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State feedback stabilization of time delay linear singular systems subject to actuator saturation

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Abstract: This paper investigates the problem of stabilization of time delay linear singular systems subject to actuator saturation. A polytopic approach is used to describe the saturation behavior. By using an augmented Lyaponuv-Krasovskii functional and adopting the delay partitioning technique, less conservative sufficient conditions are established to ensure the closed loop system to be locally admissible based on a state feedback controller. In this paper, all the conditions are transformed into minimization problem involving LMI conditions by adopting the idea of the cone complementarity algorithm. A numerical example illustrates the effectiveness of the design.

1. INTRODUCTION

The time delay singular representation, which is a mixture of algebraic and differential equations with time delay, often appear in various engineering systems, including chemical engineering systems, lossless transmission lines, etc Niculescu [2000]. Consequently, various problem of analysis and synthesis of time delay linear singular systems have been gained much attention in the past years: stability analysis Feng [2011], H_∞ control Wu [2009], robust control Chaibi [2012], observer-based control Su [2011]. On the other hand, in practical case, all actuators cannot deliver unlimited energy to physical plants. Thus, the implementation of control laws without taking into account the actuator saturation effect may have adverse effects on the performance and stability of closed-loop system. Thus, the stability analysis and controller design for systems with actuator saturation have drawn much research attention in this past decade. In Hu [2002], the problem of control has been discussed for linear systems with actuator saturation and persistent disturbances. Lv and Lin have presented in Lv [2008] an analysis of the $\mathcal{L}_2$ gain and $\mathcal{L}_\infty$ performance for singular linear systems under actuator saturation. The objective of this paper is to design a control law for stabilization of time delay linear singular systems subject to actuator saturation. The main contribution is reduction of conservatism by adopting the delay partitioned technique Feng [2011] and using an augmented Lyapunov-Krasovskii functional with triple integral term.

The reminder of this paper is organized as follows. In section 2, we give a description of time delay linear singular systems in presence of actuator saturation. In section 3, new delay dependent sufficient conditions are derived by using an augmented Lyapunov-Krasovskii functional. Then, these conditions are transformed into a minimization problem involving LMI conditions. In section 4, an illustrative example is given to demonstrate the effectiveness of the proposed result. Some conclusions are drawn in section 5.

Notations. Throughout this paper, $X \in \mathbb{R}^n$ denotes the $n-$dimensional Euclidean space, while $X \in \mathbb{R}^{n\times m}$ refers to the set of all $n \times m$ real matrices. Notation $X > 0$ (respectively, $X \geq 0$) means that matrix X is real symmetric positive definite (respectively, positive semi-definite). If not explicitly stated, all matrices are assumed to have compatible dimensions for algebraic operations. Symbol $(*)$ stands for matrix block induced by symmetry, and $sym(X)$ stands for $X + X^T$. $\varrho(M)$ denotes spectral radius the matrix M. $\mathcal{C}_{n,\bar{\tau}}$ denotes the Banach space of continuous functions mapping $[-\tau, 0]$ to $\mathbb{R}^n$. $\lambda_{max}(P)$ stand for the maximal eigenvalues of matrix P. $\|.\|$ refers to the Euclidean vector norm or spectral matrix norm and $\|\phi\|_c = sup_{-\tau\leq t\leq 0}\|\phi(t)\|$ stands for the norm of a function $\phi(t) \in \mathcal{C}_{n,\tau}$. $\rho(M)$ denotes the spectral radius of the matrix M.

2. PROBLEM FORMULATION

We consider the following time delay linear system

$$E\dot{x}(t) = Ax(t) + A_\tau x(t-\tau) + B_i sat(u(t), \bar{u}) \qquad (1)$$
$$x(t) = \phi(t), \forall t \in [-\bar{\tau}, 0],$$

where $x(t) \in \Re^{n_x}$ is the state vector; $u(t) \in \Re^{n_u}$ is the control input; τ represents the delay satisfying $0 < \tau \leq \bar{\tau}$; $\phi(t) \in \mathcal{C}_{n,\bar{\tau}}$ is a compatible vector valued initial function. sat: $\mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is a vector valued standard function defined as:

$$sat(u(t),\bar{u}) = [\, sat(u_1(t),\bar{u}_1)\;\; sat(u_2(t),\bar{u}_2) \;\; \cdots \;\; sat(u_{n_u}(t),,\bar{u}_{n_u})\,]^T \quad (2)$$

$$sat(u_l(t),\bar{u}_l) = \begin{cases} \bar{u}_l & \text{if} \quad u_l(t) > \bar{u}_l \\ u_l(t) & \text{if} \quad -\bar{u}_l \leq u_l(t) \leq \bar{u}_l \\ -\bar{u}_l & \text{if} \quad u_l(t) < -\bar{u}_l \end{cases} \quad (3)$$

where $\bar{u}$ is the saturation level, $u_l(t)$ is the lth input of $u(t)$ and $\bar{u}_l$ is the lth input of $\bar{u}$, $l = 1, \cdots, n_u$.
The matrix $E \in \mathbb{R}^{n_x \times n_x}$ may be singular and we assume that $rank(E) = n_{x1} \leq n_x$. A, A_τ and B are known real constant matrices with appropriate dimensions.
We consider the following state-feedback control law:

$$u(t) = Kx(t) \quad (4)$$

In order to obtain the main results in this paper, the following definitions and lemmas are needed:

Definition 1. Define the following subsets of $\mathbb{R}^{n_x}$.

$$\varepsilon(P,\varrho) = \{x \in \mathbb{R}^{n_x}; x^T P x \leq \varrho\}, \quad (5)$$

where P a positive definite matrix and ϱ is a positive number. $\varepsilon(P,\varrho)$ is an ellipsoid set.

$$\mathcal{L}(H,\bar{u}) = \{x \in \mathbb{R}^{n_x}; |H_l x| \leq \bar{u}_l,\, l = 1,\ldots,n_u\}, \quad (6)$$

where H_l is the lth row of the matrix $H \in \mathbb{R}^{n_u \times n_x}$. $\mathcal{L}(H,\bar{u})$ is a polyhedral set.

Definition 2. Dai [1989] Time delay descriptor system

$$\begin{cases} E\dot{x}(t) = Ax(t) + A_\tau x(t-\tau) \\ x(t) = \phi(t),\ \forall t \in [-\bar{\tau}, 0]. \end{cases} \quad (7)$$

is said to be

(1) regular if $det\Big(sE - A\Big) \neq 0$.
(2) impulse free if $deg\Big(det\Big(sE - A\Big)\Big) = rank(E)$.
(3) asymptotically stable if for any $\varepsilon > 0$ there exists a scalar $\delta(\varepsilon) > 0$ such that for any compatible initial condition, $\phi(t)$, with $\sup_{-\tau < t \leq 0} \|\phi(t)\| < \delta(\varepsilon)$, the solution $x(t)$ of (7) satisfies $\|x(t)\| < \varepsilon$ for $t \geq 0$ and $\lim_{t\to 0} x(t) = 0$.
(4) admissible if it is regular, impulse-free and asymptotically stable.

Lemma 1. Hu [2002] Let $K, H \in \mathbb{R}^{n_u \times n_x}$ be given matrix, for $x \in \mathbb{R}^{n_x}$, if $x \in \mathcal{L}(H, \boldsymbol{u})$ then

$$sat(Kx,\bar{u}) = co\{M_s K + M_s^- H, s \in [1,\eta]\}\ ; \quad \eta = 2^{n_u} \quad (8)$$

where co denotes the convex hull.

Consequently, there exist $\delta_1 \geq 0, \ldots, \delta_\eta \geq 0$ with $\sum_{s=1}^{\eta} \delta_s = 1$ such that,

$$sat(Kx,\bar{u}) = \sum_{s=1}^{\eta} \delta_s [M_s K + M_s^- H]x \quad (9)$$

Here, M_s is an n_u by n_u diagonal matrix with elements either 1 or 0 and $M_s^- = I_{n_u} - M_s$. There are 2^{n_u} possible matrices of this type. One can also consult the work of Benzaouia [2007] for more details and other extensions to linear systems with both constraints on the control and the increment or rate of the control.

Lemma 2. Sun [2009] For any constant matrix $M > 0$, any scalars τ_m and τ_M with $0 \leq \tau_m < \tau_M$, and a vector function $x(t) : [-\tau_M, -\tau_m] \to \mathbb{R}^n$ such that the integrals concerned are well defined, then the following inequalities holds

$$1)\ -\tau_r \int_{t-\tau_M}^{t-\tau_m} x^T(s) M x(s) ds \leq - \int_{t-\tau_M}^{t-\tau_m} x^T(s) ds M \int_{t-\tau_M}^{t-\tau_m} x(s) ds$$

$$2)\ -\tau_s \int_{-\tau_M}^{-\tau_m} \int_{t+\theta}^{t} x^T(s) M x(s) ds d\theta \leq - \int_{-\tau_M}^{-\tau_m} \int_{t+\theta}^{t} x^T(s) ds d\theta M \int_{-\tau_M}^{-\tau_m} \int_{t+\theta}^{t} x(s) ds d\theta$$

with $\tau_r = \tau_M - \tau_m$ and $\tau_s = \dfrac{1}{2}(\tau_M^2 - \tau_m^2)$.

Lemma 3. Wang [1992] Given matrices X, Y with compatible dimensions. Then, the following inequality holds for any matrix Q

$$XY^T + YX^T \leq XQX^T + YQ^{-1}Y^T$$

3. MAIN RESULTS

Let H be given matrices. By applying lemma 1, the saturated feedback control (4) can be written as:

$$sat(Kx(t),\bar{u}) = \sum_{s=1}^{\eta} \delta_s [M_s K + M_s^- H] x(t)\ ; \quad (10)$$

$$\delta_s \geq 0,\ \sum_{s=1}^{\eta} \delta_s = 1 \quad (11)$$

Combining (1), (4) and (10), the closed-loop system can be expressed as follows:

$$E\dot{x}(t) = \sum_{s=1}^{\eta} \delta_s [A_s x(t) + A_\tau x(t-\tau)] \quad (12)$$

with $x(t) = \phi(t)$ for $t \in [-\bar{\tau}, 0]$ and $A_s = A + B(M_s K + M_s^- H)$, $\quad s \in [1, \eta]$.

Theorem 1. Consider the time delay singular system described in (1). Given an integer $m \geq 1$, and positive scalars τ and ϱ, if there exist matrices S, G_l, $l = 1, 2, 3$, $P = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0$, $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} > 0$, $W > 0$, $\mathbb{Q} = [Q_{kl}]_{m \times m} > 0$, H such that the following conditions hold:

$$\begin{bmatrix} \Lambda_s^{11} & \Lambda_s^{12} & \Lambda_s^{13} & \Lambda_{14} \\ * & \Lambda^{22} & \Lambda^{23} & \Lambda_{24} \\ * & * & \Lambda_{33} & P_{12} \\ * & * & * & \Lambda_{44} \end{bmatrix} < 0; s = 1, \ldots, \eta \quad (13)$$

$$Q_{mm} < Q_{11} \quad (14)$$

$$\varepsilon(E^T P_{11} E, \varrho) \subset \mathcal{L}(H, \bar{u}) \quad (15)$$

in which

$$\Lambda_s^{11} = sym\big(E^T P_{12} - G_1 A_s\big) - E^T Z_{22} E + Q_{11} + \frac{\tau^2}{m^2}\Big(Z_{11} - E^T W E\Big)$$

$$\Lambda_s^{12} = \big(-E^T P_{12} + E^T Z_{22} E\big)\mathbb{I}_1 - \big(G_1 A_\tau + A_s^T G_2^T\big)\mathbb{I}_2 + \mathbb{Q}_{12}\mathbb{I}_3$$

$$\Lambda_s^{13} = G_1 - A_s^T G_3^T + E^T P_{11} + SR^T + \frac{\tau^2}{m^2} Z_{12}$$

$$\Lambda_{14} = P_{22} - E^T Z_{12}^T + \frac{\tau}{m} E^T W E$$

$$\Lambda^{22} = -\mathbb{I}_1^T (E^T Z_{22} E)\mathbb{I}_1 - \mathbb{I}_2^T sym(G_2 A_\tau)\mathbb{I}_2 + \mathbb{I}_3^T \mathbb{Q}_{22}\mathbb{I}_3 - \mathbb{Q}$$

$$\Lambda^{23} = \mathbb{I}_2^T\big(G_2 - A_\tau^T G_3^T\big)$$

$$\Lambda_{24} = \mathbb{I}_1^T\big(-P_{22} + E^T Z_{12}^T\big)$$

$$\Lambda_{33} = sym(G_3) + \frac{\tau^2}{m^2} Z_{22} + \frac{\tau^4}{4m^4} W$$

$$\Lambda_{44} = -Z_{11} - E^T W E$$

$$\mathbb{Q}_{12} = [Q_{12} \ \cdots \ Q_{1m}], \quad \mathbb{Q}_{22} = \begin{bmatrix} Q_{22} & \cdots & Q_{2m} \\ * & \ddots & \vdots \\ * & * & Q_{mm} \end{bmatrix}$$

$$\mathbb{I}_1 = \big[I_{n_x} \ 0_{n_x,(m-1)n_x}\big], \quad \mathbb{I}_2 = \big[0_{n_x,(m-1)n_x} \ I_{n_x}\big]$$

$$\mathbb{I}_3 = \big[I_{(m-1)n_x} \ 0_{(m-1)n_x,n_x}\big].$$

$R \in \mathbb{R}^{n_x \times (n_x - n_{x_1})}$ is any matrix with full column and satisfies $E^T R = 0$.

Then, the system is regular, impulse free and asymptotically stable within set $\varepsilon(E^T P_{11} E, \varrho)$ for all initial conditions satisfying

$$\begin{aligned}&(\lambda_{max}(E^T P_{11} E) + \frac{\tau^2}{m^2}\lambda_{max}(P_{22}) + \frac{\tau^2}{m^2}\lambda_{max}(P_{12}^T E E^T P_{12}) \\ &+ 1 + \tau\lambda_{max}(Q) + \frac{1}{2}\frac{\tau^3}{m^3}\lambda_{max}(Z))\|\phi\|_c^2 \\ &+ (\frac{1}{2}\frac{\tau^3}{m^3}\lambda_{max}(Z) \\ &+ \frac{1}{12}\frac{\tau^5}{m^5}\lambda_{max}(E^T W E))\|\dot{\phi}\|_c^2 \le \varrho\end{aligned} \tag{16}$$

Proof : First, we prove that the closed-loop system is regular and impulse free within the set $\varepsilon(E^T P_{11} E, \varrho)$. By using similar arguments an in the proof of theorem 3.1 in Kchaou [2013], we can demonstrate that:

The pair $(E, \sum_{s=1}^{\eta} \delta_s \widehat{A}_s)$ is regular and impulse free.

$$\rho\Big(\big(\sum_{s=1}^{\eta} \delta_s \widehat{A}_s^{22}\big)^{-1}\big(\widehat{A}_\tau^{22}\big)\Big) < 1 \tag{17}$$

Let the singular type Lyapunov Krasovskii functional be:

$$\begin{aligned}\mathbf{V}(x(t)) &= \eta_1^T(t) P \eta_1(t) + \int_{t-\frac{\tau}{m}}^{t} \Gamma^T(s)\mathbb{Q}\Gamma(s) ds \\ &+ \frac{\tau}{m}\int_{-\frac{\tau}{m}}^{0}\int_{t+\theta}^{t} \eta_2^T(s) Z \eta_2(s) ds d\theta \\ &+ \frac{\tau^2}{2m^2}\int_{-\frac{\tau}{m}}^{0}\int_{-\theta}^{0}\int_{t+\alpha}^{t} \dot{x}^T(s) E^T W E \dot{x}(s) ds d\alpha d\theta\end{aligned} \tag{18}$$

where

$$\eta_1(t) = \left[x^T(t) E^T \ \int_{t-\frac{\tau}{m}}^{t} x^T(s) ds\right]^T$$

$$\eta_2(t) = \big[x^T(t) \ (E\dot{x}(t))^T\big]^T$$

$$\Gamma(t) = \Big[\, x^T(t) \ x^T(t-\frac{\tau}{m}) \ x^T(t - 2\frac{\tau}{m}) \ \cdots \ x^T(t-(m-1)\frac{\tau}{m})\Big]^T$$

Defining the extended state vector

$$\xi(t) = \left[x^T(t) \ \Gamma^T(t-\frac{\tau}{m}) \ (E\dot{x}(t))^T \ \int_{t-\frac{\tau}{m}}^{t} x^T(s) ds\right]^T$$

As it is shown in Kchaou [2013], the derivative along the trajectories of (1) satisfies that

$$\dot{\mathbf{V}}(x(t)) \le \sum_{s=1}^{\eta} \delta_s \xi^T(t) \Lambda_s \xi(t) \tag{19}$$

If condition (13) is satisfied, the above inequality (19) is negative definite and we get $\dot{\mathbf{V}}(x(t)) \le 0$.

From $\dot{\mathbf{V}}(x(t)) \le 0$ it follows that $\mathbf{V}(x(t)) \le \mathbf{V}(\phi(t))$ and therefore

$$x(t)^T E^T P_{11} E x(t) \le \mathbf{V}(x(t)) \le \mathbf{V}(\phi(t)) \tag{20}$$

or

$$\begin{aligned}\mathbf{V}(\phi(t)) &= \eta_1^T(\phi(t)) P \eta_1(\phi(t)) + \int_{t-\frac{\tau}{m}}^{t} \Gamma^T(\phi(s))\mathbb{Q}\Gamma(\phi(s)) ds \\ &+ \frac{\tau}{m}\int_{-\frac{\tau}{m}}^{0}\int_{t+\theta}^{t} \eta_2^T(\phi(s)) Z \eta_2(\phi(s)) ds d\theta \\ &+ \frac{\tau^2}{2m^2}\int_{-\frac{\tau}{m}}^{0}\int_{-\theta}^{0}\int_{t+\alpha}^{t} \dot{\phi}^T(s) E^T W E \dot{\phi}(s) ds d\alpha d\theta\end{aligned} \tag{21}$$

Using lemmas 2-3, one can obtain

$$\begin{aligned}&\eta_1^T(\phi(t)) P \eta_1(\phi(t)) = \phi(t)^T E^T P_{11} E \phi(t) \\ &+ \int_{t-\frac{\tau}{m}}^{t} \phi(s)^T ds P_{22} \int_{t-\frac{\tau}{m}}^{t} \phi(s) ds \\ &+ \phi(t)^T E^T P_{12} \int_{t-\frac{\tau}{m}}^{t} \phi(s) ds + \int_{t-\frac{\tau}{m}}^{t} \phi(s)^T ds P_{12}^T E \phi(t) \\ &\le (\lambda_{max}(E^T P_{11} E) \\ &+ \frac{\tau^2}{m^2}\lambda_{max}(P_{22}) + \frac{\tau^2}{m^2}\lambda_{max}(P_{12}^T E E^T P_{12}) + 1)\|\phi\|_c^2\end{aligned} \tag{22}$$

$$\int_{t-\frac{\tau}{m}}^{t} \Gamma^T(\phi(s))\mathbb{Q}\Gamma(\phi(s))ds \leq \frac{\tau}{m}\lambda_{max}(Q)\|\Gamma(\phi)\|_c^2$$
$$\frac{\tau}{m}\int_{-\frac{\tau}{m}}^{0}\int_{t+\theta}^{t} \eta_2^T(\phi)Z\eta_2(\phi)dsd\theta \leq \frac{1}{2}\frac{\tau^3}{m^3}\lambda_{max}(Z)\|\eta_2\|_c^2$$
$$\frac{\tau^2}{2m^2}\int_{-\frac{\tau}{m}}^{0}\int_{-\theta}^{0}\int_{t+\alpha}^{t} \dot{\phi}^T(s)E^TWE\dot{\phi}(s)dsd\alpha d\theta \leq$$
$$\frac{1}{12}\frac{\tau^5}{m^5}\lambda_{max}(E^TWE)\|\dot{\phi}\|_c^2 \tag{23}$$

in which

$$\|\eta_2(\phi)\|_c^2 = \|\phi\|_c^2 + \lambda_{max}(E^TE)\|\dot{\phi}\|_c^2 = \|\phi\|_c^2 + \|\dot{\phi}\|_c^2$$
$$\|\Gamma(\phi)\|_c^2 = m\|\phi\|_c^2 \tag{24}$$

Hence, for all initial conditions satisfying (16), it follows that $x(t)^TE^TP_{11}Ex(t) \leq V(x(t)) \leq V(\phi(t)) \leq \varrho$.

Remark 1. The delay partitioned technique is adopted in this study, where the time delay is divided into m equal segments. It is shown in Feng [2011] that the conditions based on this approach will reduce their conservatism if the number of segments increases. Moreover, a new augmented Lyapunov-Krasovskii functional with triple integral term has been introduced to contribute to conservatism reduction.

Theorem 2. Consider the time delay singular system described in (1). Given an integer $m \geq 1$, and a positive scalars τ and ϱ, if there exist matrices S, G, $P = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0$, $R_{11} > 0$, $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} > 0$, $W > 0$, Q_{kl} $(1 \leq k \leq l \leq m)$, W^F, W^H such that (14) and the following conditions hold:

$$\begin{bmatrix} \Xi_s^{11} & \Xi^{12} & \Xi_s^{13} & \Lambda_{14} \\ * & \Xi^{22} & \Xi^{23} & \Lambda_{24} \\ * & * & \Xi^{33} & P_{12} \\ * & * & * & \Lambda_{44} \end{bmatrix} < 0; \quad s = 1, \ldots, \eta \tag{25}$$

$$\begin{bmatrix} \frac{1}{\varrho}\bar{u}_l^2 & W_l^H\mathbb{I}_5^T \\ * & (\mathbb{I}_4G\mathbb{I}_4^T)(\mathbb{I}_4P_{11}\mathbb{I}_4^T)(\mathbb{I}_4G\mathbb{I}_4^T)^T \end{bmatrix} \geq 0;$$
$$l = 1, \ldots, n_u \tag{26}$$

in which

$$\Xi_s^{11} = sym\big(E^TP_{12} - GA^T - (W^F)^TM_sB^T - (W^H)^TM_s^-B^T\big) - E^TZ_{22}E + Q_{11} + \frac{\tau^2}{m^2}\Big(Z_{11} - E^TWE\Big)$$
$$\Xi_s^{12} = \big(-E^TP_{12} + E^TZ_{22}E\big)\mathbb{I}_1 - \big(GA_\tau^T + AG^T + BM_sW^F + BM_s^-W^H\big)\mathbb{I}_2 + \mathbb{Q}_{12}\mathbb{I}_3$$
$$\Xi_s^{13} = G - AG^T - BM_sW^F - BM_s^-W^H + E^TP_{11} + SR^T + \frac{\tau^2}{m^2}Z_{12}$$
$$\Xi^{22} = -\mathbb{I}_1^T(E^TZ_{22}E)\mathbb{I}_1 - \mathbb{I}_2^T sym(GA_\tau)\mathbb{I}_2 + \mathbb{I}_3^T\mathbb{Q}_{22}\mathbb{I}_3 - \mathbb{Q}$$
$$\Xi^{23} = \mathbb{I}_2^T\big(G - A_\tau G^T\big)$$
$$\Xi_{33} = sym(G) + \frac{\tau^2}{m^2}Z_{22} + \frac{\tau^4}{4m^4}W$$
$$\mathbb{I}_5^T = [I_{n_{x1}} \;\; 0_{n_{x1},n_{x2}}]$$

$R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column and satisfies $E^TR = 0$.

Then, the system is regular, impulse free and asymptotically stable within set $\varepsilon(E^TP_{11}E, \varrho)$ for all initial conditions satisfying (16). In this case, local feedback gains K_j are given by

$$K = W^FG^{-T} \tag{27}$$

Proof :

Note that $det(sE - \sum_{s=1}^{\eta}\delta_sA_s) = det(sE^T - \sum_{s=1}^{\eta}\delta_sA_s^T)$, then the pair $(E, \sum_{s=1}^{\eta}\delta_sA_s)$ is regular and impulse free if and only if $(E^T, \sum_{s=1}^{\eta}\delta_sA_s^T)$ is regular and impulse free.

Moreover, since $\det\big(sE - \sum_{s=1}^{\eta}\delta_sA_s - e^{-\tau s}A_\tau\big) = 0$ and $\det\big(sE^T - \sum_{s=1}^{\eta}\delta_sA_s^T - e^{-\tau s}A_\tau^T\big) = 0$ have the same solution.

Hence, if we consider regularity, free of impulse and stability conditions, system (12) is equivalent to system

$$E\dot{x}(t) = \sum_{s=1}^{\eta}\delta_s[A_s^Tx(t) + A_\tau^Tx(t-\tau)] \tag{28}$$

Thus, we can replace A_s, A_τ with A_s^T, A_τ^T in (13). Furthermore, if we fix $G = G_1 = G_2 = G_3$, $W^F = KG^T$ and $W^H = HG^T$, we obtain (25).

Note that if (25) is satisfied, then

$$\Xi_{33} = sym(G) + \frac{\tau^2}{m^2}Z_{22} + \frac{\tau^4}{4m^4}W < 0$$

which implies that G is non-singular.

Let $H = [H_1 \;\; H_2]$; $H_1 \in \mathbb{R}^{n_u\times n_{x_1}}$, $H_2 \in \mathbb{R}^{n_u\times n_{x_2}}$.

The set inclusion condition $\varepsilon(E^TP_{11}E, \varrho) \subset \mathcal{L}(H, \bar{u})$ implies that $H_2 = 0$. Consequently, Constraint (15) is equivalent to

$$\varepsilon(\mathbb{I}_4P_{11}\mathbb{I}_4^T, \varrho) \subset \mathcal{L}(H_1, \bar{u}) \tag{29}$$

where $\mathbb{I}_4 = [I_{n_{x1}} \;\; 0_{n_{x1},n_{x2}}]$.

Denote H_{1l} the lth row of H_1.

On the one hand, $\forall x(t) \in \varepsilon(\mathbb{I}_4P_{11}\mathbb{I}_4^T, \varrho)$, the following inequality

$$2\bar{u}_l \geq \bar{u}_l(1 + \frac{1}{\rho}x(t)^T(\mathbb{I}_4P_{11}\mathbb{I}_4^T)x(t)) \geq 2|H_{1l}x(t)| \tag{30}$$

implies that

$$|H_{1l}x(t)| \leq \bar{u}_l \tag{31}$$

On the other hand, inequality (30) can be rewritten as follows:

$$\begin{bmatrix} 1 & \pm x(t)^T \end{bmatrix} \begin{bmatrix} \bar{u}_l & H_{1l} \\ * & \frac{1}{\rho}\bar{u}_l(\mathbb{I}_4 P_{11}\mathbb{I}_4^T) \end{bmatrix} \begin{bmatrix} 1 \\ \pm x(t) \end{bmatrix} \geq 0 \quad (32)$$

which implies that

$$\begin{bmatrix} \bar{u}_l & H_{1l} \\ * & \frac{1}{\rho}\bar{u}_l(\mathbb{I}_4 P_{11}\mathbb{I}_4^T) \end{bmatrix} \geq 0 \quad (33)$$

Pre- and post-multiplying both sides of (33) by $\begin{bmatrix} \sqrt{\frac{1}{\rho}\bar{u}_l} & 0 \\ 0 & \sqrt{\rho\frac{1}{\bar{u}_l}\mathbb{I}_4 G\mathbb{I}_4^T} \end{bmatrix}$ and its transpose, we obtain (26).
Hence, inequality (26) guarantees (15).
This completes the proof.

Note that (25) and (14) are strict LMI with respect to variables S, G, $P = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0$, $R_{11} > 0$, $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} > 0$, $W > 0$, Q_{kl} $(1 \leq k \leq l \leq m)$, W^F and W^H while, (26) are not. We need to use the standard cone complementarity numerical approach El Ghaoui [1997] to solve the matrix inequalities in theorem 2 .
It is clear that if

$$\begin{bmatrix} \frac{1}{\varrho} & W_l^H \mathbb{I}_5 \\ * & \begin{bmatrix} (\mathbb{I}_4 G\mathbb{I}_4^T)^T + (\mathbb{I}_4 G\mathbb{I}_4^T) \\ -(\mathbb{I}_4 P_{11}\mathbb{I}_4^T)^{-1} \end{bmatrix} \end{bmatrix} \geq 0; \quad l = 1, \ldots, n_u \quad (34)$$

Then the inequalities (26) of theorem 2 are satisfied.

By introducing new variable X_{11}, the original conditions (26) can be represented as

$$\begin{bmatrix} \frac{1}{\varrho} & W_l^H \mathbb{I}_5 \\ * & (\mathbb{I}_4 G\mathbb{I}_4^T)^T + (\mathbb{I}_4 G\mathbb{I}_4^T) - X_{11} \end{bmatrix} \geq 0; l = 1, \ldots, n_u \quad (35)$$

$$X_{11} = (\mathbb{I}_4 P_{11}\mathbb{I}_4^T)^{-1} \quad (36)$$

Now, using a cone complementarity problem, we suggest the following minimization problem involving LMI conditions instead of the original non-convex feasibility problem of Theorem 2.

$\underset{S_1}{min}\ Tr((\mathbb{I}_4 P_{11}\mathbb{I}_4^T)X_{11})$ subject to (14) -(25)-(35), and

$$\begin{bmatrix} X_{11} & I \\ I & \mathbb{I}_4 P_{11}\mathbb{I}_4^T \end{bmatrix} \geq 0 \quad (37)$$

where

$$S_1 = \{S, G, P = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0, R_{11} > 0,$$
$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} > 0, W > 0, Q_{kl}, W^F, W^H\}$$

4. NUMERICAL EXAMPLE

Consider the following time delay singular linear system with actuator saturation:

$$E\dot{x}(t) = Ax(t) + A_\tau x(t - \tau(t)) + Bsat(u(t), \bar{u}) \quad (38)$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -0.2 & -0.8 \end{bmatrix}$$
$$A_\tau = \begin{bmatrix} 0 & 0 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For the simulation purpose, we set $\varrho = 65$, $\tau = 0.25$, $m = 3$ and $\bar{u} = 1$. Using LMI TOOLBOX of MATLAB Gahinet [1995], yields the following feasible solution of the optimization problem (37).

$$H = [\,-0.1656 \;\; -0.0587 \;\; 0\,]$$

$$K = [\,-0.7737 \;\; -0.0732 \;\; 0.1091\,]$$

The inclusion of the ellipsoid inside the polyhedral set $(\varepsilon(E^T P_{11}E, \varrho) \subset \mathcal{L}(H, \bar{u})$ are shown in Fig. 1.

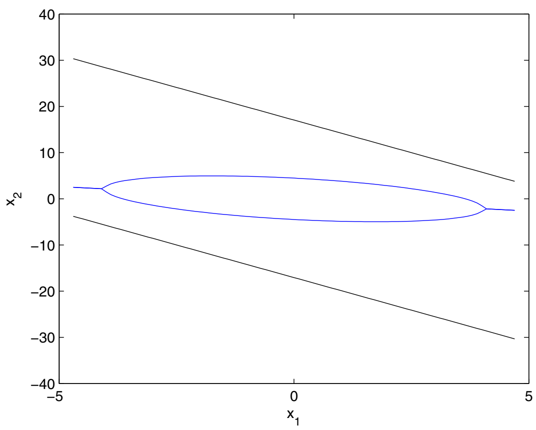

Fig. 1. The inclusion of the ellipsoid inside the polyhedral set $(\varepsilon(E^T P_{11}E, \varrho) \subset \mathcal{L}(H, \bar{u}))$

From (16), the closed loop system is regular, impulse free and stable for all initial conditions satisfying

$$28.8524\|\phi\|_c^2 + 0.0542\|\dot{\phi}\|_c^2 \leq \varrho \quad (39)$$

Now, we consider initial conditions $\phi(t) = [0.7, -1.5, -0.6]$ $\forall t \in [-0.25, 0]$. Fig. 2 shows Response of the state $x(t)$ under saturated control $u(t)$.

5. CONCLUSION

In this paper, we have studied the stabilization problem for time-delay singular linear systems with actuator saturation. An improved version of delay-dependent method has been developed for guaranteeing this class of systems to be admissible under actuator saturation. This method is based on augmented Lyapunov Krasovskii functional and by employing the delay partitioning technique. The delay-dependent result is in terms of matrix inequalities and an

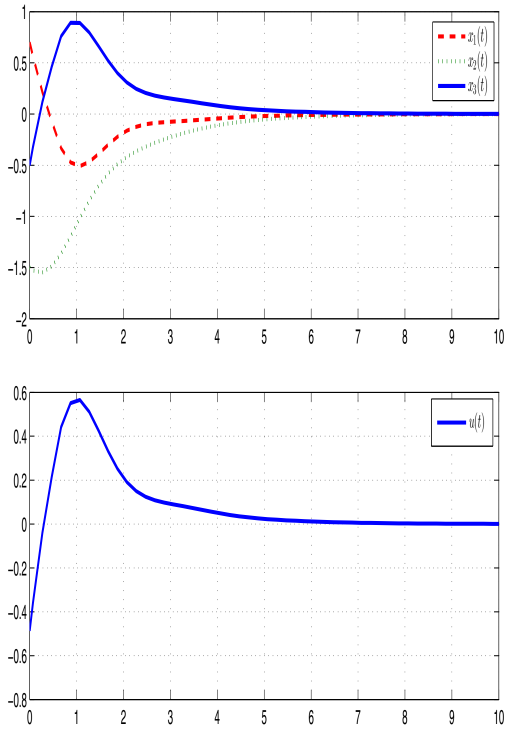

Fig. 2. Response of the state $x(t)$ under saturated control $u(t)$

effective minimization problem involving LMI conditions, adopting the idea of a so-called cone complementarity problem, is developed to solve these matrix inequalities. A numerical example has been provided to illustrate the effectiveness of the proposed design approach.

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