$
Choose a set of arbitrary controls $u^{k=0} = \{u_m^0(t),\ m = 1, 2, \ldots, M\}, t \in [0, T]$
repeat
  (for each iterative process)
  repeat
    (for each training samples $n = 1, 2, \ldots, N$)
    Compute the propagator $U_n^k(t)$ with the control strategy $u^k(t)$
  until $n = N$
  repeat
    (for each control u_m $(m = 1, 2, \ldots, M)$ of the control vector u)
    $\delta_m^k(t) = 2\,\mathrm{Im}\,(\langle\psi_{\phi_n,\varphi_n,\theta_n}(T)|\rho_{\text{target}} V_{\phi_n,\varphi_n,\theta_n}(t)|\psi_0\rangle)$
    where $\rho_{\text{target}} = |\psi_{\text{target}}\rangle\langle\psi_{\text{target}}|$ and $V_{\phi_n,\varphi_n,\theta_n}(t) = U_{\phi_n,\varphi_n,\theta_n}(T)U^\dagger_{\phi_n,\varphi_n,\theta_n}(t)f(\theta_n)H_m U_{\phi_n,\varphi_n,\theta_n}(t)$
    $u_m^{k+1}(t) = u_m^k(t) + \eta\delta_m^k(t)$
  until $m = M$
  $k = k + 1$
until the learning process ends
The optimal control strategy $u^* = \{u_m^*\} = \{u_m^k\},\ m = 1, 2, \ldots, M$

Here, we consider two two-level atoms in a cavity where there is no energy transfer between the cavity and the qubits. A similar model has been experimentally investigated in Majer et al. (2007); Filipp et al. (2011), where a microwave cavity has been used to couple the two qubits.

The parameters in atomic units are set as follows: The atomic transition frequencies are $(\omega_1, \omega_2) = (5.19, 6.45)$ and the dipole-dipole interaction parameter $\Omega_{12} = 0.0259$. The same relative relationship between the atomic transition frequencies and the dipole-dipole interaction as that in the experiment in Majer et al. (2007) has been considered. The evaluation time is $T = 2$ and the interval $[0, T]$ is discretized equally into $Q = 200$ time subintervals Δt, where $\Delta t = \frac{T}{Q} = 0.01$. The learning rate is set as $\eta = 0.1$. The initial control law is assumed to be $u_i^0 = \sin t$.

The uncertainty parameters ϕ, φ, θ are assumed to have a uniform distribution in the interval $[1-\chi, 1+\chi]$ and $\chi = 0.2$. For simplification, we assume $K(\phi_n) = \phi_n, \beta(\varphi_n) = \varphi_n, f(\theta_n) = \theta_n$. To construct a augmented system for the uncertainty parameters, we have divided the interval $[1-\chi, 1+\chi]$ into $N_\chi + 1$ subinterval and chose $N_\chi = 5$ by the same method as that used in Dong et al. (2013). This implies that number of samples is $N = N_\chi^3 = 125$ for training. Next, we present several examples of numerical results.

In the first example, we assume that all controls u_i are permitted in the Hamiltonian $H(t)$. The initial state is chosen to be $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e_1, g_2\rangle + |g_1, e_2\rangle)$ and the target state is chosen to be $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$. It is shown in Fig. 2 that the algorithm converges very quickly and a control law is found to drive the system to the target state with high fidelity. Here, the learning algorithm stops based on the rate of change in the performance index. For instance, if the rate of improvement in the last 100 iterations satisfies $J_{1000}(u) - J_{900}(u) < 0.0001$, the learning process ends. In the evaluation step, we select 300 additional samples to test the control performance. The control performance is shown in Fig 3. All the samples achieve the target state with high fidelity. Fig. 4 gives the optimal control law that is obtained in the training step. The control laws u_{1-4} represent local control, which can be achieved using electromagnetic fields where u_{1-4} is the amplitudes of these fields. The second control law u_5 can

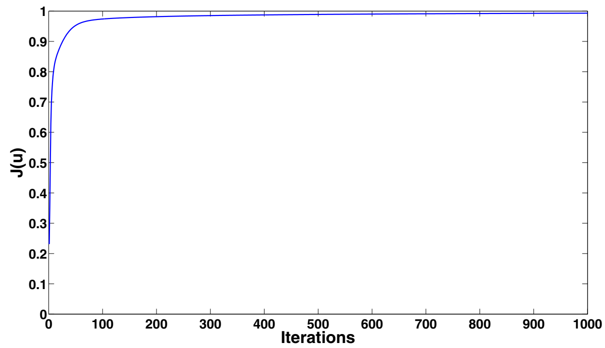

Fig. 2. Training performance $J(u)$ to find the optimal control strategy in the first example, where the initial state is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e_1, g_2\rangle + |g_1, e_2\rangle)$ and the target state $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$.

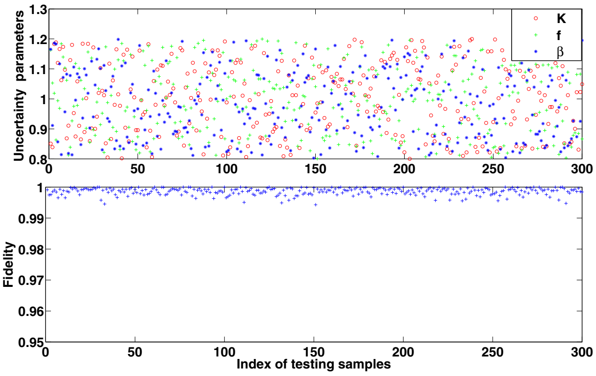

Fig. 3. The testing performance of the optimal control strategy in the first example, where the initial state is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e_1, g_2\rangle + |g_1, e_2\rangle)$ and the target state $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$.

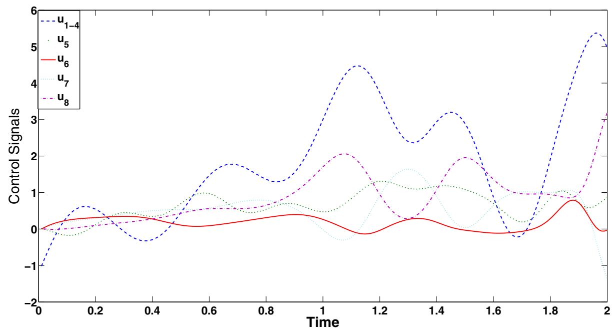

Fig. 4. The optimal control law in the first example, where the initial state is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e_1, g_2\rangle + |g_1, e_2\rangle)$ and the target state $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$.

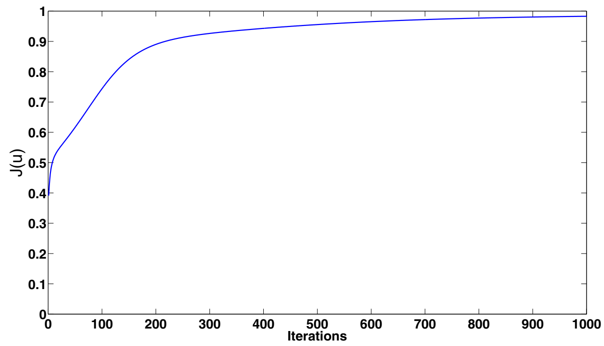

Fig. 5. Training performance $J(u)$ to find the optimal control strategy in the second example, where the initial state is $|\psi(0)\rangle = |g_1, g_2\rangle$ and the target state $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$.

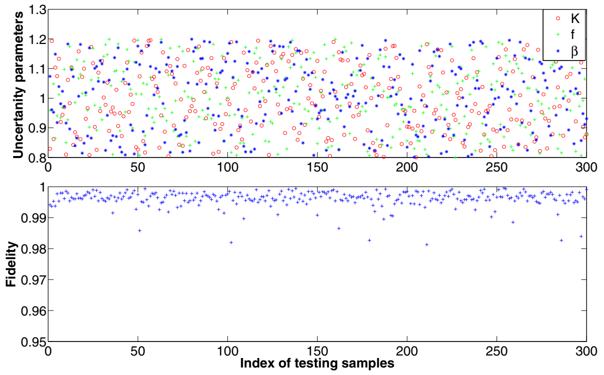

Fig. 6. The testing performance of the optimal control strategy in the second example, where the initial state is $|\psi(0)\rangle = |g_1, g_2\rangle$ and the target state $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$.

be achieved by controlling the distance between the two atoms or by changing the frequency of the driving field. The control law u_{7-8} can be achieved by changing the transition frequency of the atoms.

In the second example, the initial state is chosen to be $|\psi(0)\rangle = |g_1, g_2\rangle$ and the target state is chosen to be $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$. In this case, the learning algorithm converges more slowly than that in the first example. The performance is shown in Fig. 5. Also, the testing performance is shown in Fig. 6. It is clear that the performance in regard to fidelity level in the second example is worse than that in the first example. The control law obtained in the training step is shown in Fig. 7.

In the third example, we reduce the number of controls used in the previous cases but with the same number of iterations. We only use the controls u_{1-4} and u_6. The initial state is chosen to be $|\psi(0)\rangle = |g_1, g_2\rangle$ and the target state is chosen to be $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$. Using the same number of iterations as the above experiments, Fig. 8 shows that the performance becomes worse. Also, the fidelity of the tested samples is decreased as shown in Fig. 9. This clearly shows that reducing the number of control parameters will make the learning process converge

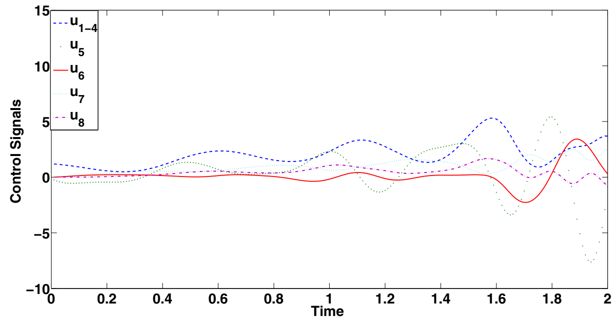

Fig. 7. The optimal control law in the second example, where the initial state is $|\psi(0)\rangle = |g_1, g_2\rangle$ and the target state $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$.

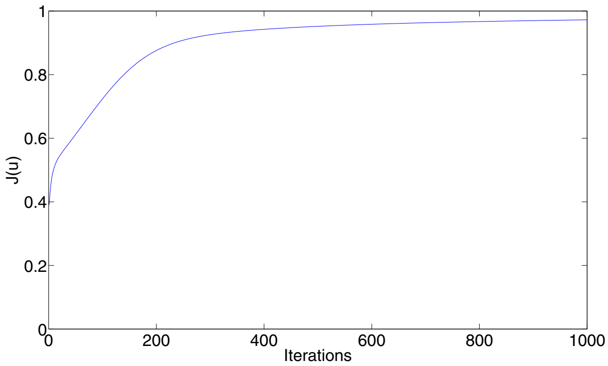

Fig. 8. Training performance $J(u)$ to find the optimal control strategy in the third example, where the initial state is $|\psi(0)\rangle = |g_1, g_2\rangle$ and the target state $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$.

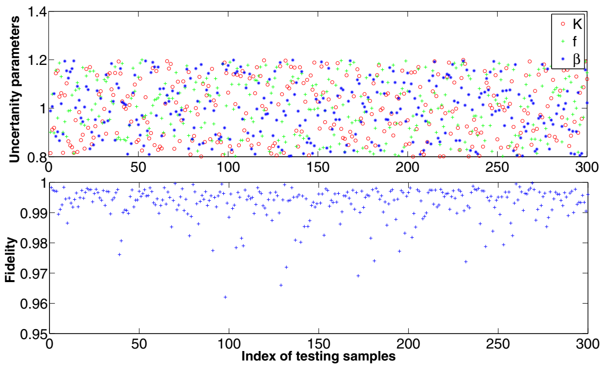

Fig. 9. The testing performance of the optimal control strategy in the third example, where the initial state is $|\psi(0)\rangle = |g_1, g_2\rangle$ and the target state $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$.

more slowly and make the performance become worse. However, less control resources are required in this case. For a practical task, it is worth considering the tradeoff between the control performance and the requirement of control resources. It is also possible to improve the training performance by increasing the number of iterations for the case with less control resources.

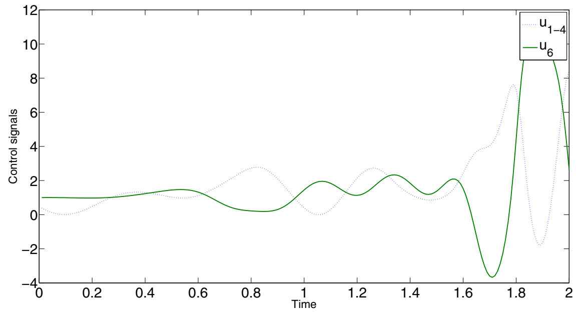

Fig. 10. The optimal control law in the third example, where the initial state is $|\psi(0)\rangle = |g_1, g_2\rangle$ and the target state $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1, e_2\rangle + |g_1, g_2\rangle)$.

5. CONCLUSION

In this paper, we present a control strategy for entanglement generation in a quantum system consisting of two two-level atoms when uncertainties exist in the Hamiltonian. An optimal control law is learned using a sampling-based learning control algorithm. The numerical results show that the fidelity depends on the initial state of the system as well as the number of control parameters used in the experiments. In the proposed method, we have assumed that the system is controllable. In fact, we can further consider the controllability of the augmented system that is closely related to performance of the proposed method. It is a significant problem for future research.

REFERENCES

G. S. Agarwal. Master-equation approach to spontaneous emission. *Phys. Rev. A*, 2: 2038–2046, Nov 1970. doi: 10.1103/PhysRevA.2.2038.

Dik Bouwmeester, Jian-Wei Pan, Klaus Mattle, Manfred Eibl, Harald Weinfurter, and Anton Zeilinger. Experimental quantum teleportation. *Nature*, 390: 575–579, 1997.

Constantin Brif, Raj Chakrabarti, and Herschel Rabitz. Control of quantum phenomena: past, present and future. *New Journal of Physics*, 12 (7): 075008, 2010.

Jianming Cai, Gian Giacomo Guerreschi, and Hans J. Briegel. Quantum control and entanglement in a chemical compass. *Phys. Rev. Lett.*, 104: 220502, Jun 2010. doi: 10.1103/PhysRevLett.104.220502.

Chunlin Chen, Daoyi Dong, Ruixing Long, Ian R. Petersen, and Herschel A. Rabitz. Sampling-based learning control of inhomogeneous quantum ensembles. *Phys. Rev. A*, 89: 023402, Feb 2014. doi: 10.1103/PhysRevA.89.023402.

D. D'Alessandro. *Introduction to Quantum Control and Dynamics*. Chapman & Hall/CRC Applied Mathematics & Nonlinear Science. Taylor & Francis, 2007.

R. G. DeVoe and R. G. Brewer. Observation of superradiant and subradiant spontaneous emission of two trapped ions. *Phys. Rev. Lett.*, 76: 2049–2052, Mar 1996. doi: 10.1103/PhysRevLett.76.2049.

D. Dong and I. R. Petersen. Quantum control theory and applications: a survey. *IET Control Theory & Applications*, 4 (12): 2651–2671, 2010.

Daoyi Dong and Ian R. Petersen. Sliding mode control of two-level quantum systems. *Automatica*, 48 (5): 725–735, 2012. doi: http://dx.doi.org/10.1016/j.automatica.2012.02.003.

Daoyi Dong, Chunlin Chen, Ruixing Long, Bo Qi, and Ian R. Petersen. Sampling-based learning control for quantum systems with Hamiltonian uncertainties. In *Prceedings of 52nd IEEE Conference on Decision and Control*, pp. 1924–1929, December 10-13, 2013, Florence, Italy.

U. Eichmann, J. C. Bergquist, J. J. Bollinger, J. M. Gilligan, W. M. Itano, D. J. Wineland, and M. G. Raizen. Young's interference experiment with light scattered from two atoms. *Phys. Rev. Lett.*, 70: 2359–2362, Apr 1993. doi: 10.1103/PhysRevLett.70.2359.

Artur K. Ekert. Quantum cryptography based on Bell's theorem. *Phys. Rev. Lett.*, 67: 661–663, Aug 1991. doi: 10.1103/PhysRevLett.67.661.

Z. Ficek and R. Tana. Entangled states and collective nonclassical effects in two-atom systems. *Physics Reports*, 372 (5): 369–443, 2002. doi: http://dx.doi.org/10.1016/S0370-1573(02)00368-X.

S. Filipp, M. Göppl, J. M. Fink, M. Baur, R. Bianchetti, L. Steffen, and A. Wallraff. Multimode mediated qubit-qubit coupling and dark-state symmetries in circuit quantum electrodynamics. *Phys. Rev. A*, 83: 063827, Jun 2011. doi: 10.1103/PhysRevA.83.063827.

D. Jaksch, J. I. Cirac, P. Zoller, S. L. Rolston, R. Côté, and M. D. Lukin. Fast quantum gates for neutral atoms. *Phys. Rev. Lett.*, 85: 2208–2211, Sep 2000. doi: 10.1103/PhysRevLett.85.2208.

M. R. James, H. I. Nurdin, and I. R. Petersen. H^∞ control of linear quantum stochastic systems. *IEEE Transactions on Automatic Control* , 53 (8): 1787–1803, 2008. doi: 10.1109/TAC.2008.929378.

J. Majer, J. M. Chow, J. M. Gambetta, Jens Koch, B. R. Johnson, J. A. Schreier, L. Frunzio, D. I. Schuster, A. A. Houck, A. Wallraff, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf. Coupling superconducting qubits via a cavity bus. *Nature*, 449: 443–447, 2007.

M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge, England: Cambridge University Press, 2000.

I. R. Petersen, V. Ugrinovskii, and M. R. James. Robust stability of uncertain linear quantum systems. *Philosophical Transactions of the Royal Society A*, 370: 5354–5363, 2012.

Anna Posazhennikova, Reinhard Birmuske, Martin Bruderer, and Wolfgang Belzig. Entanglement generation in a system of two atomic quantum dots coupled to a pool of interacting bosons. *Phys. Rev. A*, 88: 042302, Oct 2013. doi: 10.1103/PhysRevA.88.042302.

C. M. Quintana, K. D. Petersson, L. W. McFaul, S. J. Srinivasan, A. A. Houck, and J. R. Petta. Cavity-mediated entanglement generation via landau-zener interferometry. *Phys. Rev. Lett.*, 110: 173603, Apr 2013. doi: 10.1103/PhysRevLett.110.173603.

H. M. Wiseman and G. J. Milburn. *Quantum Measurement and Control*. Cambridge University Press, 2010.

Keyu Xia, Mihai Macovei, Jörg Evers, and Christoph H. Keitel. Robust coherent preparation of entangled states of two coupled flux qubits via dynamic control of the transition frequencies. *Phys. Rev. B*, 79: 024519, Jan 2009. doi: 10.1103/PhysRevB.79.024519.