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Errors-in-Variables Identification in Dynamic Networks by an Instrumental Variable Approach ⋆

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Abstract: In this paper, the objective is to obtain an estimate of a particular module embedded in a dynamic network using noisy measurements of the internal variables. This is an extension of the errors-in-variables (EIV) framework to the case of dynamic networks. The consequence of measuring the variables with noise is that the prediction error identification methods no longer result in consistent estimates. The method proposed in this paper is based on a combination of the instrumental variable approach and closed-loop prediction-error identification methods.

1. INTRODUCTION

Systems in engineering are becoming more complex and interconnected. Consider for example, power systems, telecommunication systems, flexible mechanical structures and distributed control systems. Models of these networks are important either for prediction, simulation or controller design. Fortunately sensors are becoming more ubiquitous and cheaper with the result that data can be collected from many variables in an interconnected dynamic network. Often the interconnection structure of the network is known, and the objective is to obtain an estimate of a particular module embedded in the network.

In many identification methods, the inputs are assumed to be measured without sensor noise [Ljung, 1999]. Moreover, if there is sensor noise on the inputs, the methods do not lead to consistent estimates. When sensors are used to measure the inputs, sensor noise is unavoidable and often not negligible. In this paper the following question is addressed: under what conditions is it possible to consistently identify a particular module that is embedded in a dynamic network when only noisy measurements of the internal variables of the network are available? This is an extension of the so-called Errors-in-Variables (EIV) framework to the case of dynamic networks.

The open loop EIV problem has been extensively studied (see the survey papers Söderström [2007, 2012]). The main conclusion in these papers is that prior knowledge about the system or a controlled experimental setup is required to ensure consistent estimates. In Söderström and Hong [2005] and Schoukens et al. [1997] it is shown that using periodic excitation or repeated experiments it is possible to consistently estimate the plant in an open loop setting.

The main idea behind our method is as follows. In a roughly similar vein to Söderström and Hong [2005], instead of using repeated experiments, additional (noisy) measurements generated by the dynamic network are used to deal with the sensor noise. We extend the results of Van den Hof et al. [2013] where it is shown that if noise free measurements of internal variables are known then consistent estimates of a module embedded in a network are possible to the case where only noisy measurements of the internal variables are available. The method proposed in this paper is based on a combination of the instrumental variable (IV) and prediction error identification lines of reasoning. It can be cast as a generalization of the IV method using a one-step-ahead predictor model with a Box-Jenkins model structure.

In Section 2 dynamic networks, prediction error identification and instrumental variable methods are briefly presented. In Section 3 the main results are presented.

2. BACKGROUND

In this section dynamic networks are formally defined, then prediction error identification and the basic closed-loop Instrumental Variable (IV) Method are presented.

2.1 Dynamic Networks

The networks considered in this paper is based on Van den Hof et al. [2013]. A dynamic network is built up of L elements (or nodes), related to L scalar *internal variables* w_j, $j = 1, \ldots, L$. Each internal variable can be written as:

$$w_j(t) = \sum_{k\in\mathcal{N}_j} G^0_{jk}(q)w_k(t) + r_j(t) + v_j(t) \qquad (1)$$

where G^0_{jk}, $k \in \mathcal{N}_j$ is a proper rational transfer function, q^{-1} is the delay operator $q^{-1}w_j(t) = w_j(t-1)$ and,

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- $\mathcal{N}_j$ is the set of indeces of internal variables with direct causal connections to w_j, i.e. $k \in \mathcal{N}_j$ iff $G^0_{jk} \neq 0$;
- v_j is *process noise*, that is modeled as a realization of a stationary stochastic process with rational spectral density: $v_j = H^0_j(q)e_j$ where e_j is a white noise process, and H^0_j is a monic, stable, minimum phase transfer function;
- r_j is an *external variable* that is known to the user, and can be manipulated by the user.

It may be that the disturbance and/or external variables are not present at some nodes. The network is defined by:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G^0_{12} & \cdots & G^0_{1L} \\ G^0_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G^0_{L-1\ L} \\ G^0_{L1} & \cdots & G^0_{L\ L-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_L \end{bmatrix},$$

where G^0_{jk} is non-zero if and only if $k \in \mathcal{N}_j$ for row j, and v_i (or r_i) is zero if it is not present. Using an obvious notation this results in the matrix equation:

$$w(t) = G^0(q)w(t) + r(t) + v(t) \tag{2}$$

Each internal variable is measured with some noise:

$$\tilde{w}_k(t) = w_k(t) + s_k(t)$$

where $\tilde{w}_k$ denotes the measurement of w_k, and s_k is *sensor noise*, which is represented by a stationary stochastic process with rational spectral density (s_k is not necessarily white noise).

There exists a *path* from w_i to w_j if there exist integers $n_1, \dots, n_k$ such that $G^0_{jn_1}G^0_{n_1n_2}\cdots G^0_{n_ki}$ is non-zero.

The following assumption will hold throughout the paper.

Assumption 1. General Conditions.

(a) The network is well-posed in the sense that all principal minors of $\lim_{z\to\infty}(I - G^0(z))$ are non-zero.
(b) $(I - G^0)^{-1}$ is stable.
(c) All process noise v_k, $k \in \{1, \dots, L\}$ are uncorrelated to all sensor noise s_ℓ, $\ell \in \{1, \dots, L\}$. [1]

2.2 Prediction Error Identification

In this section some key results of the *prediction error identification method* are presented. See Ljung [1999] for more details. Let w_j denote the variable which is to be predicted. The *predictor inputs* are those variables that are used to predict w_j. The set $\mathcal{D}_j$ is used to denote the set of indeces of the measurements that are chosen as predictor inputs, i.e. $\tilde{w}_k$ is a predictor input iff $k \in \mathcal{D}_j$. The one-step-ahead predictor for w_j is [Ljung, 1999]:

$$\hat{w}_j(t|t-1,\theta) = H_j^{-1}(q,\theta) \sum_{k\in\mathcal{D}_j} G_{jk}(q,\theta)\tilde{w}_k(t) + \Big(1 - H_j^{-1}(q,\theta)\Big)\tilde{w}_j(t) \tag{3}$$

where $H_j(q,\theta)$ is a (monic) noise model and $G_{jk}(q,\theta)$, $k \in \mathcal{D}_j$ are module models. The prediction error is:

$$\begin{aligned}\varepsilon_j(t,\theta) &= \tilde{w}_j(t) - \hat{w}_j(t|t-1,\theta) \\ &= H_j(q,\theta)^{-1}\Big(\tilde{w}_j(t) - \sum_{k\in\mathcal{D}_j} G_{jk}(q,\theta)\tilde{w}_k(t)\Big).\end{aligned} \tag{4}$$

Usually the parameterized transfer functions $G_{jk}(\theta)$, $k \in \mathcal{D}_j$, and $H_j(\theta)$ are estimated by minimizing the sum of squared (prediction) errors. Let $\hat{\theta}_N$ denote the estimated parameter vector. If $\hat{\theta}_N \to \theta^0$ as $N \to \infty$ with probability 1, then the obtained estimates are *consistent*.

When analyzing the consistency of a method, a notation common in the prediction error literature is:

$$\bar{\mathbb{E}}[\,\cdot\,] = \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} \mathbb{E}[\,\cdot\,]$$

where $\mathbb{E}$ denotes the expected value operator.

The method presented in this paper is based on a combination of the prediction error method and the instrumental variable philosophy. Thus, the instrumental variable technique is briefly presented in the next section.

2.3 Closed Loop Instrumental Variable Methods

The instrumental variable method is well suited to closed-loop identification because it makes effective use of the reference signal to deal with the problem that the input to the plant is correlated to the noise on the output (due to the feedback) [Gilson and Van den Hof, 2005]. We will focus our attention on the Basic Closed-Loop Instrumental Variable (BCLIV) method of Gilson and Van den Hof [2005]. A closed-loop data generating system is:

$$w_2 = G^0_{21}w_1 + v_2, \tag{5a}$$
$$w_1 = G^0_{12}w_2 + r_1. \tag{5b}$$

where there is no sensor noise. The objective is to obtain a consistent estimate of G^0_{21}. Consider an ARX model structure, i.e. the module transfer function $G_{21}(\theta)$ is parameterized as [Ljung, 1999]:

$$G_{21}(\theta) = \frac{B_{21}(\theta)}{A_{21}(\theta)} = \frac{q^{-n_k}(b^{21}_0 + \cdots + b^{21}_{n_b}q^{-n_b})}{1 + a^{21}_1 q^{-1} + \cdots + a^{21}_{n_a}q^{-n_a}} \tag{6}$$

and the noise model is parameterized as $H_2(\theta) = \frac{1}{A_{21}(\theta)}$. From (4) the prediction error is:

$$\begin{aligned}\varepsilon_2(t) &= A_{21}(q,\theta)w_2(t) - B_{21}(q,\theta)w_1(t) \\ &= w_2(t) - \phi^T_{21}(t)\theta_{21}\end{aligned} \tag{7}$$

where $\theta_{21} = [a^{21}_1 \ \cdots \ a^{21}_{n_a} \ b^{21}_0 \ \cdots \ b^{21}_{n_b}]^T$ and
$\phi^T_{21}(t) = [-w_2(t-1) \ \cdots \ -w_2(t-n_a) \ w_1(t) \ \cdots \ w_1(t-n_b)]$.

The BCLIV method is defined by the following algorithm.

Algorithm 1. Objective: obtain an estimate of G^0_{21}.

1. Choose r_1 as the *instrumental variable*. Let $z = r_1$.
2. Choose an ARX model structure and construct the prediction error (7).
3. Find a solution to the set of equations

$$\frac{1}{N}\sum_{t=0}^{N-1} \varepsilon_j(t,\theta)z(t-\tau) = 0, \ \text{for } \tau = 0, \dots, n_a + n_b \tag{8}$$

In Step 3 of Algorithm 1 a solution to a set of equations must be found. From (7) it follows that this solution can be found by linear regression.

[1] uncorrelated in the sense that the the cross-correlation function $R_{v_k s_\ell}(\tau)$ is zero for all τ.

Proposition 1. (BCLIV [Gilson and Van den Hof, 2005]). Consider the closed-loop system (5) that satisfies Assumption 1. A consistent estimate of G_{21}^0 can be obtained using Algorithm 1 if the following conditions hold:

(a) $\bar{\mathbb{E}}\big[\phi(t) \cdot [z(t) \ \cdots \ z(t-n_a-n_b)]\big]$ is nonsingular,
(b) $\bar{\mathbb{E}}[v_2(t) \cdot z(t-\tau)] = 0, \ \forall \tau \geq 0$.
(c) The parameterization is chosen flexible enough, i.e. there exists a θ such that $G_{21}(\theta) = G_{21}^0$. □

In the following section two methods are presented for identification in networks that can be cast as extensions of the BCLIV method.

3. IV METHOD EXTENDED TO DYNAMIC NETWORKS AND SENSOR NOISE

Recall, the objective considered in this paper is to obtain an estimate of a module, G_{ji}^0, embedded in a dynamic network using noisy measurements of the internal variables. This section is structured as follows: first a method is presented that is a straight-forward extension of the BCLIV method. For this method not all internal variables w_ℓ, $\ell \notin \mathcal{D}_j \cup \{j\}$ are candidate instrumental variables. Subsequently, a method is presented for which all internal variables w_ℓ, $\ell \notin \mathcal{D}_j \cup \{j\}$ are candidate instrumental variables. The key difference in the second method is that a Box-Jenkins model structure is used instead of an ARX model structure. This change is in line with closed-loop identification reasoning where is it well known that consistent estimates are possible if the noise is correctly modelled [Forssell and Ljung, 1999].

Throughout the paper it is always important to choose instrumental variables that are correlated to the predictor inputs. The presence of a correlation is key to any IV method (in Proposition 1 it ensures that Condition (a) holds). The following lemma presents conditions that ensure two internal variables are correlated.

Lemma 2. Consider a dynamic network (2) that satisfies Assumption 1. Two internal variables w_ℓ and w_k are correlated if one (or more) of the following conditions hold:

(a) There is a path from w_ℓ to an w_k.
(b) There is a path from w_k to w_ℓ.
(c) There is a variable w_n, $n \neq \ell, k$, such that there are paths from w_n to w_ℓ and w_n to w_k. □

The proof can be found in Appendix A.

To extend the BCLIV method to be able to use it in dynamic networks, a multiple input, single output ARX model structure must be used. In this case, the modules and noise model are parameterized as:

$$G_{jk}(q,\theta) = \frac{B_{jk}(q,\theta)}{A_j(q,\theta)}, \text{ and } H_j(q,\theta) = \frac{1}{A_j(q,\theta)}, \tag{9}$$

for all $k \in \mathcal{D}_j$, where all modules have the same denominator. From (1), w_j can be expressed using modules with a common denominator as follows:

$$w_j(t) = \frac{1}{A_j^0(q)} \sum_{k \in \mathcal{N}_j} \breve{B}_{jk}^0(q) w_k(t) + v_j(t)$$

where

$$A_j^0(q) = \prod_{n \in \mathcal{N}_j} A_{jn}^0(q) \text{ and } \breve{B}_{jk}^0(q) = \prod_{n \in \mathcal{N}_j \setminus k} B_{jk}^0(q) A_{jn}^0(q).$$

From (4) and (9) the prediction error is:

$$\begin{aligned} \varepsilon_j(\theta) &= A_j(q,\theta)\tilde{w}_j - \sum_{k \in \mathcal{D}_j} B_{jk}(q,\theta)\tilde{w}_k(t) \\ &= \tilde{w}_j(t) - \big[\phi_{k_1}^T(t) \ \cdots \ \phi_{k_n}^T(t) \ \phi_j^T(t)\big]\,\theta \\ &= \tilde{w}_j - \phi^T(t)\theta. \end{aligned} \tag{10}$$

where $\phi_{k_i}^T(t) = [\tilde{w}_{k_i}(t) \ \cdots \ \tilde{w}_{k_i}(t-n_b)]$, $\{k_1, \ldots, k_n\} = \mathcal{D}_j$ and $\phi_j^T(t) = [-\tilde{w}_j(t-1) \ \cdots \ -\tilde{w}_j(t-n_a)]$.

Consider the following algorithm which can be used to obtain a consistent estimate of G_{ji}^0.

Algorithm 2. Objective: obtain an estimate of G_{ji}^0.

1. Choose a measurement $\tilde{w}_\ell$ where $\ell \notin \mathcal{N}_j \cup \{j\}$ or an external variable r_m to use as the instrumental variable. The instrumental variable and w_i must be correlated (i.e. at least one of the conditions of Lemma 2 must hold). Let z denote the variable chosen as instrumental variable.
2. Choose the set of predictor inputs, $\mathcal{D}_j$, as follows: $k \in \mathcal{D}_j$ iff $k \in \mathcal{N}_j$ and at least one of the conditions of Lemma 2 is satisfied for w_k and z.
3. Choose an ARX model structure and construct the prediction error (10).
4. Find a solution to the set of equations
$$\frac{1}{N} \sum_{t=0}^{N-1} \varepsilon_j(t,\theta) z(t-\tau) = 0, \text{ for } \tau = 1, \ldots, n_\theta, \tag{11}$$
where n_θ is the number of parameters in the model.

This algorithm is very similar to that of the BCLIV (Algorithm 1). Only Steps 1 and 2 are more involved due to the increased complexity of a network vs. a closed loop. In Step 2 of Algorithm 2 only those variables w_k, $k \in \mathcal{N}_j$ that are somehow correlated to the instrumental variable z are included as predictor inputs. In the following proposition the conditions that ensure Algorithm 2 results in consistent estimates of G_{jk}^0 are presented.

Proposition 3. Consider a dynamic network as defined in Section 2.1 that satisfies Assumption 1. Let z denote the variable $\tilde{w}_\ell$ or r_ℓ chosen as the instrumental variable. A consistent estimate of G_{ji}^0 can be obtained using Algorithm 2 if the following conditions hold:

(a) If v_j is present, then there is no path from w_j to z
(b) $\bar{\mathbb{E}}\big[\phi^T(t)[z(t) \ \cdots z(t-n_\theta)]\big]$ is nonsingular, where $\phi(t)$ is defined in (10).
(c) If $z = \tilde{w}_\ell$, then the sensor noise s_ℓ is uncorrelated to all s_k , $k \in \mathcal{D}_j$.
(d) The process noise variable v_j is uncorrelated to all v_n with a path to w_j.
(e) The parameterization is flexible enough, i.e. there exists a θ such that $G_{jk}(q,\theta) = G_{jk}^0(q)$, $\forall k \in \mathcal{D}_j$. □

Condition (b) is a condition on the informativity of the data. Condition (a) puts a restriction on which internal variables are candidate instrumental variables. Depending on the interconnection structure, there may be no candidates. Condition (a) does not put a restriction on which external variables are candidate instrumental variables since, by definition there is never a path from any internal variable to an external variable. Thus all external variables are candidate instrumental variables irrespective of the interconnection structure. The idea is illustrated in the following two examples.

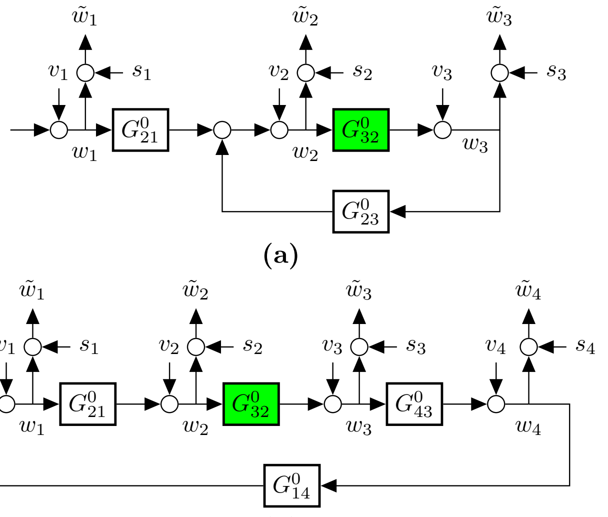

Fig. 1. Closed loop data generating systems

Example 1. Consider the data generating system shown in Fig. 1a. Suppose that the objective is to obtain a consistent estimate of G^0_{32}. Thus, $\{j\} = \{3\}$, and $\mathcal{N}_3 = \{2\}$. The only choice for the instrumental variable that satisfies Step 1 of Algorithm 2 is $z = \tilde{w}_1$. Since there is no path from w_3 to w_1, Condition (a) of Proposition 3 holds. Thus, consistent estimates of G^0_{32} are possible using Algorithm 2. □

Example 2. Consider the data generating system shown in Fig. 1b. Suppose that the objective is to obtain a consistent estimate of G^0_{32}. In this case it is not possible to satisfy Condition (a) of Proposition 3. □

In the following text a method is presented that can be used when Condition (a) of Proposition 3 does not hold.

The main reason that a feedback path from w_j to the instrumental variable w_i causes a problem is because then the projections of the predictor inputs onto the instrumental variable are correlated to the output noise. This is equivalent to the closed-loop identification problem where the plant input is correlated to the output noise. From the closed-loop identification literature, there are several methods to deal with this correlation that is induced by feedback [Forssell and Ljung, 1999, Van den Hof et al., 2013]. One method, called the Direct Method, deals with the problem by exactly modeling the noise. In the following text it is shown that this idea can be extended to the IV framework, so that measured variables that are part of a loop containing w_j can be used as instruments. Note that the idea is to exactly model the process noise term v_j, and not the sensor noise (or a sum of the two). The sensor noise is dealt with using the instrumental variable.

To exactly model the noise, a Box-Jenkins model structure is required. This amounts to the parameterization:

$$G_{jk}(q,\theta) = \frac{B_{jk}(q,\theta)}{A_{jk}(q,\theta)}, k \in \mathcal{D}_j \text{ and } H_j(q,\theta) = \frac{C_j(q,\theta)}{D_j(q,\theta)}, \quad (12)$$

where $A_{jk}(\theta)$, $B_{jk}(\theta)$, $C_j(\theta)$, $D_j(\theta)$ are polynomials in q.

Algorithm 3. Objective: obtain an estimate of G^0_{ji}.

1. Choose the set of predictor inputs as $\mathcal{D}_j = \mathcal{N}_j$.
2. For each $\tilde{w}_k$, $k \in \mathcal{D}_j \cup \{j\}$ choose a measurement $\tilde{w}_\ell$, $\ell \notin \mathcal{D}_j \cup \{j\}$ or external variable r_m that will be used as an instrumental variable. The chosen variable must be correlated to w_k (i.e. the chosen instrumental variable and w_k satisfy one of the conditions of Lemma 2). Let $\mathcal{I}_j$ and $\mathcal{X}_j$ denote the sets of indeces of internal and external variables respectively that are chosen as instrumental variables.
3. Construct the vector of instrumental variables $z(t) = [\tilde{w}_{\ell_1}(t) \cdots \tilde{w}_{\ell_n}(t)\, r_{m_1} \cdots r_{m_d}]$, where $\{\ell_1, \ldots, \ell_n\} = \mathcal{I}_j$ and $\{m_1, \ldots, m_d\} = \mathcal{X}_j$.
4. Choose a Box-Jenkins model structure, (12), and construct the prediction error (4).
5. Find a solution to the set of equations
$$\frac{1}{N}\sum_{t=0}^{N-1} \varepsilon(t,\theta) z(t-\tau) = 0, \text{ for } \tau = 1, \ldots, n_z, \quad (13)$$
where $n_z \geq n_\theta$, the number of parameters in the model.

Note from Step 2 that $\text{card}(\mathcal{D}_j \cup \{j\}) = \text{card}(\mathcal{I}_j) + \text{card}(\mathcal{X}_j)$. Also note that predictor inputs and $\tilde{w}_j$ cannot be chosen as instrumental variables, i.e. $(\mathcal{D}_j \cup \{j\}) \cap \mathcal{I}_j = \emptyset$.

There are several differences in this algorithm as compared to Algorithm 2. Firstly, a BJ model structure is used instead of an ARX model structure. Consequently, the set of equations in Step 5 are no longer linear in θ.

The second main difference that a set of instrumental variables is used, instead of just one instrumental variable. This is not a necessary choice, it is also possible to choose fewer instrumental variables. However, since the equations in (13) are no longer linear in θ, a condition to ensure the data is informative becomes more complex with fewer instrumental variables. Thus, we leave it for a future paper.

Proposition 4. Consider a dynamic network (2) that satisfies Assumption 1. A consistent estimate of G^0_{ji} is obtained using Algorithm 3 if the following conditions hold:

(a) Every w_ℓ, $\ell \in \mathcal{I}_j$ is a function of only $w_j(t-d)$, $d \geq 1$.
(b) The cross-power spectral density matrix
$$\Phi_{\mathcal{D}z}(\omega) = \begin{bmatrix} \Phi_{w_{k_1}z_1}(\omega) & \cdots & \Phi_{w_{k_1}z_n}(\omega) \\ \vdots & \ddots & \vdots \\ \Phi_{w_{k_{n-1}}z_1}(\omega) & \cdots & \Phi_{w_{k_{n-1}}z_n}(\omega) \\ \Phi_{w_j z_1}(\omega) & \cdots & \Phi_{w_j z_n}(\omega) \end{bmatrix} \quad (14)$$
is full rank for all $\omega \in [-\pi, \pi]$ where $\{k_1, \ldots, k_{n-1}\} = \mathcal{D}_j$, and z_n is the nth element of the vector of instrumental variables z.
(c) Every sensor noise s_k, $k \in \mathcal{D}_j \cup \{j\}$ is uncorrelated to every s_ℓ, $\ell \in \mathcal{I}_j$.
(d) The process noise variable v_j is uncorrelated to all v_m with a path to w_j.
(e) The parameterization is chosen flexible enough, i.e. there exists a parameter θ such that $G_{jk}(q,\theta) = G^0_{jk}(q)$, $\forall k \in \mathcal{D}_j$, and $H_j(q,\theta) = H^0_j(q)$. □

The proof can be found in the Appendix C. Condition (a) is satisfied if there is a delay in the path from w_j to the instrumental variable w_ℓ. If there is no delay from w_j to w_ℓ then the condition can be satisfied by choosing $w_\ell(t-1)$ as the instrumental variable. Condition (b) is a condition on the informativity of the data. By choosing the instrumental variables as in Step 2, it ensures that at least there will not be any rows or columns of zeros in the matrix $\Phi_{\mathcal{DI}}$ (i.e. each predictor input is correlated to at least one instrumental variable).

Example 3. Consider again the situation of Example 2. Choose, $\{j\} = \{3\}$, $\mathcal{N}_2 = \{2\}$, and $\mathcal{I}_2 = \{1, 4\}$. By Proposition 4, consistent estimates of G^0_{32} are possible. □

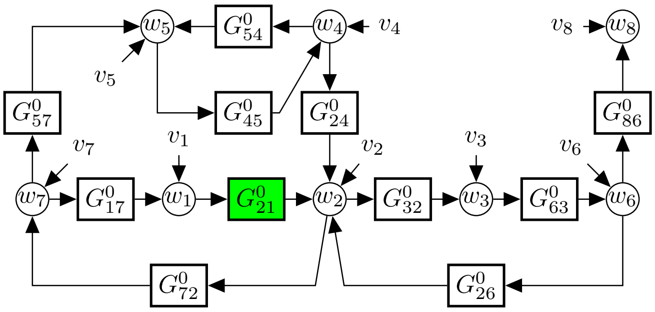

Fig. 2. Example of a dynamic network. The measurements of the internal variables are not shown. The labels of the w_is have been placed inside the summations indicating that the output of the sum is w_i.

Example 4. Consider the network shown in Fig. 2. Suppose that the objective is to obtain a consistent estimate of G^0_{21}. Thus, $\{j\} = \{2\}$, and $\mathcal{N}_2 = \{1, 4, 6\}$. By Algorithm 3 the predictor inputs must be chosen as $\tilde{w}_1$, $\tilde{w}_6$, and $\tilde{w}_4$ ($\mathcal{D}_2 = \mathcal{N}_2 = \{1, 6, 4\}$). Choose $\{\tilde{w}_3, \tilde{w}_5, \tilde{w}_7, \tilde{w}_8\}$ as the set of instrumental variables. By Proposition 4 consistent estimates of G^0_{21} are possible using Algorithm 3. □

4. CONCLUSION

In this paper a novel method is presented to obtain consistent estimates of a module embedded in a dynamic network using only noisy measurements of internal variables. In future work Algorithm 3 will be extended to allow the option to choose fewer instrumental variables.

Appendix A. PROOF OF LEMMA 2

The following lemma is used in the proof.

Lemma 5. Consider a dynamic network as defined in Section 2.1 that satisfies Assumption 1. Let $\mathcal{G}^0_{mn}$ be the (m, n)th entry of $(I - G_0)^{-1}$. If there are no paths from n to m then $\mathcal{G}^0_{mn}$ is zero. □

For a proof use Mason's Rules [Mason, 1953], or see Van den Hof et al. [2013]. Now follows the proof of Lemma 2.

Proof. The proof proceeds by showing that if none of the expressions (a)-(c) hold, then w_ℓ and w_k are uncorrelated. First both w_ℓ and w_k are expressed in terms of process noise variables. Then Lemma 5 is used to prove the result.

Using the notation of Lemma 5, w_ℓ and w_k can be expressed in terms of only process noise terms:

$$w_\ell(t) = \sum_{n=1}^{L} \mathcal{G}^0_{\ell n}(q) v_n(t) \text{ and } w_k(t) = \sum_{n=1}^{L} \mathcal{G}^0_{kn}(q) v_n(t). \quad \text{(A.1)}$$

Consider the cross-correlation between w_ℓ and w_k:

$$R_{w_\ell w_k}(\tau) = \bar{\mathbb{E}}[w_\ell(t) w_k(t - \tau)] \quad \text{(A.2)}$$

By Pareseval's theorem and substituting (A.1) into (A.2):

$$\Phi_{w_\ell w_k}(z) = \sum_{\substack{n=1 \\ n \neq \ell, k}}^{L} \mathcal{G}^0_{\ell n}(e^{j\omega}) \Phi_{v_k}(\omega) \mathcal{G}^0_{kn}(e^{-j\omega}) +$$

$$\mathcal{G}^0_{\ell k}(e^{j\omega}) \Phi_{v_k}(\omega) \mathcal{G}^0_{kk}(e^{-j\omega}) + \mathcal{G}^0_{\ell\ell}(e^{j\omega}) \Phi_{v_k}(\omega) \mathcal{G}^0_{k\ell}(e^{-j\omega})$$

By Lemma 5 if Condition (a) does not hold, $\mathcal{G}^0_{\ell m}$ is zero. Thus the first term of $\Phi_{w_\ell w_m}(z)$ is zero. Similarly, if Condition (b) does not hold the second term is zero. If Condition (c) does not hold then for each $k \in \{1, \ldots, L\} \setminus \{i, j\}$ either $\mathcal{G}^0_{\ell k}$ or $\mathcal{G}^0_{mk}$ is zero. Thus the third term of $\Phi_{w_\ell w_m}(z)$ is zero, concluding the proof.

Appendix B. PROOF OF PROPOSITION 3

Proof. Using (10) the asymptotic expression for the objective function (11) is:

$$\bar{\mathbb{E}}[\varepsilon(t) z(t - \tau)] = \bar{\mathbb{E}}\big[\big(\tilde{w}_j(t) - \theta^T \phi(t)\big) \,[\tilde{w}_i(t) \ \cdots \ \tilde{w}_i(t - n_\theta)]\big]$$

Let $x(t) = [\tilde{w}_i(t) \ \cdots \ \tilde{w}_i(t - n_\theta)]$. Then, by solving the set of equations (11) for $\tau = 1, \ldots, n_\theta$, θ can be expressed:

$$\theta^T = \bar{\mathbb{E}}[\tilde{w}_j(t) x(t)] \Big(\bar{\mathbb{E}}[\phi(t) x(t)]\Big)^{-1}$$

where $\bar{\mathbb{E}}[\phi(t) x(t)]$ is invertible by Condition (b). Expressing $\tilde{w}_j$ in terms of ϕ results in:

$$\theta^T = \theta^{0^T} + \bar{\mathbb{E}}[A^0_j(q)\big(v_j(t) + s_j(t)\big) x(t)] \Big(\bar{\mathbb{E}}[\phi(t) x(t)]\Big)^{-1} \quad \text{(B.1)}$$

where θ^0 is the parameter vector of the data generating system (Condition (e)). Thus, it must be shown that the second term of (B.1) is zero.

By Condition (c) the term involving s_j is zero. By the following reasoning, Condition (a) ensures that the term involving v_j in (B.1) is also zero. Using the notation of Lemma 5, express w_i in terms of v:

$$w_i = \sum_{k=1}^{L} \mathcal{G}^0_{nk} v_k.$$

Since there is no path from w_j to w_i, by Lemma 5 $\mathcal{G}^0_{ij}$ is zero. Thus, $w_i(t - \tau)$ is not a function of v_j, and so (B.1) reduces to $\theta^T = \theta^{0^T}$ which proves the result.

Appendix C. PROOF OF PROPOSITION 4

Proof. First simplified expressions for ε_j and z are derived. Then the two expressions are combined as

$$R_{\varepsilon z}(\tau) = \bar{\mathbb{E}}[\varepsilon(t, \theta) z(t - \tau)] \quad \text{(C.1)}$$

which is the asymptotic expression of (13). It is shown that this expression equals zero for $\tau \geq 0$ iff $G_{jk}(\theta) = G^0_{jk}$. For notational simplicity we assume throughout the proof that $\mathcal{X}_j = \emptyset$. It is easily be extended to include non-empty $\mathcal{X}_j$

Consider first an expression for the prediction error. Substitute the expressions for $\tilde{w}_j$ and $\tilde{w}_k$ into (4):

$$\varepsilon_j(\theta) = H_j^{-1}(\theta) \Big(\sum_{k \in \mathcal{N}_j} G^0_{jk} w_k + v_j + s_j - \sum_{k \in \mathcal{N}_j} G_{jk}(\theta)(w_k + s_k) \Big)$$

$$= H_j^{-1}(\theta) \sum_{k \in \mathcal{N}_j} \Delta G_{jk}(\theta) w_k + \Delta H_j(\theta) v_j + e_j$$

$$+ H_j^{-1}(\theta) \Big(s_j - \sum_{k \in \mathcal{N}_j} G_{jk}(\theta) s_k \Big) \quad \text{(C.2)}$$

where $\Delta G_{jk}(\theta) = G^0_{jk} - G_{jk}(\theta)$ and $\Delta H_j(\theta) = H_j^{-1}(\theta) - H_j^{0^{-1}}$.

Now consider the expression for the instrumental vector:

$$\begin{aligned} z(t) &= [\tilde{w}_{\ell_1}(t) \ \cdots \ \tilde{w}_{\ell_m}(t)] \\ &= [w_{\ell_1}(t) + s_{\ell_1}(t) \ \cdots \ w_{\ell_m}(t) + s_{\ell_m}(t)]. \end{aligned} \quad \text{(C.3)}$$

In the following text, an expression for $R_{\varepsilon z}(\tau)$ is derived using (C.2) and (C.3) that is valid for all $\tau \geq 0$. Subsequently, this expression is used to prove the proposition.

No measurement chosen as an instrumental variable can be a predictor input ($\mathcal{N}_j \cap \mathcal{I}_j = \emptyset$ by the statement of the algorithm). Thus, no s_k that appears in the instrumental variable vector z (C.3), will appear in the expression for ε_j, (C.2). By Condition (c) the s terms can be eliminated from (C.1) resulting in:

$$\bar{\mathbb{E}}[\varepsilon_j(t,\theta) \cdot z(t-\tau)] = \bar{\mathbb{E}}\Big[\Big(H_j^{-1}(q,\theta) \sum_{k \in \mathcal{N}_j} \Delta G_{jk}(q,\theta) w_k(t) + \Delta H_j(q,\theta) v_j(t) + e_j(t)\Big) \cdot [w_{\ell_1}(t-\tau) \ \cdots \ w_{\ell_m}(t-\tau)]\Big] \quad \text{(C.4)}$$

By Condition (a) each each w_ℓ, $\ell \in \mathcal{I}_j$ is a function of only delayed versions of v_j (and thus delayed versions of e_j). Thus, $\bar{\mathbb{E}}[e_j(t) \cdot w_n(t-\tau)] = 0$ for all $\tau \geq 0$ and $\ell \in \mathcal{I}_j$, and so from (C.4) it follows that

$$\bar{\mathbb{E}}[\varepsilon_j(t,\theta) \cdot z(t-\tau)] = \bar{\mathbb{E}}\Big[\Big(H_j^{-1}(q,\theta) \sum_{k \in \mathcal{N}_j} \Delta G_{jk}(q,\theta) w_k(t) + \Delta H_j(q,\theta) v_j(t)\Big) \cdot [w_{\ell_1}(t-\tau) \ \cdots \ w_{\ell_m}(t-\tau)]\Big]. \quad \text{(C.5)}$$

which holds for all $\tau \geq 0$. Using a vector notation (C.5) can be expressed as:

$$R_{\varepsilon z}(\tau) = \bar{\mathbb{E}}\Big[\Delta X(q,\theta)^T \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_n}(t) \\ v_j(t) \end{bmatrix} \cdot [w_{\ell_1}(t-\tau) \ \cdots \ w_{\ell_m}(t-\tau)]\Big]$$

which holds for $\tau \geq 0$ and where

$$\Delta X(q,\theta) = \begin{bmatrix} H_j^{-1}(q,\theta) \Delta G_{jk_1}(q,\theta) \\ \vdots \\ H_j^{-1}(q,\theta) \Delta G_{jk_n}(q,\theta) \\ \Delta H_j(q,\theta) \end{bmatrix}$$

and $\{k_1, \ldots, k_n\} = \mathcal{N}_j$. The variable v_j can be expressed in terms of internal variables as:

$$v_j = w_j - \sum_{k \in \mathcal{N}_j} G^0_{jk}(q) w_k$$

and so

$$\begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_n}(t) \\ v_j(t) \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ -G^0_{jk_1}(q) & \cdots & -G^0_{jk_n}(q) & 1 \end{bmatrix} \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_n}(t) \\ w_j(t) \end{bmatrix} \quad \text{(C.6)}$$

Denote the matrix in (C.6) as $J^0(q)$. Using this notation,

$$R_{\varepsilon z}(\tau) = \bar{\mathbb{E}}\Big[\Delta X(q,\theta)^T J^0(q) \begin{bmatrix} w_{k_1}(t) \\ \vdots \\ w_{k_n}(t) \\ w_j(t) \end{bmatrix} \cdot [w_{\ell_1}(t-\tau) \ \cdots \ w_{\ell_m}(t-\tau)]\Big] \quad \text{(C.7)}$$

which is valid for all $\tau \geq 0$.

Now, first consider the 'if' statement. It must be shown that if $G_{jk}(q,\theta) = G^0_{jk}$, for all $k \in \mathcal{N}_j$ and $H_j(q,\theta) = H^0_j$, then (C.1) holds. Let θ^0 denote the particular parameter vector such that $G_{jk}(q,\theta^0) = G^0_{jk}$, for all $k \in \mathcal{N}_j$, and $H_j(\theta^0) = H^0_j$. Thus $\Delta G_{jk}(\theta^0) = 0$ and $\Delta H_j(\theta^0) = 0$. From (C.7) it follows that at θ^0,

$$\bar{\mathbb{E}}[\varepsilon_j(t,\theta^0) \cdot z(t-\tau)] = 0, \ \forall \tau \geq 0.$$

Now consider the 'only if' statement. It must be shown that if (C.1) holds, then $G_{jk}(\theta) = G^0_{jk}$, for all $k \in \mathcal{N}_j$ and $H_j(\theta) = H^0_j$. Since $R_{\varepsilon z}(\tau) = 0$ for all $\tau \geq 0$, it follows that

$$R_{\varepsilon z}(\tau) R^T_{\varepsilon z}(-\tau) = 0, \ \forall \tau.$$

Thus the following equation also holds

$$\sum_{\tau=-\infty}^{\infty} R_{\varepsilon z}(\tau) R^T_{\varepsilon z}(-\tau) = 0.$$

Taking the Fourier Transform of both sides results in

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{\varepsilon z}(e^{j\omega}) \Phi^T_{\varepsilon z}(e^{-j\omega}) \mathrm{d}\omega = 0. \quad \text{(C.8)}$$

Finally, substitute (C.7) into (C.8):

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta X(e^{j\omega},\theta) J^0(e^{j\omega}) \Phi_{\mathcal{DI}}(e^{j\omega}) \cdot \Phi^T_{\mathcal{DI}}(e^{-j\omega}) J^{0^T}(e^{-j\omega}) \Delta X^T(e^{-j\omega},\theta) \mathrm{d}\omega = 0, \quad \text{(C.9)}$$

where $\Phi_{\mathcal{DI}}(e^{j\omega})$ is defined in (14). By Condition (b) it is full rank for all $\omega \in [-\pi,\pi]$. Moreover, by (C.6), the matrix $J^0(e^{j\omega})$ is also full rank for all $\omega \in [-\pi,\pi]$. Consequently,

$$J^0(e^{j\omega}) \Phi_{\mathcal{DI}}(e^{j\omega}) \Phi^T_{\mathcal{DI}}(e^{-j\omega}) J^{0^T}(e^{-j\omega})$$

in (C.9) is positive definite $\forall \omega \in [-\pi,\pi]$. Thus the only way (C.9) can hold is if $\Delta X(e^{j\omega},\theta) = 0 \ \forall \omega \in [-\pi,\pi]$. □

REFERENCES

U. Forssell and L. Ljung. Closed-loop identification revisited. *Automatica*, 35:1215–1241, 1999.

M. Gilson and P.M.J. Van den Hof. Instrumental variable methods for closed-loop system identification. *Automatica*, 41:241–249, 2005.

L. Ljung. *System Identification. Theory for the User.* Prentice Hall, 2 edition, 1999.

S. J. Mason. Feedback theory - some properties of signal flow graphs. *Proceedings of The IRE*, pages 1144–1156, September 1953.

J. Schoukens, R. Pintelon, G. Vandersteen, and G. Guillaume. Frequency-domain system identification using non-parametric noise models estimated from a small number of data sets. *Automatica*, 33:1073–1086, 1997.

T. Söderström. Errors-in-variables methods in system identification. *Automatica*, 43:939–958, 2007.

T. Söderström. System identification for the errors-in-variables problem. *Transactions of the Institute of Measurement and Control*, 34:780–792, 2012.

T. Söderström and M. Hong. Identification of dynamic errors-in-variables systems with periodic data. In *Proceedings of the 16th IFAC World Congress*, pages 809–814, Prague, Czech Republic, 2005.

P. M. J. Van den Hof, A. Dankers, P. S. C. Heuberger, and X. Bombois. Identification of dynamic models in complex networks with prediction error methods - basic methods for consistent module estimates. *Automatica*, 49:2994–3006, October 2013.