Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Mean-square Stabilization of Invariant Manifolds for SDEs

Lev Ryashko, Irina Bashkirtseva *

* *Ural Federal University, Lenina, 51, 620083, Ekaterinburg, Russia (e-mail: irina.bashkirtseva@urfu.ru).*

Abstract: We consider systems of Ito's stochastic differential equations with smooth compact invariant manifolds. The problem addressed is an exponential mean square (EMS) stabilization of these manifolds. The necessary and sufficient conditions of the stabilizability are derived on the base of the spectral criterion of the EMS-stability of invariant manifolds. We suggest methods for the design of the feedback stabilizing regulator for SDEs. Parametrical criteria of the stochastic stabilizability for limit cycles and tori are given. These criteria reduce the stabilization problem to the minimization of quadratic functionals. An analysis of the minimization problem of the quadratic functional for the case of the cycle of 2D stochastic system is presented in detail. Constructiveness of the elaborated theory is demonstrated for the stabilization of stochastically forced cycles of the Hopf system.

Keywords: stabilization, invariant manifold, cycles, tori, feedback.

1. INTRODUCTION

The stability investigation and control of oscillation systems are attractive from theoretical and engineering points of view.

It is well known that oscillations are operating modes of the modern engineering devices, chemical reactors, electronic generators and so on. From the mathematical point of view, periodic and quasi-periodic oscillations are transient regimes in the chain of bifurcations from order to chaos. Invariant manifold is a convenient general mathematical model for the stability analysis of these various nonlinear oscillations (limit cycles of various multiplicity, tori).

Most real systems operate in the presence of noise. Indeed, even weak noise can result in qualitative changes in the nonlinear system's dynamics. Control problems of nonlinear stochastic systems attract attention of many researchers [Sun (2006), Guo and Wang (2010)].

In this paper, we consider the exponential mean square stability and stabilization problem for invariant manifolds of stochastic differential equations (SDEs).

One of the most important methods of the stability analysis is the Lyapunov function technique (LFT) [Khasminskii (1980), Kushner (1967)]. LFT in the research of the stochastic stability of equilibria has been widely studied by many authors (see [Arnold (1998), Mao (1994)]). A problem of the synthesis of stochastic attractors and controlling chaos was investigated in [Chen and Yu (2003), Bashkirtseva et al. (2012)].

The orbital Lyapunov functions were used in the stability and sensitivity analysis of stochastically forced limit cycles [Ryashko (1996), Bashkirtseva and Ryashko (2004), Bashkirtseva et al. (2013)]. LFT for the stability analysis of the general invariant manifolds is considered for deterministic [Ryashko and Shnol (2003)] and stochastic [Ryashko (2007)] systems. On the base of LFT, a general spectral criterion of EMS stability of manifolds has been proved [Ryashko and Bashkirtseva (2011)].

The aim of this work is to apply this criterion to the solution of the control problem and show how it works by numerical simulations.

2. STOCHASTIC STABILITY

Consider a deterministic nonlinear system

$$dx \;=\; f(x)\,dt, \tag{1}$$

where x is $n-$vector, $f(x)$ is sufficiently smooth vector-function of the appropriate dimension. It is assumed that the system (1) has a smooth compact invariant manifold M (see for details [Fenichel (1971), Hirsch et al. (1977), Wiggins (1994)]).

Consider a function $\gamma(x)$ in a neighbourhood U of the manifold M. Here $\gamma(x)$ is a point of the manifold M that is nearest to x, $\Delta(x) = x-\gamma(x)$ is a vector of the deviation of the point x from the manifold M. It is assumed that the neighbourhood U is invariant for the system (1).

For any $x \in M$, denote by T_x the tangent subspace to M at x. Denote by N_x the orthogonal complement to T_x and by P_x the operator of the orthogonal projection onto the subspace N_x.

In this paper, we consider a randomly forced deterministic system (1) as follows:

$$dx \;=\; f(x)dt \;+\; \sum_{r=1}^{m}\sigma_r(x)dw_r(t), \tag{2}$$

where $w_r(t) \quad (r = 1, ..., m)$ are independent standard Wiener processes, $\sigma_r(x)$ are sufficiently smooth vector-functions of the appopriate dimension. To ensure M is an invariant of the stochastic system (2) we assume that

$$\sigma_r|_M \;=\; 0. \tag{3}$$

Definition 1. The manifold M is called exponentially stable in the mean square sense (EMS-stable) for the system (2) in U if there exist $K > 0,\ l > 0$ such that

$$\mathrm{E}||\Delta(x(t))||^2 \;\le\; Ke^{-lt}\mathrm{E}||\Delta(x_0)||^2,$$

where $x(t)$ is a solution of the system (2) with the initial condition $x(0) = x_0 \in U$.

Consider a space Σ of symmetrical $n \times n$ matrix functions $V(x)$ defined and sufficiently smooth on M and satisfying the following singularity condition

$$\forall x \in M\ \forall z \in T_x \quad V(x)z = 0.$$

On the space Σ, we shall consider operators:

$$\mathcal{A}[V] = \left(f, \frac{\partial V}{\partial x}\right) + F^\top V + VF,$$

$$\mathcal{S}[V] \;=\; \sum_{r=1}^{m} S_r^\top V S_r, \quad \mathcal{P} = -\mathcal{A}^{-1}\mathcal{S},$$

where

$$F(x) = \frac{\partial f}{\partial x}(x)\ ,\ S_r(x) = \frac{\partial \sigma_r}{\partial x}(x),$$

$(\cdot,\cdot)$ is the Euclidean scalar product. Note that an existence of the inverse operator $\mathcal{A}^{-1}$ follows from the exponential stability of the manifold M of the deterministic system (1).

Let $\rho(\mathcal{P})$ be a spectral radius of the operator $\mathcal{P}$.

Theorem 1. *The manifold M of the stochastic system (2) is EMS-stable if and only if*

(a) The manifold M of the deterministic system (1) is exponentially stable,

(b) The inequality $\rho(\mathcal{P}) < 1$ holds.

This theorem has been proved in [Ryashko and Bashkirtseva (2011)] on the base of the spectral theory of the positive operators [Krasnosel'skii et al. (1990)]. An analogous approach was used earlier in [Ryashko (1999)] for the stability analysis and stabilization of linear SDEs with periodic coefficients.

2.1 Stability of the Limit Cycle for 2D-system

We assume that an invariant manifold M is a limit cycle corresponding to T-periodic solution $\xi(t)$. The function $\xi(t)$ gives us a natural parametrization of the cycle orbit and defines the one-to-one correspondence between cycle points and the time interval $[0, T)$.

Using this parametrization, we introduce functions

$$F(t) = \frac{\partial f}{\partial x}(\xi(t)), \quad S_r(t) = \frac{\partial \sigma_r}{\partial x}(\xi(t))$$

defined on $[0, T]$.

In the case $n = 2$, for the spectral radius of the operator $\mathcal{P}$, one can find the following explicit formula:

$$\rho(\mathcal{P}) = -\frac{\prec \beta \succ}{\prec \alpha \succ}.$$

Here

$$\alpha(t) = p^\top(t)[F^\top(t) + F(t)]p(t),$$

$$\beta(t) = p^\top(t)\left(\sum_{r=1}^{m} S_r(t)S_r^\top(t)\right)p(t),$$

$p(t)$ is a vector orthonormal to the limit cycle M at the point $\xi(t)$, brackets $\prec \cdot \succ$ mean an integral with the time averaging:

$$\prec \alpha \succ = \frac{1}{T}\int_0^T \alpha(t)dt.$$

The inequality (famous Poincare criterion)

$$\prec \alpha \succ\ < 0$$

is a necessary and sufficient condition of the exponential stability of the limit cycle M for the deterministic system (1).

Thus, the inequality $\rho(\mathcal{P}) < 1$ written as

$$\prec \alpha + \beta \succ =$$

$$= \prec p^\top(t)\left[F^\top(t) + F(t) + \sum_{r=1}^{m} S_r(t)S_r^\top(t)\right]p(t) \succ < 0$$

is a necessary and sufficient condition of EMS-stability of the cycle M for the stochastic system (2) in 2D-case.

2.2 Stability of the Two-torus for 3D-system

Let an invariant manifold M of the system (1) for $n = 3$ be an two-dimensional toroidal surface.

Here, the following parametrization of 2-torus M is considered. Suppose some closed sufficiently smooth curve θ (equator) lies on the M (see Figure 1). This curve is defined by function $\theta(s)$ on the interval $0 \le s \le 1$ with the condition

$$\theta(0) \;=\; \theta(1).$$

Consider a solution $x(t, s)$ of the system (1) with the initial condition

$$x(0, s) \;=\; \theta(s).$$

It is supposed that the trajectory of $x(t, s)$ leaves the point $\theta(s)$ of a curve θ and after the rotation around the torus crosses this curve θ again. Let

$$T(s) \;=\; \min\{\ t > 0 \mid x(t, s) \in \theta\ \}$$

be the first return time of the trajectory $x(t, s)$ on the curve θ and $x(T(s), s)$ be the first return point. Let $\tau(s)$ be a point of the interval $[0, 1)$ where

$$\theta(\tau(s)) \;=\; x(T(s), s).$$

Here, $\tau(s)$ is the Poincare first return function for intersections of the curve θ by the phase trajectories of the system.

Torus M consists of phase trajectories $x(t, s)$ of the system (1). A function $x(t, s)$ defines one-to-one correspondence between 2-torus M points and points of the set

$$D \;=\; \{(t, s) \mid 0 \le t < T(s),\ 0 \le s < 1\}.$$

Vector-functions

$$\frac{\partial x(t, s)}{\partial t}\ ,\ \frac{\partial x(t, s)}{\partial s}$$

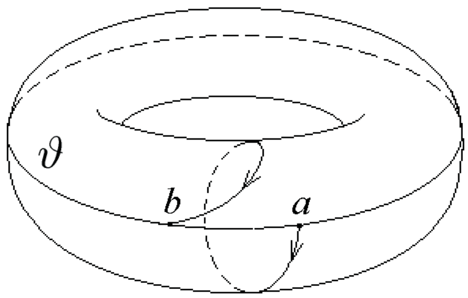

Fig. 1. Closed curve θ is an equator, $a = x(0, s) = \theta(s)$ is an initial point of the solution $x(t, s)$, $b = x(T(s), s) = \theta(\tau(s))$ is the first return point of the solution $x(t, s)$ on the curve θ.

are linearly independent. For any point $\gamma \in \mathcal{M}$, one can find $t = t(\gamma)$, $s = s(\gamma)$ such that $x(t, s) = \gamma$.

Using a parametrization of 2-torus $\mathcal{M}$ connected with a family of the solutions $x(t, s)$, one can introduce functions

$$F(t,s) = \frac{\partial f}{\partial x}(x(t,s)), \quad S_r(t,s) = \frac{\partial \sigma_r}{\partial x}(x(t,s)),$$

and $p(t, s)$ that is an orthonormal vector-function at the point $x(t, s)$ to the torus M.

In this case for the spectral radius of the operator $\mathcal{P}$, one can find the following explicit formula:

$$\rho(\mathcal{P}) = \max_s \left\{ -\frac{\prec \beta(t,s) \succ}{\prec \alpha(t,s) \succ} \right\}.$$

Here

$$\alpha(t,s) = p^\top(t,s)(F^\top(t,s) + F(t,s))p(t,s),$$

$$\beta(t,s) = p^\top(t,s) \left(\sum_{r=1}^m S_r^\top(t,s) P(t,s) S_r(t,s) \right) p(t,s),$$

brackets $\prec \cdot \succ$ mean a limit-time averaging

$$\prec \varphi \succ = \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(t) dt.$$

The inequality

$$\max_s \prec \alpha(t,s) \succ \; < 0$$

is a necessary and sufficient condition of the exponential stability of the torus M for the deterministic system (1).

The criterion $\rho(\mathcal{P}) < 1$ of exponential mean square stability of the torus M for the stochastic system (2) can be written as

$$\max_s \prec \alpha(t,s) + \beta(t,s) \succ < \; 0.$$

3. STABILIZATION

Consider a stochastic system with a control in the form

$$dx \; = \; f(x,u)dt \; + \; \sum_{r=1}^m \sigma_r(x,u) dw_r(t), \tag{4}$$

where x is $n-$dimensional state variable, u is l-dimensional vector of control inputs, $f(x, u)$, $\sigma(x, u)$ are vector-functions of the appropriate dimension, $w_r(t)$ $(r = 1, ..., m)$ are independent standard Wiener processes. It is supposed that for $u = 0$ the system (4) has an invariant manifold M.

We shall select the stabilizing regulator from the class of admissible feedbacks $u = u(x)$ satisfying conditions:

(a) $u(x)$ is sufficiently smooth and $u|_M = 0$;

(b) for the deterministic system

$$dx \; = \; f(x, u(x))dt \tag{5}$$

the manifold M is exponentially stable in the neighbourhood U of M.

Without loss of generality, we can restrict our consideration by the regulator in the following form

$$u(x) = K(\gamma(x))\Delta(x). \tag{6}$$

Here $K(x)$ is a feedback matrix coefficient.

Consider a set $\mathbf{K}$ of $l \times n$-matrices $K(x)$ satisfying the following condition: the manifold M is exponentially stable for the closed-loop deterministic system (5), (6).

For the stabilization of the closed-loop stochastic system (4), (6) we will use a spectral criterion from Theorem 1.

Consider corresponding operators

$$\mathcal{A}_K[V] = \left(f_0, \frac{\partial V}{\partial x} \right) + (F + BK)^\top V + V(F + BK),$$

$$\mathcal{S}_K[V] \; = \; \sum_{r=1}^m (C_r + H_r K)^\top V (C_r + H_r K),$$

$$\mathcal{P}_K = -\mathcal{A}_K^{-1} \mathcal{S}_K,$$

where

$$f_0 = f(x,0), \;\; F(x) = \frac{\partial f}{\partial x}(x,0), \; B(x) = \frac{\partial f}{\partial u}(x,0) \;,$$

$$C_r(x) = \frac{\partial \sigma_r}{\partial x}(x,0), \;\; H_r(x) = \frac{\partial \sigma_r}{\partial u}(x,0).$$

The Theorem 1 implies the following Theorem.

Theorem 2. *The manifold M is EMS-stabilizable for the stochastic system (4) with the feedback (6) if and only if*

(a) $\mathbf{K} \neq \emptyset$,

(b) The inequality $\inf_{K\in\mathbf{K}} \rho(\mathcal{P}_K) < 1$ *holds.*

The feedback (6) stabilizes the stochastic system (4) for any $K \in \mathbf{K}$ satisfying the inequality $\rho(\mathcal{P}_K) < 1$.

This Theorem reduces a stabilization problem to the minimization of the spectral radius of the operator $\mathcal{P}_K$.

Remark. Consider a case of the manifolds with codimension one.

For the manifold M with codim(M) = 1, projective matrix P_x has the following factorization

$$P_x = p_x p_x^\top, \tag{7}$$

where p_x is a vector that is ortonomal to the M at the point x.

It follows from (7) and the equalities

$$K(x) = K(x)P_x + K(x)(I - P_x),$$
$$P_x\Delta(x) = \Delta(x)$$

that

$$u(x) = k(\gamma(x))p^\top_{\gamma(x)}\Delta(x), \tag{8}$$

where $k(x) = K(x)p_x$. So, for the control of manifolds with codimension one it is reasonable to use more simple regulator (8).

In this case, instead of the operator $\mathcal{P}_K$ one can use an operator

$$\mathcal{P}_k = \mathcal{A}_k^{-1}\mathcal{S}_k,$$

where

$$\mathcal{A}_k[V] = \left(f_0, \frac{\partial V}{\partial x}\right) + (F + Bkp^\top)^\top V + V(F + Bkp^\top),$$

$$\mathcal{S}_k[V] = \sum_{r=1}^{m}(C_r + H_r kp^\top)^\top V(C_r + H_r kp^\top).$$

3.1 Stabilization of the Cycle for 2D-system

The cycle on a plane ($n = 2$) is a manifold with codimension one. So, due to Remark, we will use the regulator (8) and the operator $\mathcal{P}_k$ correspondingly.

The spectral radius of the operator $\mathcal{P}_k$ is the following

$$\rho(\mathcal{P}_k) = -\frac{\prec \beta_k \succ}{\prec \alpha_k \succ}.$$

Here

$$\alpha_k = p^\top\left[(F + Bkp^\top)^\top + F + Bkp^\top\right]p,$$
$$\beta_k = p^\top\left(\sum_{r=1}^{m}(C_r + H_r kp^\top)(C_r + H_r kp^\top)^\top\right)p, \tag{9}$$

$$F(t) = \frac{\partial f}{\partial x}(\xi(t), 0),\ B(t) = \frac{\partial f}{\partial u}(\xi(t), 0),$$

$$C_r(t) = \frac{\partial \sigma_r}{\partial x}(\xi(t), 0),\ \ H_r(t) = \frac{\partial \sigma_r}{\partial u}(\xi(t), 0),$$

$p(t)$ is a vector orthonormal to the limit cycle at the point $\xi(t)$.

The condition of the stabilizability

$$\inf_{k\in\mathbf{K}} \rho(\mathcal{P}_k) < 1$$

is equivalent to the inequality

$$\inf_k I(k) < 0,$$

where

$$I(k) = \prec \alpha_k + \beta_k \succ .$$

Due to (9), the functional $I(k)$ is quadratic:

$$I(k) = \prec \alpha + \beta + 2(b + c)k + k^\top Hk \succ .$$

Here

$$\alpha(t) = p^\top(t)\left[F^\top(t) + F(t)\right]p(t),\ \ b(t) = B^\top(t)p(t),$$

$$\beta(t) = \sum_{r=1}^{m} c_r^2(t),\ c(t) = \sum_{r=1}^{m} c_r(t)h_r(t),$$
$$H(t) = \sum_{r=1}^{m} h_r(t)h_r^\top(t),$$

$$c_r(t) = p^\top(t)C_r(t)p(t),\ \ h_r(t) = H_r^\top(t)p(t).$$

So, a solution of the stabilization problem is reduced to the minimizing of the quadratic functional $I(k)$.

Minimization of the quadratic functional

First consider the case of the system (4) with noise that is not depends of control input. It means that

$$H_r(t) \equiv 0,\ \ c(t) \equiv 0,\ \ H(t) \equiv 0,$$

and functional $I(k)$ is linear:

$$I(k) = \prec \alpha + \beta + 2(b + c)k \succ .$$

In this case, for the stabilizability of the system (4) for any noise, it is necessary and sufficient that for the function $b(t)$ on the interval $[0, T]$, the following holds

$$b(t) \not\equiv 0. \tag{10}$$

Indeed, for $\mu > 0$ consider the equation

$$I(k) = \prec \alpha(t) + \beta(t) + 2b(t)^\top k(t) \succ = -\mu. \tag{11}$$

Due to the condition (10), the equation(11) has an infinite set of solutions.

The additional criterion

$$\|k(t)\|^2 = \prec k^\top k \succ \to \min$$

gives the unique solution

$$k_0(t) = -\frac{\alpha(t) + \beta(t) + \frac{1}{T}\mu}{2 \prec b(t)^\top b(t) \succ}\, b(t) \tag{12}$$

with the minimal norm.

By the direct substitution one can verify the equality (11) that means the regulator with feedback coefficient (12) stabilizes the system (4).

Consider a general case of the system (2) with control dependent noise ($H(t) \neq 0$.) Here the inequality $I(k) < 0$ may not have solutions.

Below we present results of the full analysis for the case of the scalar input ($l = 1$).

For $l = 1$, the functions $b(t),\ c(t),\ H(t)$, $k(t)$ are scalar too and the functional is as follows:

$$I(k) = \prec \alpha(t) + \beta(t) + 2(b(t) + c(t))k(t) + H(t)k^2(t) \succ .$$

Let $H(t) \neq 0$ on the interval $[0, T]$. Then

$$I(k) = \prec H(t)\left(k(t) + \frac{b(t) + c(t)}{H(t)}\right)^2 - $$
$$-\frac{(b(t) + c(t))^2}{H(t)} + \alpha(t) + \beta(t) \succ .$$

For

$$k_0(t) \;=\; -\frac{b(t)+c(t)}{H(t)}, \qquad I(k_0) \;=\; \prec \alpha(t)+\beta(t)-\frac{(b(t)+c(t))^2}{H(t)} \succ \tag{13}$$

and any $k(t)$, the following inequality

$$I(k_0) \;\le\; I(k)$$

holds.

Thus, the inequality

$$\prec \alpha(t)+\beta(t) \succ \quad < \quad \prec \frac{(b(t)+c(t))^2}{H(t)} \succ \tag{14}$$

is necessary and sufficient condition for stabilizability of the system (2) by the regulator (8).

If the inequality (14) holds then the regulator (8) with the feedback coefficient $k_0(t)$ from (13) stabilizes the system (4).

3.2 Stabilization of the Two-torus for 3D-system

The two-torus for 3D-system is a manifold with codimension one. So, due to Remark, we will use the regulator (8) and investigate its stabilization capacity via the operator $\mathcal{P}_k$.

The spectral radius the of operator $\mathcal{P}_k$ for two-torus with the parametrization by family of the solutions $x(t,s)$ is the following

$$\rho(\mathcal{P}_k) = \max_s\{-\frac{\prec \beta_k(t,s) \succ}{\prec \alpha_k(t,s) \succ}\}.$$

Here

$$\begin{aligned}
&\alpha_k = p^\top\left[(F+Bkp^\top)^\top + F + Bkp^\top\right]p,\\
&\beta_k = p^\top\left(\sum_{r=1}^{m}(C_r+H_rkp^\top)(C_r+H_rkp^\top)^\top\right)p,\\
&F(t,s) = \frac{\partial f}{\partial x}(x(t,s),0),\;\; B(t,s) = \frac{\partial f}{\partial u}(x(t,s),0)\;,\\
&C_r(t,s) = \frac{\partial \sigma_r}{\partial x}(x(t,s),0),\;\;\; H_r(t,s) = \frac{\partial \sigma_r}{\partial u}(x(t,s),0),
\end{aligned} \tag{15}$$

$p(t,s)$ is a vector orthonormal to the toroidal surface at the point $x(t,s)$.

The condition of the stabilizability

$$\inf_{k\in\mathbf{K}} \rho(\mathcal{P}_k) < 1$$

is equivalent to the inequality

$$\inf_k \max_s I(k,s) \;<\; 0,$$

where

$$I(k,s) \;= \prec \alpha_k(t,s)+\beta_k(t,s) \succ .$$

The functional $I(k,s)$ is quadratic:

$$I(k,s) = \prec \alpha+\beta+2(b+c)k+k^\top Hk \succ .$$

Here

$$\begin{aligned}
&\alpha(t,s) = p^\top(t,s)\left[F^\top(t,s)+F\big(t,s)\right]p(t,s),\\
&b(t,s) = B^\top(t,s)p(t,s),\\
&\beta(t,s) = \sum_{r=1}^{m} c_r^2(t,s),\; c(t,s) = \sum_{r=1}^{m} c_r(t,s)h_r(t,s),\\
&H(t,s) = \sum_{r=1}^{m} h_r(t,s)h_r^\top(t,s),\\
&c_r(t,s) = p^\top(t,s)C_r(t,s)p(t,s),\\
&h_r(t,s) = H_r^\top(t,s)p(t,s).
\end{aligned}$$

So, a solution of the stabilization problem for two-torus is reduced to the minimax problem for the quadratic functional $I(k,s)$.

4. STABILIZATION OF CYCLES FOR THE STOCHASTIC HOPF SYSTEM

Consider stochastically forced Hopf system with control

$$\begin{aligned}
\dot{x} \;&=\; \mu x - y - (x^2+y^2)x + u + \\
&\quad +\sigma_1(x^2+y^2-\mu)\dot{w}_1(t) + \sigma_2 u\dot{w}_2(t)\\
\dot{y} \;&=\; x + \mu y - (x^2+y^2)y.
\end{aligned} \tag{16}$$

Here w_1, w_2 are standard Wiener processes, σ_1 is an intensity of state-dependent noise, and σ_2 is an intensity of control-dependent noise, u is a scalar control input.

For $u=0$, $\mu>0$, $\sigma_1=0$, this system has a limit cycle

$$x^2+y^2=\mu.$$

For this cycle, we use the parametrization

$$x=\sqrt{\mu}\cos t, y=\sqrt{\mu}\sin t.$$

The aim of the control is to stabilize this cycle in the mean square sense.

The feedback matrix in the regulator (6) for Hopf system (16) is

$$K(t)=k(t)p(t),$$

where $p(t)=(\cos t,\sin t)^\top$ and $k(t)$ is a scalar function. Functions α_k, β_k in (9) have an explicit representation

$$\alpha_k = -4\mu + 2k\cos t,\;\; \beta_k = 4\sigma_1^2\mu + k^2\sigma_2^2.$$

So, the quadratic functional $I(k) \;= \prec \alpha_k+\beta_k \succ$ is as follows

$$I(k) \;=\; 4\mu(\sigma_1^2-1) + \frac{1}{2\pi}\int_0^{2\pi}\left(2k(t)\cos t + k^2(t)\sigma_2^2\right)\,dt. \tag{17}$$

For $u=0$, a necessary and sufficient condition of the stochastic stability of the cycle is $\sigma_1^2<1$.

For numerical simulations, fix $\mu=1$, $\sigma_1=2$. Theoretically, for these parameters, the cycle $x^2+y^2=1$ of the uncontrolled ($u=0$) system (16) is stochastically unstable in the mean square sense.

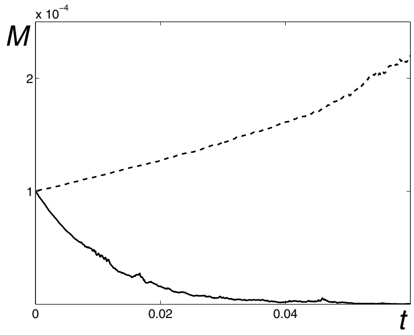

Fig. 2. Mean square deviation $M(t)$ for uncontrolled (dashed line) and controlled (solid line) stochastic Hopf system.

In Fig. 2, by dashed line, we plot a function $M(t) = \mathrm{E}\left(\sqrt{x^2(t)+y^2(t)}-1\right)^2$, where $x(t)$, $y(t)$ is a solution of the Hopf system with $u = 0$ for initial conditions $x(0) = 1.01$, $y(0) = 0$. For numerical simulations, we use Euler-Maruyama scheme with time step $\Delta t = 10^{-5}$ and averaging of 5000 random trajectories. Here, an exponential growth of the quadratic deviation of solutions from the cycle is observed.

Consider now possibilities of the stabilization. The function $k_o(t) = -\frac{\cos(t)}{\sigma_2^2}$ minimizes the functional (17). The minimal value of this functional is

$$I(k_o) = 4\mu(\sigma_1^2 - 1) - \frac{1}{2\sigma_2^2}.$$

For $\sigma_1^2 > 1$, a necessary and sufficient condition of the stabilizability can be written in a parametrical form:

$$\sigma_2^2 < \frac{1}{8\mu(\sigma_1^2 - 1)}.$$

For the considered set of parameters $\mu = 1$, $\sigma_1 = 2$, the stabilizability condition is $\sigma_2^2 < 1/24$. In this case, the feedback regulator is the following:

$$u = -\frac{x}{\sigma_2^2\sqrt{x^2+y^2}}\left(\sqrt{x^2+y^2}-1\right).$$

In Fig. 2, by solid line, we plot a function $M(t)$ for the system (16) with this regulator and $\sigma_2 = 0.1$. An exponential decrease of the quadratic deviation of solutions from the cycle demonstrates a stabilization.

5. CONCLUSION

A problem of the mean square stabilization of the general invariant compact manifolds for nonlinear stochastic systems was reduced to the minimization of the spectral radius of the corresponding operator. Constructiveness of this theory has been demonstrated for the important problem of the stabilization of the stochastically forced limit cycle and tori. The problem of the stabilization of these manifolds has been turned to the classical mathematical problem of the quadratic functional minimization. This theory was successfully applied to the stabilization of the cycles of Hopf system.

ACKNOWLEDGEMENTS

This work was partially supported by the Government of the Russian Federation 02.A03.21.0006 (27.08.2013) and RFBR (13-08-00069, 14-01-00181).

REFERENCES

L. Arnold. *Random Dynamical Systems.* Springer, Berlin, 1998.

I. A. Bashkirtseva and L. B. Ryashko. Stochastic sensitivity of 3D-cycles. *Mathematics and Computers in Simulation*, 66:55–67, 2004.

I. Bashkirtseva, G. Chen, and L. Ryashko. Stochastic equilibria control and chaos suppression for 3D systems via stochastic sensitivity synthesis. *Commun. Nonlinear Sci. Numer. Simulat.*, 17:3381–3389, 2012.

I. Bashkirtseva, G. Chen, and L. Ryashko. Stabilizing stochastically-forced oscillation generators with hard excitement: a confidence-domain control approach. *Eur. Phys. J. B*, 86:437, 2013.

G. Chen and X. Yu. *Chaos Control: Theory and Applications.* New York, Springer-Verlag, 2003.

N. Fenichel. Persistence and smoothness of invariant manifolds for flows. *Indiana University Mathematics Journal*, 2:193–226, 1971.

L. Guo and H. Wang. *S*tochastic Distribution Control System Design: A Convex Optimization Approach. Springer-Verlag, New York, 2010.

M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant Manifolds.* Springer-Verlag, 1977.

R. Z. Khasminskii. *Stochastic Stability of Differential Equations.* Sijthoff & Noordhoff, Alpen aan den Rijn, 1980.

M.A. Krasnosel'skii, Je.A. Lifshits, and A.V. Sobolev. *Positive Linear Systems: The Method of Positive Operators.* Sigma Series in Applied Mathematics 5, Helderman Verlag, Berlin, 1990.

H. J. Kushner. *Stochastic Stability and Control.* Academic Press, New York, 1967.

X. Mao. *Exponential Stability of Stochastic Differential Equations.* Marcel Dekker, 1994.

L. B. Ryashko. The stability of stochastically perturbed orbital motions. *J.Appl.Math.Mech.*, 60:579–90, 1996.

L. B. Ryashko. Stability and stabilization of SDEs with periodic oefficients. *Dynamic Systems and Applications*, 8:21–35, 1999.

L. B. Ryashko and E. E. Shnol. On exponentially attracting invariant manifolds of ODEs. *Nonlinearity*, 16:147–160, 2003.

L. B. Ryashko. Exponential mean square stability of stochastically forced invariant manifolds for nonlinear SDE. *Stochastics and dynamics*, 7:389–401, 2007.

L. Ryashko and I. Bashkirtseva. Exponential mean square stability analysis of invariant manifolds for nonlinear SDE's. Chapter 4 in N. Halidias, editor, *Stochastic Differential Equations, Series: Mathematics Research Developments*, pages 67–95. Nova Science Publishers, Inc., 2011.

J.-Q. Sun. *Stochastic Dynamics and Control.* Elsevier, 2006.

S. Wiggins. *Normally Hyperbolic Invariant Manifolds in Dynamical Systems.* Springer-Verlag, Berlin, 1994.