Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Dynamic Quantizers in Presence of Noisy Channels

Giancarlo Baldan * **Munther Dahleh** * **Alexandre Megretski** *

* *Laboratory for Information and Decision systems, Massachusetts Institute of Technology, Cambridge 02139 USA*
{gbaldan,dahleh,ameg}@mit.edu

Abstract:
In this paper, we deal with the stabilizability of a scalar plant through a noisy discrete channel. Our scheme can be regarded as a variation of the controllers based on dynamic quantizers introduced in [3] and [4], but our analysis based on Lyapunov function allows us to take into consideration more general discrete memoryless channels. Given a plant and a discrete memoryless channel we will provide, using control Lyapunov function ideas, a condition for the existence of an encoder/controller pair with a priori fixed memory that achieves stability. A construction procedure is also provided.

1. INTRODUCTION

In the last decade, substantial research has been devoted to the investigation of the interplay between communication and control when the communication between the plant and controller is not ideal.

As stressed in [1], for this framework the most fundamental question is: what is the minimal performance level of the communication link that allows the controller to somehow stabilize the plant? The answer to this question depends heavily on the figure-of-merit for the communication link and on the desired stability notion.

The earliest works in this field involved the assumption of a finite quantizer as a model for the communication link. In [7], the authors showed that a quantized control scheme for a noiseless scalar plant with parameter $|\lambda| > 1$ can achieve practical stability if and only if the rate of the quantizer (Shannon capacity) exceeds $\log_2 |\lambda|$, while [2] constructively shows that the coarsest quantizer to achieve the same stability is logarithmic. By adding memory to the encoder/decoder scheme, it is possible to design a quantized-based control scheme capable of achieving asymptotic stability, as seen in [3].

When the communication link is noisy, the stabilization problem becomes much more complicated. Although the inequality $C > H$, with C as the Shannon capacity of the channel and $H = \sum_{|\lambda_i|>1} \log_2 |\lambda_i|$ the intrinsic entropy rate of the plant, remains necessary and sufficient for almost sure asymptotic stability of a deterministic plant [5] [6], it does not hold true for other stability objectives. If the object is to achieve some moment stability, as naturally arises in situations when the plant is intrinsically noisy, then [8] shows that the Anytime and not the Shannon capacity is the correct figure-of-merit.

In this paper we will focus on almost sure asymptotic stabilization of a deterministic first order plant through a (noisy) discrete memoryless channel. In this context, the work in [6] focused more on the fundamental limit to achieve stability. The answer was that almost sure asymptotic stability for a general multidimensional plant is achievable via a discrete memoryless channel if and only if[1] $C > H$ and a construction of a finite memory controller was provided. One of the features of the analysis in [6] is that, in proving the sufficiency part, the proposed control scheme relies on codewords that are in general difficult to construct and requires an amount of memory that grows as the difference $C - H$ tends to zero thus complicating any practical implementation. Our approach, on the other end, aims at achieving almost sure asymptotic stability with an encoder/controller pair possessing a fixed a priori memory size. This restriction ensures that, once the scheme parameters have been determined, its implementation on any modern digital platform can be done easily. Moreover, in contrast to the usual analysis of dynamic quantizers algorithms [3] [4] where Lyapunov functions are only partially exploited to prove stability, our analysis fully relies on a standard Lyapunov approach that can potentially be used to assess the performances of a given controller. Based on this analysis, rather than focusing on the fundamental capacity limit to achieve stability, we will provide a sufficient condition under which the parameters defining our model can be designed and provide an algorithm to compute them that exhibits linear complexity in $|\lambda|$.

The outline of the paper is as follows: in section 2, we define the structure of the control scheme and communication link, we state precisely our design goal and we show how the analysis can be performed on a simpler two dimensional system. In section 3, we show how our scheme always achieves stability when the communication link is supposed to be ideal while in section 4 we extend the same analysis technique to the finite quantizer case. In section 5, we address the fully noisy case and present the main result of the paper in theorem 9. Finally, conclusions and ideas for further extensions are presented in section 6.

⋆ This was supported by AFOSR, grant # FA9550-09-1-0420.

[1] The necessary part holds with equality

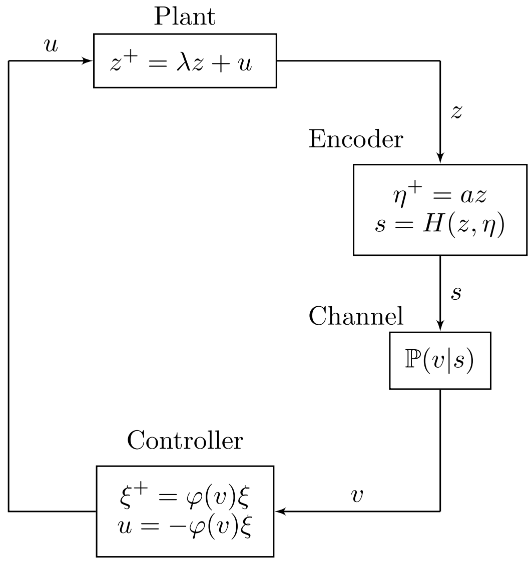

Fig. 1. The proposed control scheme.

2. MODEL DEFINITION

Our goal is to stabilize a discrete–time first order linear system where the measurements from the plant are obtained through a discrete memoryless channel.

The dynamic of the plant is given by

$$z(t+1) = \lambda z(t) + u(t),$$

where $u, z \in \mathbb{R}$, and λ is a real parameter satisfying $|\lambda| > 1$.

The channel is completely described by the input alphabet $\mathfrak{S} = \{s_1, \ldots, s_m\}$, the output alphabet $\mathfrak{V} = \{v_1, \ldots, v_n\}$, and an n by m matrix W whose entries w_{ij} describe the conditional probabilities according to:

$$\mathbb{P}(v = v_i | s = s_j) = w_{ij}.$$

In reference to figure 1, we adopted a feedback control scheme with an encoder and a controller. The encoder, placed between the plant and the channel, measures the state of the plant and codifies it into symbols drawn from $\mathfrak{S}$.

Its dynamic is described by a first order linear equation and a non–linear function taking values in $\mathfrak{S}$:

$$\begin{cases} \eta(t+1) &= az(t) \\ s(t) &= H(z(t), \eta(t)) \end{cases}, \quad (1)$$

where $\eta(t) \in \mathbb{R}$ is the state of the encoder and $a \in \mathbb{R}$ is a parameter to be determined. The coding function H is completely arbitrary and one of the subjects of our design procedure.

The controller receives the symbols from the channel and drives the plant by shaping the signal u. Its dynamic is described by two piecewise linear equations describing the evolution of the internal state and the control signal:

$$\begin{cases} \xi(t+1) &= \varphi(v(t))\xi(t) \\ u(t) &= -\varphi(v(t))\xi(t) \end{cases}, \quad (2)$$

where $\xi(t) \in \mathbb{R}$ is the state of the controller and φ is a function from $\mathfrak{V}$ to $\mathbb{R}$ defined by:

$$\varphi(v_1) = \varphi_1 \ \ldots \ \varphi(v_n) = \varphi_n$$

with φ_i real parameters to be determined.

Remark 1. It is clear from the scheme in figure 1 that $a, \varphi_1, \ldots \varphi_n$ as well as the initial condition $\xi(0)$ must be all non zero, otherwise the lack of feedback from the plant to the controller would prevent any reasonable concept of stability to be guaranteed. We also point out that our choice for the dynamic of the controller is inspired by the one used in [3] and [4].

Now that the components of our system have been accurately defined, we can precisely state what we mean by stability. In this paper we will deal exclusively with almost sure stability and, in particular, we aim to design the parameters of the control scheme so that the following is true:

$$\mathbb{P}\left[\lim_{t\to\infty} (z, \eta, \xi)(t) = 0\right] = 1 \quad \forall z(0) \in \mathbb{R}. \quad (3)$$

In the remainder of this section we will show how the analysis can be reduced to the study of a suitable two dimensional system. The intuition behind this feature is that, since the plant is noiseless, the encoder can use its extra memory to compute the input to the plant and, ultimately, the state of the controller. With this information available, the encoder then chooses the next channel input to try to compensate for errors introduced by the channel. The details of this reduction are contained in the next proposition

Proposition 2. Assume there exist a map $Q : \mathbb{R}^2 \mapsto \mathfrak{S}$ and $n+1$ real parameters $b, h_1 \ldots h_n$ such that the two dimensional Markov process described by the equation

$$\begin{bmatrix} x \\ y \end{bmatrix}(t+1) = \sqrt{|\lambda|}\,\mathrm{sgn}(\lambda) \begin{bmatrix} h_i & 0 \\ \dfrac{b}{|h_i|} & \dfrac{1}{|h_i|} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}(t) \text{ w.p. } w_{i,Q(x,y)} \quad (4)$$

satisfies $\mathbb{P}\left[\lim_{t\to\infty} (x, y)(t) = 0\right] = 1 \quad \forall (x_0, y_0) \in \mathbb{R}^2$.

Then there exists an invertible transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that the map $H = Q \circ T^{-1}$ and the sets of parameters $a = \lambda b$, $\varphi_i = \mathrm{sgn}(\lambda h_i)|\lambda| h_i^2$ guarantee condition (3).

Proof. The dynamic of the scheme shown in figure 1 is described by the equation

$$\begin{bmatrix} z \\ \eta \\ \xi \end{bmatrix}(t+1) = \begin{bmatrix} \lambda & 0 & -\varphi_i \\ a & 0 & 0 \\ 0 & 0 & \varphi_i \end{bmatrix} \begin{bmatrix} z \\ \eta \\ \xi \end{bmatrix}(t) = A_i \begin{bmatrix} z \\ \eta \\ \xi \end{bmatrix}(t),$$

where the matrix A_i, with a slight abuse of notation, is selected with probability $\mathbb{P}(v_i | H(z, \eta)) = w_{i,H(z,\eta)}$.

Let us start by noticing that, since

$$az^+ - \lambda\eta^+ + a\xi^+ = a(\lambda z - \varphi_i \xi) - \lambda a z + a\varphi_i \xi = 0 \quad \forall \varphi_i,$$

after one step the state will always lie in the unique attractive invariant subspace described by

$$az - \lambda\eta + a\xi = 0,$$

where the dynamic of ξ and η is given by

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}(t+1) = \begin{bmatrix} \varphi_i & 0 \\ -a & \lambda \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}(t) \text{ w.p. } w_{i,H(\xi,\eta)}. \quad (5)$$

If we now perform the following nonlinear transformation:

$$T : \begin{cases} \xi = -\,\mathrm{sgn}(x)x^2 \\ \eta = y|x| \end{cases} \quad (6)$$

and define $b, h_1, \ldots, h_n$ such that:

$$a = \lambda b \qquad \operatorname{sgn}(\varphi_i)\sqrt{|\varphi_i|} = \operatorname{sgn}(\lambda)\sqrt{|\lambda|}h_i,$$

the dynamic becomes:

$$\begin{bmatrix} x \\ y \end{bmatrix}(t+1) = \sqrt{|\lambda|}\operatorname{sgn}(\lambda)\begin{bmatrix} h_i & 0 \\ \frac{b}{|h_i|} & \frac{1}{|h_i|} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}(t) \quad \begin{matrix}\text{w.p.} \\ w_{i,H\circ T(x,y)}\end{matrix}.$$

Since $(x, y) \to 0$ implies $(z, \eta, \xi) \to 0$, the proof is complete as long as T is invertible. The invertibility of T in (6) is guaranteed because, given the conditions stressed in remark 1, we have $x(0), h_i \neq 0\ \forall i$ thus ensuring that x is never 0. Notice that, since the inverse transformation is unbounded in any neighbourhood of the origin, this map does not preserve stability.

Remark 3. The presence of the eigenvalue λ in equation 5 clearly shows that the original dynamic cannot be stabilized in the canonical Lyapunov sense, thus making inconclusive every approach based on storage functions. The transformation in (6) circumvents this technical difficulty by transforming the system into one for which a standard Lyapunov-function based approach has proven to be successful.

While the main result regarding the almost sure stability of our closed loop will be presented in section 5 (cfr. theorem 9), each of the following sections will present intermediate results valid under increasingly realistic assumptions on the communication link. Regardless of the particular model for the channel, we will always show that it is possible to design the parameters of (7) so that the dynamic is (stochastically) quadratically stable. In particular, given $\rho \in (0, |\lambda|)$ our aim is to design our control scheme such that a quadratic Lyapunov function V satisfying the condition

$$E[V(x^+, y^+)|(x, y)] \leq \rho V(x, y)$$

can be provided. To this end, it is convenient to define a performance parameter γ as the ratio between the required closed loop performances and the open loop instability level,

$$\gamma = \frac{\rho}{|\lambda|} \in (0, 1),$$

and scale the control signal h_i via $r_i = \sqrt{\gamma}h_i$, thus obtaining the dynamic:

$$\begin{bmatrix} x \\ y \end{bmatrix}(t+1) = \sqrt{|\lambda|}\operatorname{sgn}(\lambda)\begin{bmatrix} \frac{r_i}{\sqrt{\gamma}} & 0 \\ \frac{b\sqrt{\gamma}}{|r_i|} & \frac{\sqrt{\gamma}}{|r_i|} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}(t) \text{ w.p. } w_{i,Q(x,y)}. \tag{7}$$

Definition 4. Given $\gamma = \frac{\rho}{|\lambda|} \in (0, 1)$, we will say that the set of parameters $b, h_1 \ldots h_n$ and the map Q achieve performance parameter γ for the dynamic in (7), if there exists a quadratic Lyapunov function V such that

$$E[V(x^+, y^+)|(x, y)] \leq \rho V(x, y). \tag{8}$$

3. IDEAL CHANNEL ANALYSIS

In this section we will show how any performance parameter γ can always be obtained for the system in (7) when the channel is assumed to be noiseless and with infinite capacity. Even though this model for the communication link is far from the one we are interested in, by studying this simple scenario, we will gain insight on how to tackle more realistic formulations.

When the channel is assumed to be noiseless, the value r_i is a deterministic function of the (x, y) state. Moreover, due to the infinite capacity assumption, it is no longer quantized and can assume any value in $\mathbb{R}$ so that the dynamic in (7) can be thought as

$$\begin{bmatrix} x \\ y \end{bmatrix}(t+1) = \sqrt{|\lambda|}\operatorname{sgn}(\lambda)\begin{bmatrix} \frac{r}{\sqrt{\gamma}} & 0 \\ \frac{b\sqrt{\gamma}}{|r|} & \frac{\sqrt{\gamma}}{|r|} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}(t). \tag{9}$$

where r can be arbitrarily designed as a real function of x and y.

The main result for this section is contained in the next proposition.

Proposition 5. For any fixed $\gamma \in (0, 1)$, there exists $b \in \mathbb{R}$ and a feedback control law $r = \Psi(x, y)$ such that the system in (9) achieves the performance parameter γ.

Proof. Consider the quadratic Lyapunov function

$$V(x, y) = [\,x\ \ y\,]\,P\begin{bmatrix} x \\ y \end{bmatrix} \qquad P = \begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix},$$

where $\alpha > 1$ is a parameter to be designed. Using V as a control Lyapunov function, the inequality in (8) yields:

$$|\lambda|\frac{[\,x\ \ y\,]\begin{bmatrix} \frac{r}{\sqrt{\gamma}} & \frac{b\sqrt{\gamma}}{|r|} \\ 0 & \frac{\sqrt{\gamma}}{|r|} \end{bmatrix}\begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{r}{\sqrt{\gamma}} & 0 \\ \frac{b\sqrt{\gamma}}{|r|} & \frac{\sqrt{\gamma}}{|r|} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}{[\,x\ \ y\,]\begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}} \leq \rho$$

$$\Updownarrow$$

$$\frac{\frac{\alpha}{\gamma}r^2 + \frac{\gamma}{r^2}(b+\nu)^2 + 2\operatorname{sgn}(r)(b+\nu)}{\alpha + 2\nu + \nu^2} \leq \gamma, \tag{10}$$

where we defined ν as the ratio between y and x.

By using the value of r that minimizes the left hand side of equation (10):

$$r^* = -\operatorname{sgn}(b+\nu)\alpha^{-\frac{1}{4}}\sqrt{\gamma|b+\nu|}, \tag{11}$$

the Lyapunov inequality becomes:

$$\frac{2(\sqrt{\alpha}-1)|b+\nu|}{\alpha + 2\nu + \nu^2} \leq \gamma. \tag{12}$$

By replacing the left hand side of equation (12) with its maximum value over ν we obtain the inequality

$$\frac{\sqrt{\alpha}-1}{\sqrt{b^2 - 2b + \alpha} - |b-1|} \leq \gamma,$$

which can be further simplified if we optimize over the parameter b via

$$b^* = \operatorname{argmin}_{b\in\mathbb{R}} \frac{\sqrt{\alpha}-1}{\sqrt{b^2 - 2b + \alpha} - |b-1|} = 1,$$

thus obtaining the final inequality:

$$\frac{\sqrt{\alpha}-1}{\sqrt{\alpha-1}} = \sqrt{\frac{\sqrt{\alpha}-1}{\sqrt{\alpha}+1}} \leq \gamma. \tag{13}$$

Since the inequality in (13) can be satisfied $\forall\,\gamma$ by selecting $\alpha \in \left(1, \frac{(1+\gamma^2)^2}{(1-\gamma^2)^2}\right]$, the proof is complete.

4. FINITE QUANTIZER ANALYSIS

In this section we will assume that the communication link between the plant and the controller is a noiseless finite quantizer. Under this assumption, the original dynamic in (7) does not contain any stochastic component and reduces to:

$$\begin{bmatrix} x \\ y \end{bmatrix}(t+1) = \sqrt{|\lambda|}\,\mathrm{sgn}(\lambda) \begin{bmatrix} \frac{r_i}{\sqrt{\gamma}} & 0 \\ \frac{b\sqrt{\gamma}}{|r_i|} & \frac{\sqrt{\gamma}}{|r_i|} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}(t) \quad r_i = Q(x, y), \tag{14}$$

where Q assumes only a finite set of values. As in the previous section, we will show that, for any performance parameter γ, there exists b and a finite–valued function $Q(x, y)$ achieving performance γ.

The approach is an extension of the one presented in the previous section. In particular we will select a particular Lyapunov function and design the parameters of the control scheme so that the Lyapunov inequality is satisfied. Due to the increasing complexity, we will assume:

$$b = 1, \quad V(x, y) = [\,x \ \ y\,]\, P \begin{bmatrix} x \\ y \end{bmatrix} \quad P = \begin{bmatrix} \frac{1+\gamma^4}{(1-\gamma^2)^2} & 1 \\ 1 & 1 \end{bmatrix}, \tag{15}$$

instead of optimizing over the values of b and P as we did in the previous section. The particular choices in (15) are motivated by the fact that $b = 1$ has been proven optimal in the ideal case and we expect it to be asymptotically optimal when $\gamma \to 0$, while P corresponds to a choice of α in the middle [2] of the interval of values satisfying (13).

Based on the assumptions in (15), the Lyapunov inequality in (10), which is obtained when the channel is ideal, becomes:

$$\begin{aligned} F_\gamma(\nu, r) \; = \; & \frac{1+\gamma^4}{\gamma(1-\gamma^2)^2} r^2 + \frac{\gamma}{r^2}(1+\nu)^2 + 2\,\mathrm{sgn}(r)(1+\nu) \\ & - \gamma\left(\frac{1+\gamma^4}{(1-\gamma^2)^2} + 2\nu + \nu^2\right) \le 0. \end{aligned} \tag{16}$$

If we can show that the set

$$S_\gamma = \left\{(\nu, r) \in \mathbb{R}^2 | F_\gamma(\nu, r) \le 0\right\}$$

contains the graph of a function $r = \Psi(\nu)$ assuming only a finite set of values, then the feedback loop $r_i = Q(x, y) = \Psi(y/x)$ would guarantee the required performance for the dynamic in (14). This is the basis for the next proposition.

Proposition 6. For any fixed $\gamma \in (0, 1)$ the set S_γ is symmetric around the point $(-1, 0)$ and its restriction to the region $r \ge 0$ can be described as

$$S_\gamma \cap \{r \ge 0\} = \{\underline{r}(\nu) \le r \le \bar{r}(\nu),\ \nu \le \nu_L \vee \nu \ge \nu_H\}, \tag{17}$$

where

$$\underline{r}(\nu) = \sqrt{\mu - \sqrt{\mu^2 - \tfrac{\gamma^2(1-\gamma^2)^2}{1+\gamma^4}(1+\nu)^2}},$$

$$\bar{r}(\nu) = \sqrt{\mu + \sqrt{\mu^2 - \tfrac{\gamma^2(1-\gamma^2)^2}{1+\gamma^4}(1+\nu)^2}},$$

$$\mu = \tfrac{\gamma^2}{2} + \tfrac{\gamma(1-\gamma^2)^2}{1+\gamma^4}\left(\tfrac{\gamma}{2}\nu^2 + (\gamma-1)\nu - 1\right),$$

$$\nu_L = -1 + \tfrac{\sqrt{1+\gamma^4}+1-\gamma^2}{\gamma(1-\gamma^2)} - \sqrt{\left(\tfrac{\sqrt{1+\gamma^4}+1-\gamma^2}{\gamma(1-\gamma^2)}\right)^2 - \tfrac{2\gamma^2}{(1-\gamma^2)^2}},$$

$$\nu_H = -1 + \tfrac{\sqrt{1+\gamma^4}+1-\gamma^2}{\gamma(1-\gamma^2)} + \sqrt{\left(\tfrac{\sqrt{1+\gamma^4}+1-\gamma^2}{\gamma(1-\gamma^2)}\right)^2 - \tfrac{2\gamma^2}{(1-\gamma^2)^2}}.$$

Furthermore S_γ will always contain the graph of a finite-valued function $r = \Psi(\nu)$, thus proving that a performance level γ can always be achieved with a finite quantizer.

Proof. The symmetry around the point $(-1, 0)$ is easily proved since $F_\gamma(-\nu-2, -r) = F_\gamma(\nu, r)$. When restricted to the positive r half plane, the Lyapunov inequality in (16) becomes

$$\frac{1+\gamma^4}{\gamma(1-\gamma^2)^2} r^2 + \frac{\gamma}{r^2}(1+\nu)^2 + 2(1+\nu) - \gamma\left(\frac{1+\gamma^4}{(1-\gamma^2)^2} + 2\nu + \nu^2\right) \le 0,$$

which is substantially a polynomial equation of degree two in r whose roots are given by $\underline{r}(\nu) \le \bar{r}(\nu)$ and whose discriminant is non-negative only when $\nu \le \nu_L \vee \nu \ge \nu_H$. This proves the form of S restricted to $r \ge 0$.

Finally, a study of $\underline{r}(\nu)$ and $\bar{r}(\nu)$ aided by a sum of square decomposition software reveals the key properties:

- $\lim_{\nu\to-\infty} \underline{r}(\nu) = 1, \quad \lim_{\nu\to-\infty} \bar{r}(\nu) = +\infty$,
- $\nu_L > -1$,
- $(\bar{r}^2 - \underline{r}^2)^2 \ge \frac{\gamma^8}{20(1+\gamma^4)^2} > 0 \quad \forall \nu \le -1$,

from which we conclude the existence of a finite-valued function $r = \Psi(\nu)$ whose graph is contained in S for $r <= -1$ and the same holds true $\forall r$ once the symmetry of S_γ around $(-1, 0)$ is exploited.

An example of the set S_γ together with the functions $\underline{r}$ and $\bar{r}$ is presented in figure 2.

Once the existence of a finite-valued function achieving performance γ is established, the next question is how to properly design it. Ideally one would like to provide the map that utilizes the smallest possible number of levels but this seems to be a substantially difficult problem. Instead, we provide a greedy algorithm that iteratively constructs a possible finite-valued function $r = \Psi(\nu)$ achieving performance γ. The procedure is explained in Algorithm 1 and its correctness relies on the fact that both $\underline{r}$ and $\bar{r}$ are monotone decreasing for $\nu \le -1$, thus allowing us to fill that region by starting at $\nu = -1$ and creating steps backward toward $-\infty$. By virtue of the symmetry of S_γ, the same procedure can also fill the region $\nu \ge -1$.

Algorithm 1 Design a finite partition $\{I_i\}$ of $\mathbb{R}$ and a set of values r_i such that the map $r = \Psi(\nu) = r_i$ if $\nu \in I_i$ achieves performance $\gamma \in (0, 1)$

$n \Leftarrow 1; \quad \nu \Leftarrow -1; \quad r \Leftarrow \bar{r}(-1) = \frac{\sqrt{2}\gamma^2}{\sqrt{1+\gamma^4}} = r_0$
while $r < 1$ **do**
  $\nu^+ \Leftarrow \underline{r}^{-1}(r)$
  $I_{-n} \Leftarrow (\nu^+, \nu]; \quad r_{-n} \Leftarrow r$
  $I_n \Leftarrow (-2-\nu, -2-\nu^+]; \quad r_n \Leftarrow -r$
  $r \Leftarrow \bar{r}(\nu^+); \quad \nu \Leftarrow \nu^+; \quad n \Leftarrow n+1$
end while
$I_{-n} \Leftarrow (-\infty, \nu]; \quad r_{-n} \Leftarrow 1$
$I_n \Leftarrow (-2-\nu, +\infty); \quad r_n \Leftarrow -1$

[2] This choice also turns out to be optimal for $\gamma \to 0$.

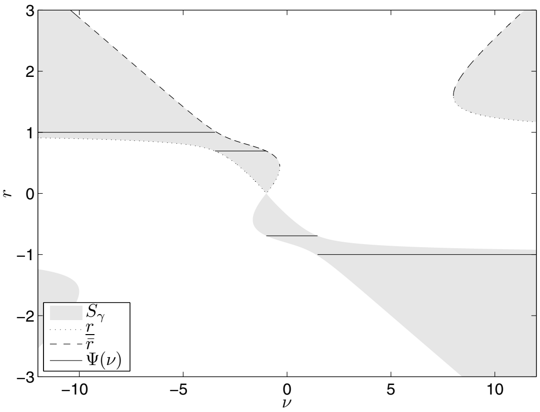

Fig. 2. The set S_γ, $\underline{r}$, $\bar{r}$ and the finite-valued function $r = \Psi(\nu)$ constructed using the proposed algorithm for $\gamma = 0.75$.

The number of levels generated by Algorithm 1 is given by $N_g = 2(N + 1)$ where N is the number of times the algorithm executes the while cycle and its execution always reaches the stopping condition because of the above mentioned properties of $\underline{r}$ and $\bar{r}$. An example of the map $r = \Psi(\nu)$ obtained applying the proposed greedy algorithm is shown in figure 2.

The complexity of our algorithm is clearly proportional to the number of levels generated N_g. The next proposition shows that this number can always be estimated and grows at most linearly with $1/\gamma$.

Proposition 7. The number of quantization levels N_g, generated by Algorithm 1 for a given performance parameter γ, is bounded by the following function:

$$\hat{N}=\begin{cases} 4+2\left\lceil\dfrac{\operatorname{atan2}\left((1-2q-pq)\sqrt{\Delta};\, p^2q+qp+4q+p-2q^2-1\right)}{\operatorname{atan}\frac{\sqrt{\Delta}}{p+1}}\right\rceil & \gamma\leq\dfrac{4}{5} \\ 4+2\left\lceil\dfrac{\log\frac{\sigma_2}{\sigma_1}\frac{(\sigma_1-k)(q-\sigma_1)}{(\sigma_2-k)(q-\sigma_2)}}{\log\frac{\sigma_2}{\sigma_1}}\right\rceil & \gamma>\dfrac{4}{5} \end{cases}, \tag{18}$$

where $p = \frac{-\gamma^2-\frac{3}{2}\gamma+1}{\sqrt{\gamma^4+1}}$, $q = \frac{\sqrt{2}\gamma^2}{\sqrt{\gamma^4+1}}$, $\Delta = 4q - (p-1)^2$, $k = \frac{\sqrt{2}\gamma^2}{\sqrt{2}\gamma^2+\sqrt{1+\gamma^4}}\left(\frac{1-\gamma^2}{\sqrt{1+\gamma^4}} + \sqrt{\frac{2\sqrt{2}\gamma^2}{\sqrt{1+\gamma^4}} - \frac{2\gamma^2(\gamma^2+1)}{\gamma^4+1}}\right)$, $\sigma_1 = \frac{1-\sqrt{1-4k}}{2}$ and $\sigma_2 = \frac{1+\sqrt{1-4k}}{2}$.

Furthermore

$$\lim_{\gamma\to 0^+}\gamma\hat{N}(\gamma)=\frac{(128\sqrt{2}+72)\sqrt{16\sqrt{2}-9}\left(\pi-\operatorname{atan}\frac{1}{3}\sqrt{16\sqrt{2}-9}\right)}{431}\approx 4.9, \tag{19}$$

thus proving that the required number of levels grows linearly with γ^{-1}.

Proof. See Appendix A.

Remark 8. We would like to mention that, while there is no performance level γ that can be achieved using a two level quantizer, a three level quantizer is capable of achieving some performance level but requires an ad-hoc design different from the procedure presented in Algorithm 1. Finally, the number of levels required can be sensibly lowered close to the theoretical bound of $1/\gamma$ if we use a less conservative and more cumbersome analysis approach. Some of these ideas and results will be presented in an upcoming paper.

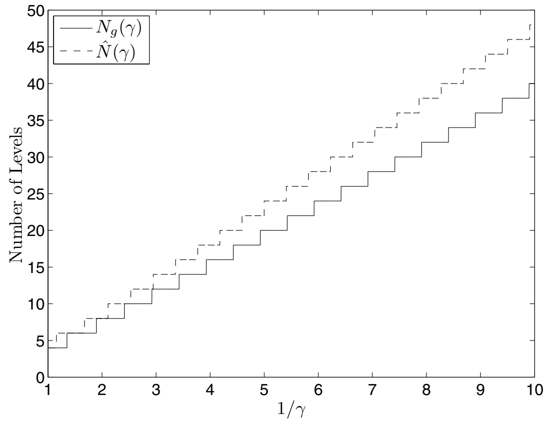

Fig. 3. The minimum number of levels N_g required by our proposed greedy algorithm and the analytic upper bound $\hat{N}$ as a function of the required performances.

5. NOISY CHANNEL ANALYSIS

In this section we present the main result of our paper. The channel is now assumed to be a discrete memoryless channel with input alphabet $\mathfrak{S}$, output alphabet $\mathfrak{V}$ and conditional probability matrix W. We will characterize a set of such channels for which the existence of a set of parameters $b, r_1, \ldots, r_n$ and a map Q guaranteeing almost sure asymptotic stability of (7) can be proved.

Theorem 9. Consider a discrete memoryless channel of cardinality $n = |\mathfrak{S}| = |\mathfrak{V}| \geq 4$ and conditional probability matrix W. Define the maximum probability of error as

$$\delta_e = \max_{i=1,\ldots n} 1 - w_{ii},$$

and

$$\gamma^* = \max\left\{\hat{N}^{-1}(n), \left(\frac{3\delta_e}{1-\delta_e}\right)^{1/4}\right\},$$

where $\hat{N}^{-1}(n)=\inf\{\gamma \,|\, \hat{N}(\gamma)\leq n\}$ and $\hat{N}$ is defined in (18). If

$$\gamma^*(1-\delta_e) + \frac{1}{(\gamma^*)^3}\delta_e < \frac{1}{|\lambda|},$$

then the dynamic in (7), and hence our original feedback loop, can be almost surely asymptotically stabilized by the controller designed using Algorithm 1 with parameter $\gamma = \gamma^*$.

Proof. The proof is based on the dynamic obtained when the parameters are designed according to Algorithm 1 for some performance level γ. Utilizing the same quadratic Lyapunov function with matrix P as in (15), we would like to prove the inequality [3]

$$E[V(x^+, y^+)|(x, y)] \leq \bar{\rho}V(x, y), \tag{20}$$

for a given $\bar{\rho}$.

[3] A function V satisfying (20) is called a super-martingale and it guarantees almost sure stability as shown in [10]

Independently of the current state, we know that when the channel does not make a transmission error, the parameters are designed so that

$$E[V(x^+, y^+)|(x, y), \text{ no error}] \leq \rho V(x, y). \quad (21)$$

When there is a communication error, since we cannot predict which symbol will be received, we can upper bound the ratio V^+/V via the following:

$$\begin{aligned} \frac{V^+}{V} &\leq \max_{\substack{|r|\in[r_0, 1] \\ (x, y)\in\mathbb{R}^2}} |\lambda| \frac{[x \ y] \begin{bmatrix} \frac{r}{\sqrt{\gamma}} & \frac{\sqrt{\gamma}}{|r|} \\ 0 & \frac{\sqrt{\gamma}}{|r|} \end{bmatrix} P \begin{bmatrix} \frac{r}{\sqrt{\gamma}} & 0 \\ \frac{\sqrt{\gamma}}{|r|} & \frac{\sqrt{\gamma}}{|r|} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}{[x \ y] P \begin{bmatrix} x \\ y \end{bmatrix}} \\ &= |\lambda| \frac{1 + 3\gamma^4 + \sqrt{9\gamma^8 - 16\gamma^6 + 6\gamma^4 + 1}}{4\gamma^3} \\ &\leq \frac{|\lambda|}{\gamma^3}. \end{aligned} \quad (22)$$

Combining the inequalities (21) and (22) with the known bound on the probability of a transmission error, we obtain:

$$E[V(x^+, y^+)|(x, y)] \leq \left[(1 - \delta_e)\rho + \delta_e \frac{|\lambda|}{\gamma^3}\right] V(x, y).$$

To enforce the inequality in (20), it is then sufficient that $(1 - \delta_e)\rho + \delta_e \frac{|\lambda|}{\gamma^3} \leq \bar{\rho}$, or, equivalently:

$$(1 - \delta_e)\gamma + \delta_e \frac{1}{\gamma^3} \leq \bar{\gamma}. \quad (23)$$

Since γ can be chosen arbitrarily given that $N_g(\gamma) \leq \hat{N}(\gamma) \leq n$ and the constrained minimum of the left hand side of (23) is achieved for $\gamma = \gamma^*$, the thesis is proven.

Remark 10. When the proposed algorithm generate less control signals r_i than the actual number of symbols, the result still holds provided that the remaining levels are selected in the set $[-1, -r_0] \cup [r_0, 1]$. Along the same idea, theorem 9 can be also applied to channels with $|\mathfrak{S}| \neq |\mathfrak{V}|$ provided that the extra levels are generated in the same set.

Remark 11. Since theorem 9 relies on the noiseless design for the controller parameters, the result is overconservative. In particular we observe that the bottleneck is given by the probability of error since achieving high performance parameters ($|\lambda| \to \infty$) requires that $\delta_e \to 0$.

Example 12. Consider a plant with parameter $\lambda = -1.1$ and a discrete memoryless channel described by the conditional probability matrix:

$$W = \begin{bmatrix} 0.98 & 0.01 & 0.04 & 0 \\ 0 & 0.95 & 0 & 0.03 \\ 0.01 & 0.03 & 0.96 & 0.02 \\ 0.01 & 0.01 & 0 & 0.95 \end{bmatrix}$$

In this case we have $n = 4$, $\delta_e = 0.05$, $\hat{N}^{-1}(4) \approx 0.865$ and $\gamma^* = 0.865$. Since

$$\gamma^*(1 - \delta_e) + \frac{1}{(\gamma^*)^3}\delta_e = 0.899 < \frac{1}{|\lambda|} = 0.909,$$

it is possible to design, using our proposed algorithm, a set of parameters such that the closed loop in figure 1 is almost surely asymptotically stable. In this case the parameters obtained are: $a = -1.1$, $\varphi_1 = -1.27$, $\varphi_2 = -0.91$, $\varphi_3 = 0.91$, $\varphi_4 = 1.27$, while the map H is given by:

$$H = s1 \text{ if } \tfrac{\eta}{z-\eta} \leq -6.87; \quad H = s2 \text{ if } -6.87 < \tfrac{\eta}{z-\eta} \leq -1;$$
$$H = s3 \text{ if } -1 < \tfrac{\eta}{z-\eta} \leq 4.87; \quad H = s4 \text{ if } \tfrac{\eta}{z-\eta} \geq 4.87.$$

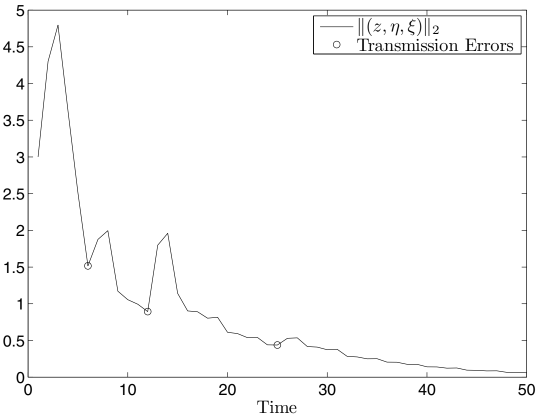

Fig. 4. One possible trajectory obtained for the designed closed loop with initial condition $(z(0), \eta(0), \xi(0)) = (2, 2, 1)$

An example of a possible trajectory obtained with the designed controller is shown in figure 4. We point out that the initial overshoot is not due to a communication error but to the fact that the system is not formally stable but only convergent.

6. CONCLUSIONS AND FUTURE WORKS

In this paper we presented a new scheme to asymptotically stabilize a one dimensional plant when the measurements are collected via a discrete memoryless channel. The novelty of our scheme lies in the a priori restriction on the amount of memory required, which is kept independent of the channel and the plant characteristics. Moreover the analysis completely based on a Lyapunov function has never be proposed for the type of controllers under investigation. We characterized a set of channels for which we can guarantee stabilizability through our scheme and show how its parameters can be designed via an algorithm of linear complexity in the level of instability of the plant.

We pointed out that the current design relies on the noiseless solution even when the channel makes mistakes. In this case, however, a better design technique capable of capturing the effect of the noise needs to be investigated.

We are also currently trying to extend these results to multidimensional SISO plants where observability issues arise.

REFERENCES

[1] G. N. Nair, F. Fagnani, S. Zampieri and R. J. Evans, *Feedback Control Under Data Rate Constraints: An Overview*, Proceedings of the IEEE, volume 95, issue 1, pages 108-137, 2007.

[2] N. Elia and S. K. Mitter, Stabilization of Linear Systems With Limited Information, IEEE Transaction

on automatic control, volume 46, issue 9, pages 1384-1400, 2001.

[3] R. W. Brockett and D. Liberzon, *Quantized Feedback Stabilization of Linear Systems*, IEEE Transaction on automatic control, volume 45, issue 7, pages 1279-1289, 2000.

[4] M. Fu and L. Xie, *Finite-level quantized feedback control for linear systems*, IEEE Transaction on automatic control, volume 54, issue 5, pages 1165-1170, 2009.

[5] G. N. Nair and R. J. Evans, *Exponential stabilisability of finite–dimensional linear systems with limited data rates*, Automatica, volume 49, pages 585-593, 2003.

[6] A. S. Matveev and A. V. Savkin, *An analogue of Shannon information theory for detection and stabilization via noisy discrete communication channels*, SIAM journal on control and optimization, volume 46, issue 4, pages 1323-1367, 2007.

[7] W. S. Wong and R. W. Brockett, *Systems with finite communication bandwidth constraints II: stabilization with limited information feedback*, IEEE Transaction on automatic control, volume 44, issue 5, pages 1049-1053, 1999.

[8] A. Sahai and S. Mitter, *The Necessity and Sufficiency of Anytime Capacity for Stabilization of a Linear System Over a Noisy Communication Link- Part I: Scalar Systems*, IEEE Transactions on information theory, volume 52, issue 8, pages 3369–3395, 2006.

[9] S. Tatikonda and S. Mitter, *Control Under Communication Constraints*, IEEE Transactions on automatic control, volume 49, issue 7, pages 1056–1068, 2004.

[10] R. S. Bucy, *Stability and Positive Supermartingales*, Journal of Differential Equations, volume 1, pages 151–155, 1965.

Appendix A. PROOF OF THE BOUND ON N_G

The number of levels required when using the proposed greedy algorithm is given by $2(N+1)$ where N is the number of iterations the algorithm performs before reaching the stopping condition.

From the description of the algorithm and the definitions of $\underline{r}$ and $\bar{r}$, it is straightforward to verify that the value r_i selected during a generic iteration is a function of the value r_{i-1} selected the previous iteration via an update map $r_i = f_\gamma(r_{i-1})$ where f_γ is given by:

$$f_\gamma(r) = \frac{r\frac{1-\gamma^2}{\sqrt{1+\gamma^4}} + \sqrt{r^4 - \frac{2\gamma^2(\gamma^2+1)}{\gamma^4+1}r^2 + \frac{2\gamma^4}{\gamma^4+1}}}{1-r^2}.$$

The total number of iterations can then be expressed as the smallest integer N such that

$$f_\gamma^{(N)}(r_0) \geq 1, \qquad r_0 = \frac{\sqrt{2}\gamma^2}{\sqrt{\gamma^4+1}}.$$

Since the iterations of f_γ are prohibitive to handle, an upper bound on N can be obtained by providing another map g_γ such that $g_\gamma \leq f_\gamma \quad \forall r \in [0,\,1]$ and then finding the minimum N such that $g_\gamma^{(N)}(r_0) \geq 1$. We will bound f_γ with two different lower bounds depending on γ.

Let us first consider the case $\gamma \in \left(0,\, \frac{4}{5}\right]$ and the map

$$g_\gamma(r) = \frac{\frac{-\gamma^2 - \frac{3}{2}\gamma + 1}{\sqrt{\gamma^4+1}} r + \frac{\sqrt{2}\gamma^2}{\sqrt{\gamma^4+1}}}{1-r} = \frac{pr+q}{1-r}. \qquad \text{(A.1)}$$

After some basic algebraic manipulation, the irrational inequality $f_\gamma \geq g_\gamma$ can be reduced to checking the non-negativity of the polynomial

$$\begin{aligned} P(r,\gamma) \;=\; & -\left(3\gamma^2 + \frac{\gamma}{4} - 3\right) r^3 - 4\gamma^3 + 3\sqrt{2}\gamma^2 \\ & + \frac{\sqrt{2}}{4}(\gamma+2)(3\sqrt{2} - 4\gamma)(1-2\gamma)r^2 \\ & + \frac{2-\sqrt{2}}{8}\gamma\left(48(1+\sqrt{2})\gamma - 16\gamma^2 - 50 - 33\sqrt{2}\right) r \end{aligned}$$

in the region $(r,\gamma) \in [0,\,1] \times [0,\,4/5]$. Since this condition is easily verified, for example via sum of squares decomposition, the proposed g_γ is indeed a lower bound for $f_\gamma \quad \forall \gamma \in (0,\,4/5]$.

The particularly easy form of g_γ in (A.1) allows us to write a closed form for its N-th iteration. In particular, keeping in mind that $\Delta = 4q - (p-1)^2 \geq 0$, we obtain

$$g_\gamma^{(N+1)}(r) = \frac{\left[p\cos N\theta + \frac{p^2-p-2q}{\sqrt{\Delta}}\sin N\theta\right] r + q\left[\cos N\theta + \frac{p+1}{\sqrt{\Delta}}\sin N\theta\right]}{-\left[\cos N\theta + \frac{p+1}{\sqrt{\Delta}}\sin N\theta\right] r + \left[\cos N\theta + \frac{1-p-2q}{\sqrt{\Delta}}\sin N\theta\right]},$$

where $\theta = \text{atan}\,\frac{\sqrt{\Delta}}{p+1}$.

Finally, after noticing that $r_0 = q$, in this case the inequality $g_\gamma^{(N)}(r_0) \geq 1$ leads to

$$(pq+2q-1)\sqrt{\Delta}\cos(N-1)\theta + (q(p^2+p+4)+p-2q^2-1)\sin(N-1)\theta \geq 0,$$

which is satisfied when

$$(N-1)\theta \geq \text{atan2}\left((1-2q-pq)\sqrt{\Delta};\, q(p^2+p+4)+p-2q^2-1\right),$$

thus proving the proposed bound when $\gamma \in (0,\,4/5]$.

In the case $\gamma \in (4/5,\,1)$, an easy lower bound valid for $r \in (r_0,\,1]$ can be obtained as follows:

$$\begin{aligned} f_\gamma(r) \;&=\; \frac{r}{r+1}\,\frac{\frac{1-\gamma^2}{\sqrt{1+\gamma^4}} + \sqrt{r^2 + \frac{2\gamma^4}{r^2(1+\gamma^4)} - \frac{2\gamma^2(\gamma^2+1)}{\gamma^4+1}}}{1-r} \\ &\geq\; \frac{r_0}{r_0+1}\,\frac{\frac{1-\gamma^2}{\sqrt{1+\gamma^4}} + \sqrt{\frac{2\sqrt{2}\gamma^2}{\sqrt{1+\gamma^4}} - \frac{2\gamma^2(\gamma^2+1)}{\gamma^4+1}}}{1-r} \\ &=\; \frac{k}{1-r} = g_\gamma(r). \end{aligned}$$

Once again, the simple structure of g_γ allows us to compute its N-th iteration. After some basic calculations we obtained:

$$g_\gamma^{(N)}(r) = \frac{k}{\sigma_1}\,\frac{(1-(\sigma_2/\sigma_1)^{N-1})\frac{r}{\sigma_1} + ((\sigma_2/\sigma_1)^N - 1)}{(1-(\sigma_2/\sigma_1)^N)\frac{r}{\sigma_1} + ((\sigma_2/\sigma_1)^{N+1} - 1)},$$

and the inequality $g_\gamma^{(N)}(r_0) \geq 1$ can easily be reduced to

$$N \geq \frac{\log \frac{\sigma_2}{\sigma_1}\frac{(\sigma_1-k)(r_0-\sigma_1)}{(\sigma_2-k)(r_0-\sigma_2)}}{\log\frac{\sigma_2}{\sigma_1}},$$

thus proving the proposed bound for $\gamma \in (4/5,\,1)$.