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On the Computation of the Response of Perturbed Discrete Time Descriptor Systems ⋆

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Abstract: Descriptor systems provide the natural framework for the study of a wide variety of physical, electrical, mechanical, economical and social systems. In this paper, the response of a Linear Time Invariant (LTI), descriptor system in discrete-time over a finite time interval is examined, whose coefficient matrix on the right hand side of the descriptor equation has been perturbed by a constant matrix. The response of the perturbed system is explicitly computed using a modified version of the well known Weierstrass canonical form and a simplified approximation formula is derived. A numerical example illustrates the findings.

Keywords: descriptor systems, perturbation, response approximation

1. INTRODUCTION

In the literature of system and control theory, linear time invariant descriptor (also known as generalized state space or differential-algebraic) systems play a key role in the dynamic behavior and modeling process of different type of physical, electrical, mechanical, economical and social systems. In the last four decades, the development of theoretical and numerical methods that can be directly applied to the descriptor systems have been the subject of a large number of research projects and many different mathematical directions have been taken.

In this paper, the Linear Time Invariant (LTI) discrete-time descriptor system of the form

$$Ex_{k+1} = Ax_k + Bu_k,\ k = 0, 1, 2, \dots, L \qquad (1)$$

is considered to be the nominal (or base) system, where $E, A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The vectors x_k and u_k are respectively the state (descriptor) and input vector sequences of the system described by (1).

If E is a singular matrix, that is if $rankE = r < n$, the system (1) may be non − causal depending on the index of the matrix E. A well known result (see Dai [1989]), which provides a characterization of the causality of (1), states that the system (1) is causal if and only if $\deg|\lambda E - A| = rankE$.

Practically speaking, when the system (1) is causal, its state can be decomposed into two parts corresponding to a pure state space system and set of algebraic constraints (see for instance Lewis [1986], Dai [1989]). Under the assumption of causality and regularity, with appropriate choice of initial conditions, equation (1) is uniquely solvable in a forward fashion for $k \in \mathbb{N}$.

However, the non causal case is also of particular interest when it comes to applications such as economical models (see Luenberger [1977]), the solution of discrete time Riccati equation (see Bender and Laub [1987]), or systems where the variable k is spatial rather than temporal. In this case the usual approach (see for instance Luenberger [1977], Lewis [1984], Dai [1989]) is to consider the response of (1) over a finite time interval that is $k = 0, 1, 2, \dots, L$ for some fixed $L \in \mathbb{N}$, given admissible initial and terminal conditions x_0, x_L along with the input u_k for every k in the above interval.

For the purposes of this paper, it will be assumed that the matrix pencil $\lambda E - A$ is regular, i.e. that the determinant $|\lambda E - A|$ is not identically zero. Furthermore, it will be assumed that the matrix pencil $\lambda E - A$ has no eigenvalues on the unit circle.

We are particularly interested in the state response of the perturbed version of (1)

$$Ex_{k+1} = (A + \Delta A)x_k + Bu_k, \qquad (2)$$

where $\Delta A \in \mathbb{R}^{n\times n}$ is the unstructured perturbation of the matrix A. Following a similar approach to Qiu and Davison [1992], Fang and Chang [1993], Fang et al. [1994] and Lin et al. [2003], we are only concerned with the perturbation of A. This can be justified by the fact that the matrix E usually a “structure” rather than a “parameter matrix”. Furthermore, as shown in Qiu and Davison [1992], under certain assumptions, a perturbation

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problem where both E, A are subject to variations, can be be transformed to an equivalent one where only the matrix A is perturbed. Finally, (see Nye [1985], Pantelous et al. [2011]) we shall assume that the matrices E, A have already been appropriately scaled, thus it is reasonable to assume that the magnitude of the perturbation is small compared to unity, that is $\|\Delta A\| = O(\epsilon)$ for $\epsilon \to 0$.

The relationship between (1) and (2) is of considerable high importance in numerical analysis and theory of uncertainties since it evaluates the derived accuracy of any particular method used to construct the solution of the nominal system, see Nye [1985] and Pantelous et al. [2011]).

In what follows, unless otherwise stated, the norm $\|.\|$ will denote the matrix 2 - norm. By sp(.) and $\rho(.)$, we shall denote the spectrum and spectral radius of the matrix in the brackets respectively.

2. COMPUTATION OF THE STATE RESPONSE

In this section, the preliminary results for the computation of the state response of the perturbed system (2) will be considered using elements of matrix pencil theory. As it will be clear later, a slightly modified version of the classical Weierstrass canonical form of the matrix pencil $\lambda E - A$ corresponding to the unperturbed system (1) will be used eventually.

Since the pencil $\lambda E - A$ has been assumed to be regular, we have the following types of elementary divisors:

- Elementary divisors of the form$(\lambda - a)^{\pi}$, $a \in \mathbb{C}$, called *finite elementary divisors*, and
- Finite elementary divisors of the "dual" pencil $E - \hat{\lambda}A$, of the form $\hat{\lambda}^q$, which give rise to the *infinite elementary divisors* of $\lambda E - A$.

Denote also the $m \times m$ identity matrix by I_m, the $m \times m$ nilpotent matrix with ones on the super-diagonal and zeros elsewhere by N_m (index of nilpotency m), and the $m \times m$ Jordan block $aI_m + N_m$ corresponding to eigenvalue a by $J_m(a)$.

Then, the Weierstrass form $\lambda E_w - A_w$ of the regular pencil $\lambda E - A$ is defined by

$$\lambda E_w - A_w = \begin{bmatrix} \lambda I_p - J_p & 0 \\ 0 & \lambda N_q - I_q \end{bmatrix}, \tag{3}$$

where the matrix $\lambda I_p - J_p$ consists of blocks of the form

$$\lambda I_p - J_p = \begin{bmatrix} \lambda I_{p_1} - J_{p_1}(a_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda I_{p_v} - J_{p_v}(a_v) \end{bmatrix}, \tag{4}$$

corresponding to the finite elementary divisors of $\lambda E - A$, of the form $(\lambda - a_1)^{p_1}, \ldots, (\lambda - a_\nu)^{p_\nu}$, where $\sum_{j=1}^{\nu} p_j = p$. The second block $\lambda N_q - I_q$ corresponds to the infinite elementary divisors $\hat{\lambda}^{q_1}, \ldots, \hat{\lambda}^{q_\sigma}$, where $\sum_{j=1}^{\sigma} q_j = q$, of $\lambda E - A$ and consists of blocks of the form

$$\lambda N_q - I_q = \begin{bmatrix} \lambda N_{q_1} - I_{q_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda N_{q_\sigma} - I_{q_\sigma} \end{bmatrix}. \tag{5}$$

In this respect, there exist P, Q square invertible matrices transforming the pencil $\lambda E - A$ to its Weierstrass canonical form (see for instance Gantmacher [1959]), that is

$$PEQ = \begin{bmatrix} I_p & 0 \\ 0 & N_q \end{bmatrix}, \qquad PAQ = \begin{bmatrix} J_p & 0 \\ 0 & I_q \end{bmatrix}, \tag{6}$$

for I_p, J_p, N_q and I_q, where

$$I_p = \begin{bmatrix} I_{p_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{p_v} \end{bmatrix}, \ J_p = \begin{bmatrix} J_{p_1}(a_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{p_v}(a_v) \end{bmatrix},$$

$$N_q = \begin{bmatrix} N_{q_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & N_{q_\sigma} \end{bmatrix}, \text{and } I_q = \begin{bmatrix} I_{q_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{q_\sigma} \end{bmatrix}$$

Note that $\sum_{j=1}^{\nu} p_j = p$ and $\sum_{j=1}^{\sigma} q_j = q$, where $p + q = n$. With the following lemma, a very useful, modified version of (6) is derived.

Lemma 1. Given a regular matrix pencil $\lambda E - A$, with no eigenvalues on the unit circle, there exist square invertible matrices U, V such that

$$UEV = \begin{bmatrix} I_\mu & 0 \\ 0 & J_B \end{bmatrix}, \qquad UAV = \begin{bmatrix} J_F & 0 \\ 0 & I_{n-\mu} \end{bmatrix}, \tag{7}$$

where $J_F \in \mathbb{R}^{\mu \times \mu}$, $J_B \in \mathbb{R}^{(n-\mu)\times(n-\mu)}$ are in Jordan canonical form and satisfy $\rho(J_F) < 1$ and $\rho(J_B) < 1$.

Proof. In the standard decomposition (6), we rearrange the blocks J_μ, using an appropriate permutation similarity matrix S to obtain

$$SJ_pS^{-1} = \begin{bmatrix} J_F & 0 \\ 0 & \bar{J}_B \end{bmatrix},$$

such that $\forall a_i \in \text{sp}(J_F)$, $|a_i| < 1$, while $\forall a_i \in \text{sp}(\bar{J}_B)$, $|a_i| > 1$. Thus, $\rho(J_F) \leq 1$ and $\bar{J}_B$ is invertible and $\rho(\bar{J}_B^{-1}) < 1$.

Now, let M be a square invertible matrix, such that $M\bar{J}_B^{-1}M^{-1} = \widetilde{J}_B$, where $\widetilde{J}_B$ is in Jordan canonical form. If we set

$$U = \text{diag}[I_\mu, M\bar{J}_B^{-1}, I_q]\,\text{diag}[S, I_q]\,Q,$$
$$V = P\,\text{diag}[S^{-1}, I_q]\,\text{diag}[I_\mu, M^{-1}, I_q],$$

it is easy to verify that (7) holds for

$$J_B = \begin{bmatrix} \widetilde{J}_B & 0 \\ 0 & N \end{bmatrix}.$$

Taking into account the definitions of $\widetilde{J}_B$ and N, it is trivial to verify that $\rho(J_B) < 1$. □

Using the above result we set accordingly

$$UB = \begin{bmatrix} B_F \\ B_B \end{bmatrix}, \qquad \begin{bmatrix} y_k \\ z_k \end{bmatrix} = V^{-1}x_k. \tag{8}$$

With the above notation and the assumption of absence of unitary eigenvalues on $\lambda E - A$, the descriptor system (1) can be decomposed into a forward and a backward subsystem

$$y_{k+1} = J_F y_k + B_F u_k \tag{9}$$
$$J_B z_{k+1} = z_k + B_B u_k. \tag{10}$$

In view of the above decomposition, it is natural to treat the problem as a two point boundary condition problem over the finite time interval $[0, L]$. The subsystem corresponding to (9) expresses the propagation of the initial conditions y_0 in the forward direction, while (10) reflects the propagation of the final conditions z_L backwards. Notice that both forward and backward transition matrices J_F,J_B, are by construction Schur stable.

Following an approach similar to that of Dai [1989], the following result can be derived,

Lemma 2. The responses of subsystems (9) and (10) are respectively

$$y_k = J_F^k y_0 + \sum_{i=0}^{k-1} J_F^{k-i-1} B_F u_i \tag{11}$$

and

$$z_k = J_B^{L-k} z_L - \sum_{i=0}^{L-k-1} J_B^i B_F u_{k+i}, \tag{12}$$

where the initial conditions, y_0 and z_L, can be arbitrarily chosen. Furthermore, the overall state response of (1) is given by

$$x_k = V_1 \left(J_F^k y_0 + \sum_{i=0}^{k-1} J_F^{k-i-1} B_F u_i \right) + V_2 \left(J_B^{L-k} z_L - \sum_{i=0}^{L-k-1} J_B^i B_F u_{k+i} \right), \tag{13}$$

where V_1, V_2 are block columns of the matrix V of appropriate dimensions.

Proof. Equation (9) is essentially an ordinary state space system, so after some simple algebraic calculations, it gives rise to a response of the form (11). On the other hand for equation (10), the formula (12) is derived by propagating backwardly the final condition z_L; see also Dai [1989]. The overall response (13) is then easily given in the view of (8). □

We now focus on the computation of the response of the perturbed equation (2). Let us assume that

$$U \Delta A V = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}, \tag{14}$$

where the dimensions of Δ_{ij} in the above partitioning are in accordance to the ones of the matrices in the decomposition (7).

Applying the decomposition (7) into the perturbed descriptor system (2), the following two coupled subsystems derive:

$$y_{k+1} = (J_F + \Delta_{11}) y_k + \Delta_{12} z_k + B_F u_k \tag{15}$$

$$J_B z_{k+1} = \Delta_{21} y_k + (I + \Delta_{22}) z_k + B_B u_k \tag{16}$$

In view of the above decomposition, the following result is derived.

Lemma 3. The response of (15) is given by

$$y_k = (J_F + \Delta_{11})^k y_0 + \sum_{i=0}^{k-1} (J_F + \Delta_{11})^{k-i-1} (\Delta_{12} z_i + B_F u_i). \tag{17}$$

Assuming that $(I + \Delta_{22})$ is invertible, the response of (16) is given respectively by

$$z_k = \left[(I + \Delta_{22})^{-1} J_B \right]^{L-k} z_L - \sum_{j=0}^{L-k-1} \left[(I + \Delta_{22})^{-1} J_B \right]^j (\Delta_{21} y_{k+j} + B_B u_{k+j}). \tag{18}$$

Proof. The response formulas (15) and (16) derive straightforwardly from (11) and (12) respectively, if we consider the term $\Delta_{12} z_k + B_F u_k$ in (15) as the input term $B_F u_k$ in (9) and the term $(I + \Delta_{22}) z_k + B_B u_k$ in (16) as the input term $B_F u_k$ in (10), accordingly. □

Practically speaking, now the new challenge with both equations (17) and (18), is raised by the fact that the two responses are given in an implicit, coupled form, and consequently their solution can not be calculated only in terms of the boundary conditions and their inputs. Fortunately enough, this difficulty is disappearing with the proposed approximations and eventually two interesting formulas for the computation of the response are obtained as it will present in the next section.

3. APPROXIMATE RESPONSE

In this section we elaborate an approximate response formula for the decomposition of the perturbed descriptor system (2).

Lemma 4. The forward response of equation (15) is given by

$$y_k = (J_F + \Delta_{11})^k y_0 + \sum_{i=0}^{k-1} F(k-i-1, L-i) z_L - \sum_{i=0}^{k-1} \sum_{j=0}^{L-i-1} F(k-i-1, j) (\Delta_{21} y_{i+j} + B_B u_{i+j}) + \sum_{i=0}^{k-1} (J_F + \Delta_{11})^{k-i-1} B_F u_i, \tag{19}$$

where

$$F(i, j) = (J_F + \Delta_{11})^i \Delta_{12} \left[(I + \Delta_{22})^{-1} J_B \right]^j. \tag{20}$$

The backward response of equation (16) is given by

$$z_k = \left[(I + \Delta_{22})^{-1} J_B \right]^{L-k} z_L + - \sum_{j=0}^{L-k-1} B(j, k+j) y_0 - \sum_{j=0}^{L-k-1} \sum_{i=0}^{k+j-1} B(j, k+j-i-1) (\Delta_{12} z_i + B_F u_i) - \sum_{j=0}^{L-k-1} \left[(I + \Delta_{22})^{-1} J_B \right]^j B_B u_{k+j}, \tag{21}$$

where

$$B(i, j) = \left[(I + \Delta_{22})^{-1} J_B \right]^i \Delta_{21} (J_F + \Delta_{11})^j. \tag{22}$$

Proof. Equation (19) is obtained by substituting (18) into (17), while (21) can be obtained by applying the reverse substitution. □

In view of the discussion so far, and particularly in the introduction, we assume that $\|\Delta A\| = O(\epsilon)$, for $\epsilon \to 0$, thus since

$$\left\| \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \right\| \leq \|U\| \, \|\Delta A\| \, \|V\|$$

it reasonable to assume that the $\|\Delta_{ij}\| = O(\epsilon)$. Now, since $\epsilon \ll 1$, it follows that $\|\Delta_{ij}\| < 1$ so the matrix $(I + \Delta_{22})$ is invertible (see for instance lemma 2.1, Demmel [1997]) and

$$(I + \Delta_{22})^{-1} = \sum_{i=0}^{\infty} (-\Delta_{22})^i, \tag{23}$$

hence

$$\left\|(I + \Delta_{22})^{-1} - (I - \Delta_{22})\right\| = O(\epsilon^2). \tag{24}$$

Now, since $\epsilon \ll 1$, we can neglect second order terms and apply the following approximation

$$(I + \Delta_{22})^{-1} \approx I - \Delta_{22}. \tag{25}$$

Similar approximations can be considered for the matrices $F(i,j)$ and $B(i,j)$. Thus, the following lemma derives as a consequence of these approximations.

Lemma 5. For every $i, j \in \mathbb{N}$, the following two expressions hold:

$$\left\|F(i,j) - J_F^i \Delta_{12} J_B^j\right\| = O(\epsilon^2), \tag{26}$$

and

$$\left\|B(i,j) - J_B^i \Delta_{21} J_F^j\right\| = O(\epsilon^2). \tag{27}$$

Proof. If $i = j = 0$, then $F(0,0) - \Delta_{12} = 0$, so (26) holds trivially. Let at least one of the indices i, j be greater than 0. Taking into account the definition of $F(i,j)$ which has been derived from the expressions (20) and (23), it is easy to see that

$$\begin{aligned} F(i,j) &= (J_F + \Delta_{11})^i \Delta_{12} \left[(I + \Delta_{22})^{-1} J_B\right]^j \\ &= J_F^i \Delta_{12} J_B^j + R(i,j), \end{aligned}$$

where $R(i,j)$ consists of terms, involving multiplicatively at least one of the matrices Δ_{11}, Δ_{12} along with the matrix Δ_{12}. Hence, since $\|\Delta_{ij}\| = O(\epsilon)$, we get

$$\left\|F(i,j) - J_F^i \Delta_{12} J_B^j\right\| = \|R(i,j)\| = O(\epsilon^2).$$

Equation (27) follows similarly.□

In view of equations (26) and (27), if we choose to neglect the second order terms, we can adopt the following approximations

$$F(i,j) \approx J_F^i \Delta_{12} J_B^j, \tag{28}$$

$$B(i,j) \approx J_B^i \Delta_{21} J_F^j. \tag{29}$$

Furthermore, it can be easily verified that $\|F(i,j)\Delta_{21}\| = O(\epsilon^2)$ and $\|B(i,j)\Delta_{12}\| = O(\epsilon^2)$, so we can apply the approximations

$$F(i,j)\Delta_{21} \approx 0, \qquad B(i,j)\Delta_{12} \approx 0. \tag{30}$$

Taking into account the whole discussion so far, we can state here our main result.

Theorem 6. If $\|\Delta_{ij}\| = O(\epsilon)$, for $\epsilon \to 0$ and $i, j = 1, 2$, the approximate response of (2) is given by

$$\hat{x}_k = V_1 \hat{y}_k + V_2 \hat{z}_k \tag{31}$$

where

$$\begin{aligned} \hat{y}_k = {} & (J_F + \Delta_{11})^k y_0 + \sum_{i=0}^{k-1} J_F^{k-i-1} \Delta_{12} J_B^{L-i} z_L \\ & - \sum_{i=0}^{k-1} \sum_{j=0}^{L-i-1} J_F^{k-i-1} \Delta_{12} J_B^j B_B u_{i+j} \\ & + \sum_{i=0}^{k-1} (J_F + \Delta_{11})^{k-i-1} B_F u_i \end{aligned} \tag{32}$$

and

$$\begin{aligned} \hat{z}_k = {} & \left[(I - \Delta_{22}) J_B\right]^{L-k} z_L - \sum_{j=0}^{L-k-1} J_B^j \Delta_{21} J_F^{k+j} y_0 \\ & - \sum_{j=0}^{L-k-1} \sum_{i=0}^{k+j-1} J_B^j \Delta_{21} J_F^{k+j-i-1} B_F u_i \\ & - \sum_{j=0}^{L-k-1} \left[(I - \Delta_{22}) J_B\right]^j B_B u_{k+j} \end{aligned} \tag{33}$$

where V_1, V_2 are block columns of appropriate dimensions of the transformation matrix V in (7).

Proof. The approximate forward and backward responses $\hat{y}_k, \hat{z}_k$ in (32) and (33) follow by taking into account (25), (28), (29) and (30) combined with (19) and (21). It is clear that

$$\|\hat{y}_k - y_k\| = O(\epsilon^2), \qquad \|\hat{z}_k - z_k\| = O(\epsilon^2), \tag{34}$$

hence we can write

$$\hat{y}_k \approx y_k, \qquad \hat{z}_k \approx z_k.$$

The entire approximate response $\hat{x}_k$, which is expressed in the original coordinate system, can be recovered using transformation (8). □

At this part of the section, with the following example we illustrate the main findings of this paper.

Example 7. Consider the LTI descriptor system (2) with

$$E = \begin{pmatrix} 12.6 & -4.32 & 6. & 10.44 & 3.24 \\ 7.8 & 0.72 & 9. & 2.76 & -0.84 \\ 17.04 & -7.2 & 14.4 & 24. & 8.04 \\ 4.44 & -2.88 & 2.4 & 7.56 & 3. \\ 0.84 & 5.76 & 13.2 & -2.52 & -3.84 \end{pmatrix},$$

$$A = \begin{pmatrix} 22.62 & -8.4 & 2.16 & 13.62 & 7.32 \\ 14.34 & -5.4 & -7.86 & -4.62 & 1.08 \\ 22.68 & -2.64 & 51.96 & 36.6 & 7.32 \\ 20.46 & -8.16 & 5.88 & 11.58 & 6.6 \\ -6.3 & 3.84 & 1.56 & -12.06 & -5.52 \end{pmatrix},$$

$$B = \begin{pmatrix} -6.6 \\ -9.48 \\ 14.04 \\ 0.6 \\ -6.48 \end{pmatrix},$$

and the perturbation matrix is given by

$$\Delta A = 10^{-2} \times \begin{pmatrix} -0.8 & 0.9 & 0.8 & 0.2 & 0. \\ 0.7 & -0.5 & -0.7 & 0.7 & 0.4 \\ -1. & -0.7 & -0.9 & -0.5 & 1. \\ -0.4 & 0.1 & 0.4 & 0.1 & 0.7 \\ 1. & 0.1 & -1. & 0.5 & -0.8 \end{pmatrix}.$$

Applying the decomposition (7), we obtain

$$U = \begin{pmatrix} 0.5693 & -1.073 & -0.3348 & 0.4568 & 0.7946 \\ -0.2852 & -0.01720 & -0.05354 & 0.2500 & -0.1537 \\ 0.4688 & -0.7158 & -0.09896 & 0.2256 & 0.4302 \\ 0.2295 & -0.2240 & -0.1644 & 0.3421 & 0.1656 \\ 0.2126 & -0.2581 & -0.01831 & -0.1020 & 0.1179 \end{pmatrix},$$

$$V = \left(\begin{array}{cc|ccc} 0.1190 & -0.3514 & -0.2216 & 0.3333 & 0.5556 \\ 0.6940 & -0.8433 & -0.4335 & 1.306 & 2.648 \\ 0.08571 & 0.1405 & 0.4127 & -0.3000 & -1.000 \\ -0.3857 & -0.1405 & -0.4266 & 0 & 1.000 \\ 1.000 & 0 & 1.000 & 1.000 & 0 \end{array}\right),$$

$$J_F = \begin{pmatrix} 0.5 & 1 \\ 0 & 0.5 \end{pmatrix}, \qquad J_B = \begin{pmatrix} 0.33333 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} B_F \\ B_B \end{pmatrix} = \left(\begin{array}{c} -3.15973 \\ 2.4395 \\ \hline -0.349468 \\ -2.56728 \\ -0.0383349 \end{array}\right).$$

According to (7) the perturbation matrix can be decomposed into the following four submatrices,

$$\Delta_{11} = 10^{-2} \times \begin{pmatrix} -0.0461645 & -0.966604 \\ 0.143594 & 0.189139 \end{pmatrix},$$

$$\Delta_{12} = 10^{-2} \times \begin{pmatrix} -1.25155 & 0.403384 & 2.6623 \\ 0.396645 & -0.0452686 & -0.703222 \end{pmatrix},$$

$$\Delta_{21} = 10^{-3} \times \begin{pmatrix} 1.11215 & -5.12927 \\ 1.76967 & -3.04589 \\ -0.266996 & -1.80678 \end{pmatrix},$$

and

$$\Delta_{22} = 10^{-3} \times \begin{pmatrix} -5.61819 & 2.89327 & 14.3386 \\ -1.72403 & 2.98434 & 8.29726 \\ -2.73531 & 0.293767 & 5.01235 \end{pmatrix}.$$

For the input $u_k = (0.9)^k \sin(0.3k)$, and the initial and terminal conditions $y_0 = (0\ 0)^T$, $z_{50} = (0\ 0\ 0)^T$ respectively, and $L = 50$, the state response of (2) can be approximated by (33). The approximate response $\hat{x}_k$ of the perturbed system (2) can be compared with the response x_k of the unperturbed system (1).

In figures 1 - 5 we compare the approximate response $\hat{x}_k^i$ of the perturbed system (2) to the response x_k^i of the unperturbed system (1), for $i = 1, 2, \ldots, 5$. The results are in accordance to our results

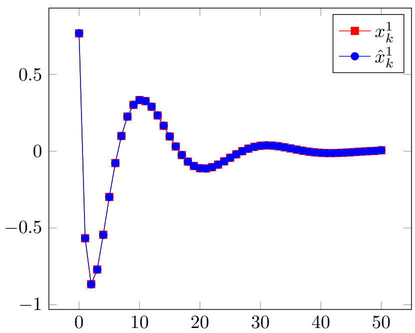

Fig. 1. Approximate perturbed $\hat{x}_k^1$ vs. unperturbed x_k^1

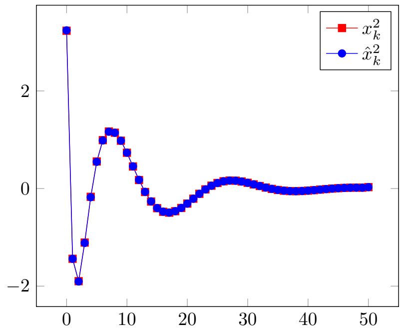

Fig. 2. Approximate perturbed $\hat{x}_k^2$ vs. unperturbed x_k^2

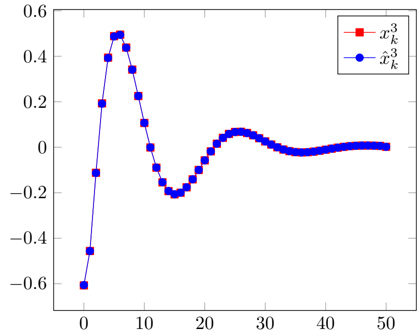

Fig. 3. Approximate perturbed $\hat{x}_k^3$ vs. unperturbed x_k^3

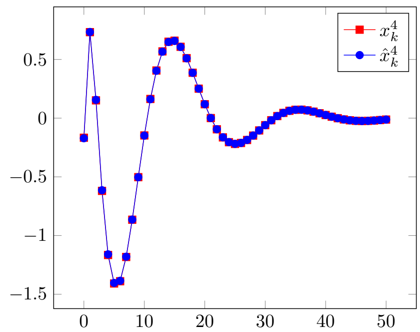

Fig. 4. Approximate perturbed $\hat{x}_k^4$ vs. unperturbed x_k^4

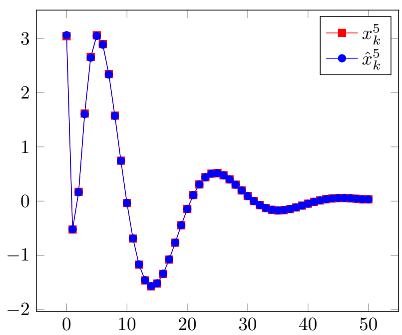

Fig. 5. Approximate perturbed $\hat{x}_k^5$ vs. unperturbed x_k^5

Finally, figure 6 shows the norm of the difference $\hat{x}_k - x_k$.

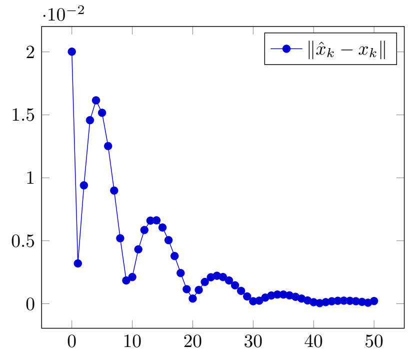

Fig. 6. Norm of the difference $\hat{x}_k - x_k$

We would like to stress out that in the above plots the comparison is made between the approximate response given by formula (31) and the unperturbed response (13). The norm of the difference $\hat{x}_k - x_k$, is as expected of order 10^{-2}. This should be contrasted to result of relations (34) according to which the approximation error should be of order 10^{-4}. Such a comparison would probably be more appropriate, but the computation of the exact perturbed response using the result of Lemma 4 is quite complex in view of the coupling between the forward and backward subsystems.

4. CONCLUSIONS - FURTHER RESEARCH DIRECTIONS

In this paper, an approximate formula for the computation of the state response of a LTI descriptor system is calculated, whose coefficient matrix A is subject to a relatively small perturbation ΔA. The approximation formula is derived using a modified version of the well known Weierstrass decomposition. Using this particular decomposition we obtain a separation of the descriptor variable space into two subspaces, corresponding to forward and backward subsystems with stable transition matrices. In this setting, the approximation formula is obtained by making certain assumptions on the magnitude of the perturbation matrix. This process is of considerable importance in numerical analysis since it has a direct bearing upon the accuracy of any particular method used to construct the solution of the base system.

As directions for further research in the subject, the following topics will be considered

- Investigation of the perturbation problem for the more general class of possibly singular linear descriptor systems with consistent and non-consistent initial conditions. In this case, a modified version of the Kronecker canonical form could be employed.
- Comparison of the results of the present paper to those derived using the Drazin inverse approach for the solution of the base system. In the literature of descriptor (regular and singular) systems, both Drazin inverse and matrix pencil theory approaches have been extensively used for the solution of such kind of systems. Inevitably, a comparison between these two methods has its own merit.
- Finally, fuhrer numerical aspects should be considered. Orthogonal transformations, used in the computation of the generalized Schur form of the matrix pencil, would result in a decomposition that is more appropriate for the study of the error produced by the perturbation ΔA. Such an approach is expected to provide more accurate error bounds for the perturbed response.

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