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Analytical Solutions to a Class of Feedback Systems on SO(n) ⋆

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Abstract: This paper provides analytical solutions to the closed loop system kinematics for a class of almost globally asymptotically stable feedback laws on SO(n). The resulting closed loop kinematics are solved for the respective matrices as functions of time, the initial conditions and the gain parameters of the control laws. The analytical solutions provide insight into the transient dynamics of the system and can be used to prove almost global attractiveness of the identity matrix. We consider an application of these results towards model predictive control where the transient phase of the system is utilized to attempt to complete a task of secondary importance by choosing the gain parameters as functions of time and the initial conditions.

Keywords: Attitude control; nonlinear systems; Lie groups.

1. INTRODUCTION

The stabilization of the attitude of a rigid-body is a well-known problem in the area of nonlinear control. Its applications range from attitude control of satellites to robotic manipulation. The nonlinear state equations and the topology of the underlying state space SO(3) make this problem challenging. The choice of parameterization used to represent SO(3) has important implications for control performance [N.A. Chaturvedi et al., 2011, S.P. Bhat and D.S. Bernstein, 2000, C.G. Mayhew et al., 2011a]. It is for example known that global stability on SO(3) cannot be achieved by means of a continuous, time-invariant feedback [S.P. Bhat and D.S. Bernstein, 2000]. The literature does however provide results such as almost global asymptotical stability through continuous time-invariant feedback [N.A. Chaturvedi et al., 2011, Sanyal et al., 2009], almost semi-global stability [Lee, 2012], or global stability by means of a hybrid control approach [C.G. Mayhew et al., 2011b].

The solutions of a closed-loop system gives a detailed picture of its transients and asymptotical behavior and may hence be of use in control applications. Let us divide the literature on analytical solutions to attitude dynamics into two categories. Firstly, in a number of works the solutions are obtained during the control design process, *e.g.* using exact linearization [Dwyer III, 1984] or optimal control design principles such as the one by Pontryagin [Spindler, 1998]. Secondly, there are works that focus on solving the equations defining rigid-body dynamics under a set of specific assumptions [Elipe and Lanchares, 2008, M.A. Ayoubi and J.M. Longuski, 2009, A.V. Doroshin, 2012]. Our paper falls into this second category.

There is a literature on the kinematics and dynamics of n-dimensional rigid-bodies. Part of this work is mainly of theoretical concern, but it does also cover the special cases of $n \in \{2, 3\}$ which are of interest from an applied point of view. This literature includes works on attitude stabilization [D.H.S. Maithripala et al., 2006], attitude synchronization [Lageman et al., 2009] and generalized equations of motion [J.E. Hurtado and A.J. Sinclair, 2004] on SO(n). It also includes our previous paper [Markdahl et al., 2013], which we shall comment on shortly.

The main contribution of this paper is to provide analytical solutions to differential equations representing closed feedback loops on SO(n). Recent work on this problem include Markdahl et al. [2012, 2013]. Other works such as those previously referenced by Elipe and Lanchares [2008], M.A. Ayoubi and J.M. Longuski [2009], A.V. Doroshin [2012] are related in spirit but do address somewhat different problems. The work Markdahl et al. [2012] considers the solutions to the closed-loop kinematics of a feedback law on SO(3). An application towards model predictive control (MPC) is proposed but left unexplored. The more general problem of solving two differential equations on SO(n) is considered in Markdahl et al. [2013]. An application towards the problem of continuous time actuation under discrete-time sensing is considered. This paper generalizes the results of Markdahl et al. [2013] to a much wider class of feedback laws and also explores the applications towards MPC proposed in Markdahl et al. [2012].

NOMENCLATURE

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$. The spectrum of $\mathbf{A}$ is written as $\sigma(\mathbf{A})$. The commutator of $\mathbf{A}$ and $\mathbf{B}$ is defined by $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$. The set of invertible matrices is denoted by GL(n). The special orthogonal group is denoted by $\mathrm{SO}(n) = \{\mathbf{R} \in \mathrm{GL}(n) \,|\, \mathbf{R}^{-1} = \mathbf{R}^\top, \det \mathbf{R} = 1\}$ (in this paper we write $\mathbf{R}^{-1}$ instead of $\mathbf{R}^\top$). The Lie algebra of SO(n) is denoted by $\mathfrak{so}(n) = \{\mathbf{S} \in \mathbb{R}^{n\times n} \,|\, \mathbf{S}^\top = -\mathbf{S}\}$. In this paper, we typically use $\mathbf{S}$ to denote the matrix $\mathrm{Log}\,\mathbf{R} \in \mathfrak{so}(n)$. We also use the notation

$$\mathrm{N}(n) = \{\mathbf{R} \in \mathrm{SO}(n) \,|\, -1 \in \sigma(\mathbf{R})\}. \quad (1)$$

Observe that N(n) is a null set, *i.e.* a set of measure zero in SO(n). It can be shown that $\mathrm{N}(3) = \{\mathbf{R} \in \mathrm{SO}(3) \,|\, \mathbf{R}^{-1} = \mathbf{R}\}\backslash\{\mathbf{I}\}$, but such a relation does not hold in higher

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dimensions. Finally, we denote the set of positive definite matrices by $\mathrm{S}^{++}(n) = \{\mathbf{P} \in \mathrm{GL}(n) \,|\, \mathbf{P}^\top = \mathbf{P},\, \sigma(\mathbf{P}) \subset \mathbb{R}^{++}\}$.

2. PROBLEM STATEMENT

Consider the problems of kinematic level attitude tracking for a fully actuated rigid body. The attitude of a rigid body is represented by means of the relation between two orthonormal coordinate frames, one inertial fixed and one body fixed. The problem of stabilization is that of designing a control law which makes the closed loop system governing the body fixed frame converge to the desired attitude which we without any loss of generality take to be orientation of the inertial fixed frame. Let $\mathbf{R} \in \mathrm{SO}(n)$ be a matrix that transforms the inertial fixed frame into the body fixed frame. This matrix can be interpreted as an attitude error to be used in feedback. The problem of tracking a desired attitude curve in $\mathrm{SO}(n)$ subject to kinematic level actuation can be shown to be equivalent to the stabilization problem if the velocity along the desired curve at the current time is known.

Let $\mathbf{R} \in \mathrm{SO}(n)$ be a matrix that represents the attitude error. The inertial fixed frame kinematics of $\mathbf{R}$ are

$$\dot{\mathbf{R}} = \mathbf{\Omega}\mathbf{R}, \tag{2}$$

where $\mathbf{\Omega} \in \mathfrak{so}(n)$. The attitude stabilization problem concerns the design of $\mathbf{\Omega}$. In this paper we consider the problem of finding the analytical solutions to (2) for a number of given feedback laws $\mathbf{\Omega}$. We also derive stability properties of the closed loop system by means of the analytical solutions and consider the potential applicative use of such solutions in model predictive control problems.

3. MATHEMATICAL PRELIMINARIES

By $\mathrm{Log} : \mathrm{S}(n) \to \mathbb{R}^{n\times n}$ of a matrix $\mathbf{A} \in \mathrm{S}(n) \subset \mathrm{GL}(n)$ we refer to the principal matrix logarithm which has the following property: if $\sigma(\mathrm{Log}\,\mathbf{A}) \cup \mathbb{R}^- = \emptyset$, then $\mathfrak{Im}\,\sigma(\mathrm{Log}\,\mathbf{A}) \subset \{z \in i\mathbb{R} \,|\, |z| < \pi\}$. In particular, this holds for $\mathbf{A} \in \mathrm{SO}(n)\backslash\mathrm{N}(n)$ which are mapped to $\mathfrak{so}(n)$. Note that we do not give the set $\mathrm{S}(n)$ explicitly since its characterization is somewhat involved. Also note that the logarithm of a normal matrix $\mathbf{A} = \mathbf{U}^*\mathbf{\Lambda}\mathbf{U}$, where $\mathbf{U} \in \mathrm{U}(n)$ and $\cdot^*$ denote complex conjugation, can be calculated as $\mathrm{Log}\,\mathbf{A} = \mathbf{U}^*\,\mathrm{Log}(\mathbf{\Lambda})\mathbf{U}$.

By the kth root of a normal matrix $\mathbf{A} = \mathbf{U}^*\mathbf{\Lambda}\mathbf{U}$ we refer to its principal root, the normal matrix $\mathbf{A}^{\frac{1}{k}} = \mathbf{U}^*\mathbf{\Lambda}^{\frac{1}{k}}\mathbf{U}$. Consider $\mathbf{R} \in \mathrm{SO}(n)$. The principal root satisfies $-1 \notin \sigma(\mathbf{R}^{\frac{1}{k}})$ if $-1 \notin \sigma(\mathbf{R})$ since $\sigma(\mathbf{R}) = \{\lambda^{\frac{1}{k}} \,|\, \lambda \in \sigma(\mathbf{R})\}$. The kth root of a matrix $\mathbf{A}$ generated by an exponential map, *i.e.* a matrix such that $\mathbf{A} = \exp\mathbf{B}$ for some $\mathbf{B} \in \mathbb{R}^{n\times n}$, can be calculated as $\mathbf{A}^{\frac{1}{k}} = \exp(\frac{1}{k}\mathbf{B})$. This shows that $\mathbf{A}^{\frac{1}{k}}$ is also generated by an exponential map; in particular it holds that if $\mathbf{A} \in \mathrm{SO}(n)$, then $\mathbf{A}^{\frac{1}{k}} \in \mathrm{SO}(n)$.

4. CONTROL LAWS

In this section we outline a number of well-known attitude control laws for which we shall obtain the trajectories of the closed-loop systems in §5.

Algorithm 1. (Positive definite gain matrix). The control law and the resulting closed loop system are respectively given by

$$\mathbf{\Omega}_1 = \mathbf{P}\mathbf{R}^{-1} - \mathbf{R}\mathbf{P}, \quad \dot{\mathbf{R}} = \mathbf{P} - \mathbf{R}\mathbf{P}\mathbf{R}, \tag{3}$$

where $\mathbf{P} \in \mathrm{S}^{++}(n)$.

This algorithm is well-known. The analytical solutions to (3) in the special case of $\mathbf{P} = \mathbf{I}$ is studied in Markdahl et al. [2013].

Algorithm 2. Let $\mathbf{F} : \mathfrak{so}(n) \to \mathfrak{so}(n)$ satisfy $[\mathbf{F}(\mathbf{S}), \mathbf{S}] = \mathbf{0}$ and be such that the origin is a globally asymptotically stable equilibrium of

$$\dot{\mathbf{S}} = \mathbf{F}(\mathbf{S}), \tag{4}$$

where (4) also has a known, unique solution. The input matrix and the resulting closed loop system are respectively given by

$$\mathbf{\Omega}_2 = \mathbf{F}(\mathrm{Log}\,\mathbf{R}), \quad \dot{\mathbf{R}} = \mathbf{F}(\mathrm{Log}\,\mathbf{R})\mathbf{R}, \tag{5}$$

where $\mathrm{Log} : \mathrm{SO}(n) \to \mathfrak{so}(n)$ denotes the matrix logarithm.

Example 3. (Matrix logarithm). An important special case of Algorithm 2 is the geodesic feedback $\mathbf{\Omega}_3 = \mathbf{F}(\mathbf{S}) = -\mathbf{S}$ [Bullo and Murray, 1995].

Example 4. (Matrix root). Algorithm 1 with $\mathbf{P} = \mathbf{I}$ results in $[\mathbf{\Omega}_1, \log\mathbf{R}] = \mathbf{0}$, even when we replace $\mathbf{R}$ by its kth root $\mathbf{R}^{\frac{1}{k}}$ for $k \in \mathbb{N}$ as defined in §3. The input matrix and the resulting closed loop system for this control law are given by

$$\mathbf{\Omega}_4 = k(\mathbf{R}^{-\frac{1}{k}} - \mathbf{R}^{\frac{1}{k}}), \quad \dot{\mathbf{R}} = k(\mathbf{R}^{1-\frac{1}{k}} - \mathbf{R}^{1+\frac{1}{k}}), \tag{6}$$

respectively. The proportional gain factor k is used to scale the time dependence of $\mathbf{R}$.

The feedback of Example 4 is related to Algorithm 1 as $\mathbf{\Omega}_4 = \mathbf{\Omega}_1$ when $k = 1$ and $\mathbf{P} = \mathbf{I}$. It is related to the feedback of Example 3 as $\lim_{k\to\infty}\mathbf{\Omega}_4 = 2\mathbf{\Omega}_3$.

Example 5. (Cayley transform). Another special case of Algorithm 2 is the Cayley transform and the higher order Cayley transforms. The input matrix is given by

$$\mathbf{\Omega}_5 = k(\mathbf{I} - \mathbf{R}^{\frac{1}{k}})(\mathbf{I} + \mathbf{R}^{\frac{1}{k}})^{-1}, \tag{7}$$

where a scalar gain factor have been introduced. The closed loop system is

$$\dot{\mathbf{R}} = k(\mathbf{I} - \mathbf{R}^{\frac{1}{k}})(\mathbf{I} + \mathbf{R}^{\frac{1}{k}})^{-1}\mathbf{R}. \tag{8}$$

The control law of Example 5 is related to the Rodriguez parameters and the modified Rodriguez parameters in the cases of $k = 1$ and $k = 2$ respectively (and to the parametrizations obtained from higher order Cayley transforms in the case of general k [Tsiotras et al., 1997]). Note that $\lim_{k\to\infty}\mathbf{\Omega}_5 = \frac{1}{2}\mathbf{\Omega}_3$. It follows that $\lim_{k\to\infty}\mathbf{\Omega}_5 = \frac{1}{4}\lim_{k\to\infty}\mathbf{\Omega}_4$.

Algorithm 1 and 2 differ in several respects. Algorithm 1 have a constant positive definite gain matrix that can be tuned for desired performance. It provides a continuous feedback but has a low input norm for rotations that are far from the identity, the disadvantage of which is slow convergence in the case of large errors [Lee, 2012]. Example 3 provides a geodesic control law. The feedback laws of Example 3 and 4 have input norms that are increasing functions of $\|\mathbf{S}\|_2$. This property can *e.g.* be useful in attitude control of satellites that are required to make large angle maneuvers [Lee, 2012]. The input norm is however not defined for $\mathbf{R} \in \mathrm{N}(n)$. The feedback law of Example 5 has the property that the input norm grows unbounded as $\mathbf{R}$ approaches the set of symmetric matrices in the case of $k = 1$. Fig. 1 illustrates some of these considerations for $\mathbf{R} \in \mathrm{SO}(3)$.

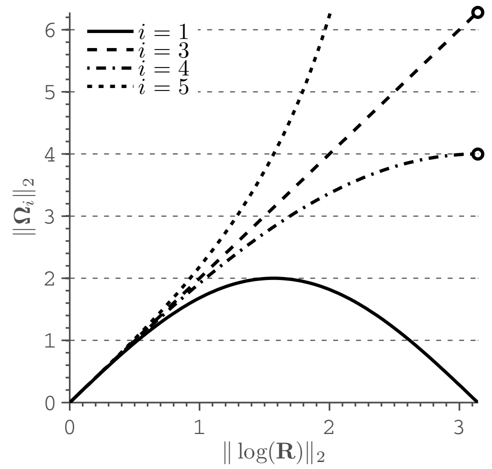

Fig. 1. The norm of the input signal $\|\mathbf{\Omega}_i\|_2$ as a function of the geodesic distance from $\mathbf{R}$ to $\mathbf{I}$ on SO(3). The matrix $\mathbf{P} = \mathbf{I}$ in $\mathbf{\Omega}_1$, $k = 2$ in $\mathbf{\Omega}_4$, and $k = 1$ in $\mathbf{\Omega}_5$. The curves have been scaled to have equal tangent at the origin.

5. MAIN RESULTS

It is possible to establish global existence and uniqueness of the exact solutions, see Lemma 14 in Appendix B. This allows us to draw conclusions regarding control performance from the analytical solutions. We prove that the region of attraction of $\mathbf{I}$ for the closed loop systems generated by Algorithm 1–2 is $\mathrm{SO}(n)\backslash\mathrm{N}(n)$, *i.e.* that this equilibrium is almost globally attractive. We also return to Example 3–5. Let $\mathbf{R}_0$ denote the state value at the initial time. In the remainder of this section we make the assumption that $\mathbf{R}_0 \notin \mathrm{N}(n)$.

Theorem 6. The trajectories of the closed-loop system (3) resulting from Algorithm 1 are given by

$$\mathbf{R}(t) = (\sinh(\mathbf{P}t) + \cosh(\mathbf{P}t)\mathbf{R}_0) \cdot (\cosh(\mathbf{P}t) + \sinh(\mathbf{P}t)\,\mathbf{R}_0)^{-1}. \tag{9}$$

The equilibrium $\mathbf{R} = \mathbf{I}$ is almost globally attractive and locally exponentially stable.

Proof: Equation (3) is a matrix valued differential Ricatti equation that can be solved using the adjoint equations technique. Introduce two matrices $\mathbf{X}, \mathbf{Y} \in \mathrm{GL}(n)$ that satisfy

$$\dot{\mathbf{X}} = \mathbf{P}\mathbf{Y}, \quad \dot{\mathbf{Y}} = \mathbf{P}\mathbf{X}. \tag{10}$$

with initial conditions $\mathbf{X}(0) = \mathbf{I}$, $\mathbf{Y}(0) = \mathbf{R}_0$. Note that

$$\mathbf{R} = \mathbf{Y}\mathbf{X}^{-1} \tag{11}$$

since $\mathbf{R}(0) = \mathbf{Y}(0)\mathbf{X}^{-1}(0) = \mathbf{R}_0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Y}\mathbf{X}^{-1} = \dot{\mathbf{Y}}\mathbf{X}^{-1} - \mathbf{Y}\mathbf{X}^{-1}\dot{\mathbf{X}}\mathbf{X}^{-1} = \mathbf{P} - \mathbf{R}\mathbf{P}\mathbf{R} = \dot{\mathbf{R}}.$$

Equation (10) is linear and has the transition matrix

$$\exp\left(\begin{bmatrix} \mathbf{0} & \mathbf{P} \\ \mathbf{P} & \mathbf{0} \end{bmatrix} t\right) = \begin{bmatrix} \cosh(\mathbf{P}t) & \sinh(\mathbf{P}t) \\ \sinh(\mathbf{P}t) & \cosh(\mathbf{P}t) \end{bmatrix}.$$

By plugging the expressions for $\mathbf{X}$ and $\mathbf{Y}$ into (11) we find $\mathbf{R}$.

We assume that $\mathbf{R}_0 \notin \mathrm{N}(n)$. The induced Euclidean norm is sub-multiplicative whereby

$$\|\mathbf{Y}\mathbf{X}^{-1} - \mathbf{I}\|_2 = \|(\mathbf{Y} - \mathbf{X})\mathbf{X}^{-1}\|_2 \leq \|\mathbf{Y} - \mathbf{X}\|_2 \cdot \|\mathbf{X}^{-1}\|_2.$$

That $\lim_{t\to\infty} \|\mathbf{Y}\mathbf{X}^{-1} - \mathbf{I}\|_2 = 0$ hence follows from

$$\lim_{t\to\infty} \mathbf{Y} - \mathbf{X} = \lim_{t\to\infty} \exp(-\mathbf{P}t)(\mathbf{I} - \mathbf{R}_0) = \mathbf{0},$$

$$\lim_{t\to\infty} \mathbf{X}^{-1} = \lim_{t\to\infty} (\mathbf{I} + \tanh(\mathbf{P}t)\mathbf{R}_0)^{-1} \cosh^{-1}(\mathbf{P}t) = \mathbf{0}.$$

The last limit is proved in Appendix A (it also requires the assumption of $\mathbf{R}_0 \notin \mathrm{N}(n)$). Hence we have shown that $\mathbf{I}$ attracts all system trajectories such that $\mathbf{R}_0 \in \mathrm{SO}(n)\backslash\mathrm{N}(n)$.

It remains to show that $\mathbf{I}$ is a locally exponentially stable equilibrium of $\mathbf{R}$. We use the first method of Lyapunov. Set $\mathbf{E} = \mathbf{R} - \mathbf{I}$. Let $\mathbf{Z}$ be the matrix corresponding to the linearization of $\mathbf{E}$. Then

$$\dot{\mathbf{Z}} = -\mathbf{P}\mathbf{Z} - \mathbf{Z}\mathbf{P}, \tag{12}$$

with $\mathbf{Z}(0) = \mathbf{Z}_0 = \mathbf{R}_0 - \mathbf{I}$. The system (12) is exponentially stable due to $\mathbf{P} \in \mathrm{S}^{++}(n)$. ■

Consider Algorithm 2. Let $\mathbf{\Phi}(\mathbf{S}_0, t)$ denote the flow on $\mathfrak{so}(n)$, *i.e.* $\mathbf{\Phi}(\mathbf{S}_0, t) = \mathbf{S}(t)$, where $\mathbf{S}(t)$ is the value at time t to the unique solution of (4) with initial value $\mathbf{S}_0 \in \mathfrak{so}(n)$.

Theorem 7. The trajectories of (5) are given by

$$\mathbf{R}(t) = \exp(\mathbf{\Phi}(\operatorname{Log}\mathbf{R}_0, t)). \tag{13}$$

Moreover, the equilibrium $\mathbf{R} = \mathbf{I}$ is almost globally asymptotically stable.

Proof. The proof is mainly by verification. Note that $\mathbf{R}(0) = \mathbf{R}_0$. Since $[\mathbf{\Omega}_2, \mathbf{S}] = \mathbf{0}$, it follows that $[\dot{\mathbf{S}}, \mathbf{S}] = \mathbf{0}$, see Lemma 15 in Appendix B. Hence

$$\mathbf{\Omega}_2\mathbf{R} = \dot{\mathbf{R}} = \frac{\mathrm{d}}{\mathrm{d}t}\exp(\mathbf{S}) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\frac{1}{i!}\mathbf{S}^i = \dot{\mathbf{S}}\mathbf{R}.$$

By multiplying the above identity by $\mathbf{R}^{-1}$ from the right, we are left with

$$\dot{\mathbf{S}} = \mathbf{\Omega}_2 = \mathbf{F}(\mathbf{S}). \tag{14}$$

The solution to (14) is given by $\mathbf{S}(t) = \mathbf{\Phi}(\mathbf{S}_0, t)$ as defined earlier. The expression (13) for $\mathbf{R}(t)$ is obtained from the exponential mapping and

$$\begin{aligned}\dot{\mathbf{R}} &= \dot{\mathbf{\Phi}}(\operatorname{Log}\mathbf{R}_0, t)\exp(\mathbf{\Phi}(\operatorname{Log}\mathbf{R}_0, t)) \\ &= \dot{\mathbf{S}}\mathbf{R} = \mathbf{F}(\mathbf{S})\mathbf{R}.\end{aligned}$$

Since the zero matrix is a globally asymptotically stable equilibrium of (4), we find that

$$\lim_{t\to\infty} \mathbf{\Phi}(\operatorname{Log}\mathbf{R}_0, t) = \mathbf{0}, \quad \lim_{t\to\infty} \mathbf{R}(t) = \mathbf{I},$$

i.e. the identity matrix is almost globally attractive. The almost global region of attraction follows from $\operatorname{Log}\mathbf{R}_0$ being defined for $\mathbf{R}_0 \in \mathrm{SO}(n)/\mathrm{N}(n)$. The identity matrix being a stable equilibrium of (5), follows from the stability of (4), and the continuity of the exponential mapping. ■

Example 3. (Cont'd). Consider the case of $\mathbf{F}(\mathbf{S}) = -\mathbf{S}$. Note that $\mathbf{F}(\mathbf{S}) \in \mathfrak{so}(n)$ and $[\mathbf{F}(\mathbf{S}), \mathbf{S}] = \mathbf{0}$. Moreover, the system

$$\dot{\mathbf{S}} = -\mathbf{S}$$

is globally exponentially stable. The conditions of Algorithm 2 are hence fulfilled and we obtain the geodesic control law $\boldsymbol{\Omega} = -\operatorname{Log}(\mathbf{R})$, with solution given by

$$\mathbf{R}(t) = \exp(\mathrm{e}^{-t}\operatorname{Log}(\mathbf{R}_0)).$$

Example 4. (Cont'd). The trajectories of (6), the closed loop system in Example 4, are given by

$$\mathbf{R}(t) = (\tanh(t)\,\mathbf{I} + \mathbf{R}_0^{\frac{1}{k}})^k(\mathbf{I} + \tanh(t)\,\mathbf{R}_0^{\frac{1}{k}})^{-k}, \qquad (15)$$

where the unique unitary kth root of $\mathbf{R}_0$ is calculated using the spectral decomposition. The equilibrium $\mathbf{R} = \mathbf{I}$ is almost globally attractive and locally exponentially stable.

To prove this, introduce the variable $\mathbf{X} = \mathbf{R}^{\frac{1}{k}} \in \mathrm{SO}(n)$. Then

$$\begin{aligned}\dot{\mathbf{X}} &= \frac{1}{k}\dot{\mathbf{R}}\mathbf{R}^{\frac{1}{k}-1} = \frac{1}{k}k(\mathbf{R}^{1-\frac{1}{k}} - \mathbf{R}^{1+\frac{1}{k}})\mathbf{R}^{\frac{1}{k}-1} \\ &= \mathbf{I} - \mathbf{R}^{\frac{2}{k}} = \mathbf{I} - \mathbf{X}^2, \qquad (16)\end{aligned}$$

which also results from setting $\mathbf{P} = \mathbf{I}$ in Algorithm 1. By reversing the change of variables in the solution for $\mathbf{X}$ given by Theorem 6 we obtain (15).

The attractiveness and stability properties of $\mathbf{I}$ as an equilibrium of $\mathbf{R}$ also follows from those of $\mathbf{X}$ obtained from Theorem 6.

Example 5. (Cont'd). The trajectories of system (7) generated by Algorithm 5 are given by

$$\mathbf{R}(t) = \exp(2k\operatorname{atanh}\mathbf{C}(t)),$$

where

$$\mathbf{C}(t) = \sinh\left(\frac{1}{2k}\mathbf{S}_0\right)\left(\sinh^2\left(\frac{1}{2k}\mathbf{S}_0\right) + \mathrm{e}^t\,\mathbf{I}\right)^{-\frac{1}{2}},$$

and $\mathbf{S}_0 = \operatorname{Log}\mathbf{R}_0$. Moreover, the equilibrium $\mathbf{R} = \mathbf{I}$ is asymptotically stable.

Let us prove this. That $\mathbf{C}(t)$ and $\operatorname{atanh}\mathbf{C}(t)$ are well-defined follows from Lemma 17 in Appendix B. Change variables from $\mathbf{R}$ to $\mathbf{X} = \frac{1}{2k}\operatorname{Log}\mathbf{R}$ where the scaling is just a matter of notational convenience. Note that

$$\boldsymbol{\Omega}_5 = -k\tanh\mathbf{X},$$

whereby $[\mathbf{X}, \boldsymbol{\Omega}_5] = \mathbf{0}$ and $\dot{\mathbf{X}} = \frac{1}{2k}\boldsymbol{\Omega}_5$. It will hence suffice to study $\mathbf{X}$.

As an intermediate step, consider the evolution of

$$\mathbf{C}(t) = \sinh(\mathbf{X}_0)\left(\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I}\right)^{-\frac{1}{2}} \qquad (17)$$

given by

$$\begin{aligned}\dot{\mathbf{C}}(t) &= -\frac{1}{2}\sinh(\mathbf{X}_0)(\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I})^{-\frac{3}{2}}\,\mathrm{e}^t \\ &= -\frac{1}{2}\mathbf{C}(t)(\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I})^{-1}\,\mathrm{e}^t \\ &= -\frac{1}{2}\mathbf{C}(t)(\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I})^{-1}\cdot \\ &\qquad (\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I} - \sinh^2\mathbf{X}_0) \\ &= -\frac{1}{2}\mathbf{C}(t)(\mathbf{I} - \mathbf{C}(t)^2).\end{aligned}$$

It remains to verify that

$$\mathbf{X}(t) = \operatorname{atanh}\mathbf{C}(t)$$

solves $\dot{\mathbf{X}} = -\frac{1}{2}\tanh\mathbf{X}$. Note that

$$\mathbf{X}(0) = \operatorname{atanh}(\tanh(\mathbf{X}_0)) = \mathbf{X}_0.$$

What is more

$$\begin{aligned}\dot{\mathbf{X}}(t) &= (\mathbf{I} - \mathbf{C}^2(t))^{-1}\dot{\mathbf{C}}(t) \\ &= -\frac{1}{2}\mathbf{C}(t) = -\frac{1}{2}\tanh\mathbf{X}(t),\end{aligned}$$

where we used the previous result concerning $\dot{\mathbf{C}}(t)$. To prove that $\mathbf{I}$ is almost globally attractive, note that

$$\lim_{t\to\infty}\mathbf{C}(t) = \sinh(\mathbf{X}_0)\lim_{t\to\infty}(\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I})^{-\frac{1}{2}} = \mathbf{0},$$

which follows from

$$\sigma(\mathbf{C}(t)) = \{i\sin(\lambda)(\mathrm{e}^t - \sin^2\lambda)^{-\frac{1}{2}} \in i\mathbb{R} \mid i\lambda \in \sigma(\mathbf{X}_0)\}.$$

Hence $\lim_{t\to\infty}\mathbf{R}(t) = \mathbf{I}$ for all $\mathbf{R}_0 \in \mathrm{SO}(n)/\mathrm{N}(n)$.

To prove stability we use the first method of Lyapunov. Let $\mathbf{Y} \in \mathbb{R}^{n\times n}$ be a variable corresponding to the linearization of $\dot{\mathbf{X}} = -\frac{1}{2}\tanh\mathbf{X}$ around $\mathbf{X} = \mathbf{0}$. Then

$$\dot{\mathbf{Y}} = -\frac{1}{2}\mathbf{Y}$$

is an exponentially stable system. The stability of $\mathbf{I}$ as an equilibrium of $\mathbf{R}$ follows from the continuity of the exponential map.

6. APPLICATIONS

Analytical solutions to the closed loop attitude kinematics can be used as an alternative to the zero-order hold approach to implementing a continuous control law using feedback based on an output of the full state that is temporarily unavailable [Markdahl et al., 2013]. They can also be used to pose a model predictive control (MPC) problem in terms of the feedback gain parameters of the control law [Markdahl et al., 2012]. In Markdahl et al. [2012] there are two strictly positive gain parameters. In this paper Algorithm 1 provide six gain parameters given by the matrix $\mathbf{P} \in \mathrm{S}^{++}(3)$. The potential gain from using optimization techniques in lieu with the analytical solutions should hence be greater than in Markdahl et al. [2012].

Before posing the MPC problem we consider a switched feedback control based on Algorithm (1), where a time-dependence is introduced by replacing the gain matrix $\mathbf{P}$ by a piece-wise constant function of time $\mathbf{P}(t)$.

Algorithm 8. Consider a feedback

$$\boldsymbol{\Omega}_3 = \mathbf{P}(t)\mathbf{R}^{-1} - \mathbf{R}\mathbf{P}(t),$$

where $\mathbf{P}(t)$ is a matrix valued switching signal. Let the switching times be given by $\{t_k\}_{k=0}^{\infty}$ with $t_0 = 0$. The matrix $\mathbf{P}(t) \in \mathrm{S}^{++}(3)$ switches at each t_k and is constant on $[t_k, t_{k+1})$ for all $k \in \mathbb{N}$, has a strictly positive dwell time Δt, and satisfies $\mathbf{P}(t) \succeq \varepsilon\mathbf{I}$ for some strictly positive constant ε. The closed loop system is

$$\dot{\mathbf{R}} = \mathbf{P}(t) - \mathbf{R}\mathbf{P}(t)\mathbf{R}. \qquad (18)$$

Proposition 9. Suppose the system (2) with $\mathbf{R}_0 \in \mathrm{SO}(3)$ is governed by Algorithm 8. Then the identity matrix is a uniformly asymptotically stable equilibrium of $\mathbf{R}$.

Proof. The proof is omitted.

Note that (18) has a solution given by Theorem 6 on $I_0 = [t_0, t_1)$ since $P(t)$ is constant on I_0. Suppose that (18) has a solution on $I_k = [t_k, t_{k+1})$. Then set $\mathbf{R}_{k+1} = \lim_{t\uparrow t_k}\mathbf{R}(t)$. Since $\mathbf{P}(t)$ is constant on I_{k+1} we can use $\mathbf{R}_{k+1}$ as an initial condition to obtain a solution on I_{k+1} that satisfies $\lim_{t\uparrow t_{k+1}} \mathbf{R}(t) = \lim_{t\downarrow t_{k+1}} \mathbf{R}(t)$. By proceeding inductively, we obtain a solution to (18) on $\mathbb{R}^+ = \cup_{k\in\mathbb{N}}[t_k, t_{k+1})$.

Let us state the MPC problem.

Problem 10. (MPC). Let a set of time instances $\{t_i\}_{i=0}^m \subset \mathbb{R}^+ \cup \{\infty\}$ where $t_{i+1} - t_i \geq \Delta t$ and an initial condition $\mathbf{R}_0 \in \mathrm{SO}(3)$ be given. Suppose the dynamics (2) are governed by Algorithm 8. Consider the problem of optimizing a function f with respect to the input $\mathbf{P}(t) = \mathbf{P}_i \in \mathrm{S}^{++}(3)$, $t \in [t_i, t_{i+1}]$, *i.e.* to solve

$$\begin{aligned}
&\min_{\mathbf{P}_i, \forall i \in M} f(\mathbf{R}(t), \mathbf{P}_0, \ldots, \mathbf{P}_m), \\
&\mathbf{R}(t) = (\sinh(\mathbf{P}_i(t-t_i)) + \cosh(\mathbf{P}_i(t-t_i))\mathbf{R}(t_i)) \\
&\qquad (\cosh(\mathbf{P}_i(t-t_i)) + \sinh(\mathbf{P}_i(t-t_i))\mathbf{R}(t_i))^{-1}, \\
&\quad t \in [t_i, t_{i+1}),\ \forall i \in M, \qquad (19)\\
&\mathbf{P}_i \succeq \varepsilon \mathbf{I},\ \forall i \in M, \qquad (20)
\end{aligned}$$

where $M = \{0, \ldots, m\}$.

The constraint (19) is obtained from solving the closed loop system generated by Algorithm 8. The constraint (20) is imposed in Algorithm 8 to ensure convergence under arbitrary switching. It then follows that $\lim_{t\to\infty} \mathbf{R}(t) = \mathbf{I}$ for any feasible solution $\{\mathbf{P}_i\}_{i=0}^m$ to the MPC problem. This frees the specification of f from any concerns regarding the stabilization of the identity matrix. Hence f could be chosen to optimize some secondary objective.

What the MPC problem does is to utilize the transient phase of the system evolution to carry out a task of secondary importance. The MPC problem could also be posed with (19) replaced by (18). The benefit gained by using the analytical solution obtained from Theorem 6 as compared to not having access to them is to eliminate the computational cost of solving (18) numerically.

Example 11. Consider the problem of stabilizing the attitude of a camera while at some point in time wishing to see a desired view corresponding to the camera orientation $\mathbf{R}_d$. A possible choice of f that attempts to achieve this is

$$f(\mathbf{R}(t), \mathbf{P}) = \inf_{t\in[0,\infty)} \|\mathbf{R}(t) - \mathbf{R}_d\|_2^2.$$

Note that the problem in addressed in Example 11 cannot be solved by tracking a curve in $\mathrm{SO}(n)$ that interpolates the points $\mathbf{R}_0$, $\mathbf{R}_d$, and $\mathbf{I}$. The key idea is to utilize the transient phase of the system. This can however also be done for the transient in the case of trajectory tracking problems.

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Appendix A. MATRIX HYPERBOLIC FUNCTIONS

Definition 12. The matrix valued hyperbolic functions cosine, sine, and tangent of $\mathbf{A} \in \mathbb{R}^{n\times n}$ can be defined via the matrix exponential as

$$\cosh \mathbf{A} = \frac{1}{2}\left(\exp(\mathbf{A}) + \exp(-\mathbf{A})\right),$$
$$\sinh \mathbf{A} = \frac{1}{2}\left(\exp(\mathbf{A}) - \exp(-\mathbf{A})\right),$$
$$\tanh \mathbf{A} = \sinh(\mathbf{A})\cosh^{-1}(\mathbf{A}).$$

The hyperbolic arctangent, atanh, is defined by

$$\operatorname{atanh} \mathbf{A} = \frac{1}{2}\operatorname{Log}(\mathbf{I}+\mathbf{A}) - \frac{1}{2}\operatorname{Log}(\mathbf{I}-\mathbf{A}),$$

for arguments $\mathbf{A}$ such that $\pm 1 \notin \sigma(\mathbf{A})$.

Lemma 13. The matrices $\cosh^{-1}\mathbf{P}$ and $\tanh \mathbf{P}$ are well defined for $\mathbf{P} \in \mathrm{S}^{++}(n)$ and satisfies

$$\lim_{t\to\infty} \cosh^{-1}(\mathbf{P}t) = \mathbf{0}, \quad \lim_{t\to\infty} \tanh(\mathbf{P}t) = \mathbf{I}.$$

Proof. For any eigenpair $(\lambda, \mathbf{v})$ of $\mathbf{P}$ there is an eigenpair $(\cosh(\lambda t), \mathbf{v})$ of $\cosh(\mathbf{P}t)$. Since $\mathbf{P}$ is positive definite, the algebraic and geometric multiplicities of all its eigenvalues are equal. Moreover, cosh is invertible when restricted to $\mathbb{R}^+$. It follows that there is a one-to-one correspondence between eigenpairs of $\mathbf{P}$ and $\cosh(\mathbf{P}t)$. Hence $\cosh(\mathbf{P}t)$ is nonsingular.

The first limit follows from the corresponding limit for the eigenvalues. For the second limit, note that $[\cosh \mathbf{P}, \sinh \mathbf{P}] = \mathbf{0}$. Commutativity of diagonalizable matrices implies simultaneous diagonalizability. The spectral theorem gives $\mathbf{P} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$ for some orthogonal matrix $\mathbf{Q}$. Then

$$\cosh \mathbf{P} = \mathbf{Q}\cosh(\mathbf{\Lambda})\mathbf{Q}^{-1},$$
$$\sinh \mathbf{P} = \mathbf{Q}\sinh(\mathbf{\Lambda})\mathbf{Q}^{-1},$$
$$\tanh \mathbf{P} = \mathbf{Q}\tanh(\mathbf{\Lambda})\mathbf{Q}^{-1},$$

and the limit follows from $\lim_{t\to\infty} \tanh(t) = 1$. ■

Appendix B. LEMMAS

Lemma 14. The equations (3), (5), (6), and (8) have unique solutions that belong to $\mathrm{SO}(n)$ for all $t \in \mathbb{R}^+$.

Proof: The proof in the case of (3) is similar to that in Markdahl et al. [2013]. The assumptions made in Algorithm 3 ensures uniqueness of the solution $\mathbf{S}(t)$ to (4) and hence of $\mathbf{R}(t)$ to (5). In the case of (6), we can use the change of variables $\mathbf{X} = \mathbf{R}^{\frac{1}{k}}$ from the proof of Proposition 4, to obtain (16). The uniqueness of the solution to (3) and the uniqueness of unitary roots of $\mathbf{R} \notin \mathrm{N}(n)$ imply that the solution $\mathbf{X}(t)$ to (16) is also unique. In the case of (8) we can make the change of variables $\mathbf{Y} = (\mathbf{I} + \mathbf{R}^{\frac{1}{k}})^{-1}$. Then

$$\dot{\mathbf{Y}} = \mathbf{Y}(\mathbf{I}-\mathbf{Y})(\mathbf{I}-2\mathbf{Y}),$$

with the right-hand side being a polynomial in $\mathbf{Y}$. Uniqueness follows by reasoning as done in Markdahl et al. [2013]. By reversing the change of variables for $\mathbf{R}_0 \notin \mathrm{N}(n)$ we prove uniqueness of $\mathbf{R}(t)$. ■

Lemma 15. The statements $[\dot{\mathbf{S}}, \mathbf{S}] = \mathbf{0}$ and $[\mathbf{\Omega}, \mathbf{S}] = \mathbf{0}$ are equivalent. Moreover, they imply that $\dot{\mathbf{S}} = \mathbf{\Omega}$.

Proof. From $[\dot{\mathbf{S}}, \mathbf{S}] = \mathbf{0}$ we get

$$\mathbf{\Omega}\mathbf{R} = \dot{\mathbf{R}} = \dot{\mathbf{S}}\mathbf{R}.$$

Canceling $\mathbf{R}$ yields $\dot{\mathbf{S}} = \mathbf{\Omega}$ which results in $[\mathbf{\Omega}, \mathbf{S}] = \mathbf{0}$.

Conversely, suppose $[\mathbf{\Omega}, \mathbf{S}] = \mathbf{0}$. Since $\mathbf{R}$ is a normal matrix it has a spectral factorization given by $\mathbf{R} = \mathbf{U}^*\mathbf{\Lambda}\mathbf{U}$, where $\mathbf{U}$ is a unitary matrix and $\mathbf{\Lambda}$ a diagonal matrix. From $[\mathbf{\Omega}, \mathbf{S}] = \mathbf{0}$ we get $[\mathbf{\Omega}, \mathbf{R}] = \mathbf{0}$ which implies that $\mathbf{\Omega} = \mathbf{U}^*\mathbf{\Xi}\mathbf{U}$, where $\mathbf{\Xi}$ is a diagonal matrix. Then

$$\begin{aligned}
\dot{\mathbf{R}} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}^*\mathbf{\Lambda}\mathbf{U} = \dot{\mathbf{U}}^*\mathbf{\Lambda}\mathbf{U} + \mathbf{U}^*\dot{\mathbf{\Lambda}}\mathbf{U} + \mathbf{U}^*\mathbf{\Lambda}\dot{\mathbf{U}} \\
&= \mathbf{U}^*(\mathbf{U}\dot{\mathbf{U}}^*\mathbf{\Lambda} + \dot{\mathbf{\Lambda}} + \mathbf{\Lambda}\dot{\mathbf{U}}\mathbf{U}^*)\mathbf{U} \\
&= \mathbf{U}^*([\mathbf{\Lambda}, \dot{\mathbf{U}}\mathbf{U}^*] + \dot{\mathbf{\Lambda}})\mathbf{U}, \\
\mathbf{\Omega}\mathbf{R} &= \mathbf{U}^*\mathbf{\Xi}\mathbf{U}\mathbf{U}^*\mathbf{\Lambda}\mathbf{U} = \mathbf{U}^*\mathbf{\Xi}\mathbf{\Lambda}\mathbf{U}.
\end{aligned}$$

Taken together, we have

$$[\mathbf{\Lambda}, \dot{\mathbf{U}}\mathbf{U}^*] + \dot{\mathbf{\Lambda}} = \mathbf{\Xi}\mathbf{\Lambda}.$$

where both $\dot{\mathbf{\Lambda}}$ and $\mathbf{\Xi}\mathbf{\Lambda}$ are diagonal matrices. The commutator has zero diagonal and hence $[\mathbf{\Lambda}, \dot{\mathbf{U}}\mathbf{U}^*] = \mathbf{0}$. It follows that

$$\begin{aligned}
\dot{\mathbf{S}} &= \dot{\mathbf{U}}^*\operatorname{Log}(\mathbf{\Lambda})\mathbf{U} + \mathbf{U}^*(\dot{\operatorname{Log}}\,\mathbf{\Lambda})\mathbf{U} + \mathbf{U}^*\operatorname{Log}(\mathbf{\Lambda})\dot{\mathbf{U}} \\
&= \mathbf{U}^*[\operatorname{Log}\mathbf{\Lambda}, \dot{\mathbf{U}}\mathbf{U}^*]\mathbf{U} + \mathbf{U}^*(\dot{\operatorname{Log}}\,\mathbf{\Lambda})\mathbf{U} \\
&= \mathbf{U}^*\mathbf{\Lambda}^{-1}\dot{\mathbf{\Lambda}}\mathbf{U} = \mathbf{U}^*\mathbf{\Lambda}^{-1}\mathbf{U}\mathbf{U}^*\dot{\mathbf{\Lambda}}\mathbf{U} \\
&= \mathbf{R}^{-1}\dot{\mathbf{R}} = \mathbf{R}^{-1}\mathbf{\Omega}\mathbf{R} = \mathbf{\Omega},
\end{aligned}$$

which results in $[\dot{\mathbf{S}}, \mathbf{S}] = \mathbf{0}$ (that $[\operatorname{Log}\mathbf{\Lambda}, \dot{\mathbf{U}}\mathbf{U}^*] = \mathbf{0}$ follows from $[\mathbf{\Lambda}, \dot{\mathbf{U}}\mathbf{U}^*] = \mathbf{0}$ using the Taylor series definition of the matrix logarithm).

Remark 16. This result is important because it allows us to replace the assumption of $[\dot{\mathbf{S}}, \mathbf{S}] = \mathbf{0}$ with $[\mathbf{\Omega}, \mathbf{S}] = \mathbf{0}$. The latter assumption is preferable since we assume $\mathbf{\Omega}$ to be the control input, *i.e.* we can design $\mathbf{\Omega}$. It is not, however, possible to chose $\dot{\mathbf{S}}$ in general.

Lemma 17. The expression for $\mathbf{C}(t)$ given by (17) is well-defined for all $t \in \mathbb{R}^+$, and so is $\operatorname{atanh}\mathbf{C}(t)$.

Proof. Since

$$\sigma(\mathbf{R}) \subset \{z \in \mathbb{C} \,|\, |z| = 1\},$$

we may obtain $\mathbf{S} = \operatorname{Log}\mathbf{R}$ for $\mathbf{R} \notin \mathrm{N}(n)$ using the principal logarithm. Then

$$\sigma(\mathbf{S}) = \{i\lambda \in i\mathbb{R} \,|\, |\lambda| < \pi,\, \mathrm{e}^{i\lambda} \in \sigma(\mathbf{R})\}.$$

Since $\mathbf{X} = \frac{1}{2k}\mathbf{S}$ we find that all $\lambda \in \sigma(\mathbf{X})$ satisfy $|\lambda| < \frac{1}{2}\pi$. It follows that

$$\sigma(\sinh^2\mathbf{X} + \mathrm{e}^t\mathbf{I}) = \{-\sin^2\lambda + \mathrm{e}^t \in \mathbb{R}^{++} \,|\, i\lambda \in \sigma(\mathbf{X})\}.$$

These eigenvalues are strictly positive, *i.e.* $\sinh^2\mathbf{X} + \mathrm{e}^t\mathbf{I}$ is nonsingular. It is also normal, whereby its principal square root can be calculated as detailed in §3. This shows $\mathbf{C}(t)$ to be well-defined.

Recall the definition of atanh given in Appendix A. Note that $\mathbf{C}(t)$ is skew-symmetric, implying that $\sigma(\mathbf{C}(t)) \in i\mathbb{R}$. Hence $\operatorname{atanh}\mathbf{C}(t)$ is well-defined. ■