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Computation of Continuous-Time Probabilistic Invariant Sets and Ultimate Bounds

Ernesto Kofman ** **José A. De Doná** * **Maria M. Seron** *
Noelia Pizzi **

* *School of Electrical Engineering and Computer Science and CDSC, The University of Newcastle, Australia. (emails: Jose.Dedona, Maria.Seron@newcastle.edu.au)*
** *CIFASIS - CONICET, FCEIA - UNR, Argentina. (email: kofman, pizzi@cifasis-conicet.gov.ar)*

Abstract: The concepts of ultimate bounds and invariant sets play a key role in several control theory problems, as they replace the notion of asymptotic stability in the presence of unknown disturbances. However, when the disturbances are unbounded, as in the case of Gaussian white noise, no ultimate bounds nor invariant sets can in general be found. To overcome this limitation we introduced, in previous work, the notions of probabilistic ultimate bound (PUB) and probabilistic invariant set (PIS) for discrete-time systems. This article extends the notions of PUB and PIS to continuous-time systems, studying their main properties and providing tools for their calculation. An application of these concepts to robust control design is also presented.

Keywords: Invariant sets; Ultimate bounds; Stochastic differential equations; Linear systems; Probabilistic methods

1. INTRODUCTION

Dynamical systems under the influence of non-vanishing unknown disturbances cannot achieve asymptotic stability. However, under certain conditions, the ultimate boundedness of the system trajectories can be guaranteed and invariant sets can be found. Consequently, the notions of ultimate bounds and invariant sets play a key role in control systems theory and design.

A necessary condition to ensure the existence of ultimate bounds and invariant sets is that the disturbances be bounded. However, in systems theory, disturbances are often represented by unbounded signals such as Gaussian white noise, in which case ultimate bounds and invariant sets cannot be obtained in a classical sense. To overcome this problem, the authors have introduced in Kofman et al. (2011, 2012) the notions of *probabilistic ultimate bound* (PUB) and *probabilistic invariant set* (PIS), as sets where the trajectories converge to and stay in with a given probability. The concepts in Kofman et al. (2011, 2012) are limited to the discrete-time domain. Ultimate boundedness and invariance are also important concepts in continuous-time systems (see Blanchini (1999) and the references therein), and they experience the same limitations regarding unbounded disturbances.

Motivated by these facts, this work firstly extends the notions, properties and tools for PUB and PIS developed in Kofman et al. (2011, 2012) to the continuous-time domain. While in the case of PUB the extension is almost straightforward, the concept of probabilistic invariance in continuous time needs to be redefined because of the limitations imposed by the infinite-bandwidth nature of continuous-time white noise disturbances (see, e.g., the discussions in Åström (1970)). Finally, the problem of designing a controller so that the closed loop system under white noise disturbances has a desired PUB is studied. The work is organized as follows: Section 2 introduces the concepts of continuous time PUB and PIS and establishes their basic properties. Then, Sections 3 and 4 present closed formulas for the calculation of PUB and PIS, respectively. Section 5 develops the technique for control design and Section 6 illustrates the results with a numerical example.

2. BACKGROUND AND DEFINITIONS

We consider a continuous-time LTI system given by the following stochastic differential equation

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + \mathrm{d}w(t) \tag{1}$$

with $x(t), w(t) \in \mathbb{R}^n$. We assume that A is a Hurwitz matrix and that the disturbance vector $w(t)$ is a stochastic process whose increments are uncorrelated with zero mean values (e.g., in the case of a normal distribution the disturbance is given by a Wiener process).

2.1 Expected Value and Covariance of $x(t)$

We denote $\Sigma_w dt \triangleq \mathrm{cov}[\mathrm{d}w(t)] = \mathrm{E}[\mathrm{d}w(t)\mathrm{d}w^T(t)]$ the incremental covariance of $w(t)$ and we define

$$\Sigma_x(t) \triangleq \mathrm{cov}[x(t)] = \mathrm{E}[(x(t) - E[x(t)])(x(t) - E[x(t)])^T] \tag{2}$$

Both, Σ_w and $\Sigma_x(t)$ are symmetric positive semidefinite matrices. The expected value $\mu_x(t) = \mathrm{E}[x(t)]$ can be computed (see e.g. Åström (1970), Theorem 6.1, page 66) as the solution of $\dot{\mu}_x(t) = A\mu_x(t)$. We assume that the initial state $x(t_0)$ is known, then $\mu_x(t_0) = x(t_0)$ and the previous equation has the solution

$$\mu_x(t) = e^{A(t-t_0)}x(t_0) \tag{3}$$

The covariance matrix $\Sigma_x(t)$ verifies (see e.g. Åström (1970), Theorem 6.1, page 66) the following differential equation:

$$\dot{\Sigma}_x(t) = A\Sigma_x(t) + \Sigma_x(t)A^T + \Sigma_w \tag{4}$$

with $\Sigma_x(t_0) = 0$ (since $x(t_0)$ is known). Since A is a Hurwitz matrix, the latter expression converges as $t \to \infty$. Then, defining $\Sigma_x \triangleq \lim_{t\to\infty} \Sigma_x(t)$ we have from Eq.(4) that Σ_x can be obtained from the Lyapunov equation

$$A\Sigma_x + \Sigma_x A^T = -\Sigma_w \tag{5}$$

2.2 Definition of PUB and γ-PIS

We next define the two notions that concern this article.

Definition 1. (Probabilistic Ultimate Bounds). Let $0 < p \le 1$ and let $S \subset \mathbb{R}^n$. We say that S is a PUB with probability p for system (1) if for every initial state $x(t_0) = x_0 \in \mathbb{R}^n$ there exists $T = T(x_0) \in \mathbb{R}$ such that the probability $\Pr[x(t) \in S] \ge p$ for each $t \ge t_0 + T$.

For the definition of PIS, we first define the product of a scalar $\gamma \ge 0$ and a set S as $\gamma S \triangleq \{\gamma x : x \in S\}$. Notice that when $0 \le \gamma \le 1$, and provided that S is a *star–shaped set with respect to the origin*,[1] it results $\gamma S \subseteq S$.

Definition 2. (γ-Probabilistic Invariant Sets). Let $0 < p \le 1$, $0 < \gamma \le 1$ and let $S \subset \mathbb{R}^n$ be a star–shaped set with respect to the origin. We say that S is a γ–PIS with probability p for system (1) if for any $x(t_0) \in \gamma S$ the probability $\Pr[x(t) \in S] \ge p$ for each $t > t_0$.

Remark 3. The definition of PUB for discrete and continuous time systems are almost identical. However, PIS for discrete-time systems were defined to ensure that the states belonging to any trajectory starting in the set remain in the set with a given probability. By choosing a sufficiently large set, the contractivity of the system's dynamics at the boundary of the set dominates the noise and the probability of the trajectory leaving the set at the next step can be made arbitrarily small. In continuous time, however, this is not possible. Irrespective of the contractivity, when a trajectory starts at time t_0 at the border of the set, taking t sufficiently close to t_0 the dynamics is always dominated by the white noise due to its infinite-bandwidth nature. Thus, at those instants of time, the probability of leaving the set only depends on the noise and becomes independent of the size of the set. In order to overcome this fundamental difficulty, the initial states of a PIS are restricted in Definition 2 to a subset γS, with γ less than one.

2.3 Some properties of PUB and γ–PIS

The main properties of continuous-time PUB are almost identical to their discrete-time counterparts, i.e., Lemmas 3, 4, 5, 8, as well as Corollaries 7 and 10 in Kofman et al. (2012). Thus, the proofs of the corresponding lemmas are omitted.

[1] A set $S \subset R^n$ is star shaped, or a star domain, with respect to the origin if $x \in S \Rightarrow \gamma x \in S$ for all $0 \le \gamma \le 1$

Lemma 4. If S is a PUB (γ–PIS) with probability p for (1), then it is also a PUB (γ–PIS) with probability $\tilde{p} \ge 0$ for any $\tilde{p} < p$.

Lemma 5. (γ-PIS⇒PUB). Let $S_0 \subset \mathbb{R}^n$ be a γ–PIS for (1) with probability p which contains the origin. Given $\varepsilon > 0$ we define $S_\varepsilon = \{x : \mathrm{dist}(x, S_0) \le \varepsilon\}$. Then, S_ε is a PUB for (1) with probability p.

Lemma 6. (Intersection of PUB). Let S_1 be a PUB with probability p_1 for system (1) and let S_2 be a PUB with probability p_2 for the same system, with $p_1 + p_2 > 1$. Then, the set $S = S_1 \cap S_2$ is a PUB with probability $p = p_1 + p_2 - 1$.

The proofs of Lemmas 4–6 are almost identical to those of Lemmas 3–5 in Kofman et al. (2012). Induction on Lemma 6 results in the following corollary.

Corollary 7. (Intersection of several PUB). Let $\{S_i\}_{i=1}^r$ be a collection of PUB for system (1) with probabilities p_i, $i = 1, \ldots, r$, respectively, with $\sum_{i=1}^r p_i > (r-1)$. Then, the set $S = \cap_{i=1}^r S_i$ is a PUB with probability $p = \sum_{i=1}^r p_i - (r-1)$.

Lemma 8. (Intersection of γ–PIS). Let S_1 be a γ_1–PIS with probability p_1 for system (1) and let S_2 be a γ_2–PIS with probability p_2 for the same system, with $p_1 + p_2 > 1$. Then, the set $S = S_1 \cap S_2$ is a γ–PIS with probability $p = p_1 + p_2 - 1$ where $\gamma = \min\{\gamma_1, \gamma_2\}$.

Proof. Notice that $\gamma S \subseteq \gamma S_i \subseteq \gamma_i S_i$, $i = 1, 2$. Then, given an initial state $x(t_0) \in \gamma S$, we have that $x(t_0) \in \gamma_i S_i$, $i = 1, 2$. Then, for any $t > t_0$, and $i = 1, 2$, we have

$$\Pr[x(t) \in S_i] \ge p_i \Rightarrow \Pr[x(t) \notin S_i] \le 1 - p_i \Rightarrow$$
$$\Pr[x(t) \notin S_1 \vee x(t) \notin S_2] \le \Pr[x(t) \notin S_1] + \Pr[x(t) \notin S_2] \le 2 - p_1 - p_2$$

Finally,

$$\begin{aligned}\Pr[x(t) \in S] &= \Pr[x(t) \in S_1 \wedge x(t) \in S_2] \\ &= 1 - \Pr[x(t) \notin S_1 \vee x(t) \notin S_2] \ge p_1 + p_2 - 1\end{aligned}$$

which concludes the proof. □

Induction on Lemma 8 results in the following corollary.

Corollary 9. (Intersection of several γ-PIS). Let $\{S_i\}_{i=1}^r$ be a collection of γ_i–PIS for system (1) with probabilities p_i, $i = 1, \ldots, r$, respectively, with $\sum_{i=1}^r p_i > (r-1)$. Then, the set $S = \cap_{i=1}^r S_i$ is a γ–PIS with probability $p = \sum_{i=1}^r p_i - (r-1)$ where $\gamma = \min\{\gamma_i : i = 1, \ldots, r\}$.

Lemma 10. (Union of PUB). Let S_1 be a PUB with probability p_1 for system (1) and let S_2 be a PUB with probability p_2 for the same system, then the set $S_1 \cup S_2$ is a PUB with probability $p = \max\{p_1, p_2\}$.

Corollary 11. (Union of several PUB). Let $\{S_i\}_{i=1}^r$ be a collection of PUB for system (1) with probabilities p_i, $i = 1, \ldots, r$. Then, the set $S = \cup_{i=1}^r S_i$ is a PUB with probability $p = \max\{p_i : i = 1, \ldots, r\}$.

Lemma 12. (Union of γ–PIS). Let S_1 be a γ_1–PIS with probability p_1 and S_2 be a γ_2–PIS with probability p_2 for system (1), then the set $S_1 \cup S_2$ is a γ–PIS with probability $p = \min\{p_1, p_2\}$ where $\gamma = \min\{\gamma_1, \gamma_2\}$.

Corollary 13. (Union of several γ-PIS). Let $\{S_i\}_{i=1}^r$ be a collection of γ_i–PIS for system (1) with probabilities p_i, $i = 1, \ldots, r$. Then, the set $S = \cup_{i=1}^r S_i$ is a γ–PIS with probability $p = \min\{p_i : i = 1, \ldots, r\}$ where $\gamma = \min\{\gamma_i : i = 1, \ldots, r\}$.

The proof of Lemma 10 is identical to its discrete time counterpart given by Lemma 8 of Kofman et al. (2012). The proof of Lemma 12 combines that of Lemma 8 above and Lemma 9 in Kofman et al. (2012) for discrete-time systems. Corollaries 11 and 13 are the result of applying induction on Lemmas 10 and 12, respectively.

Remark 14. When $p_i = \gamma_i = 1$, $i = 1, \ldots, r$, Corollaries 9 and 13 say that the intersection and the union of *deterministic* invariant sets are *deterministic* invariant sets, which is a well known result.

3. PUB COMPUTATION

We develop first a method to compute Probabilistic Ultimate Bounds for (1) based on Chebyshev's inequality which can be used for stochastic processes $w(t)$ with arbitrary distributions. We will then give tighter bounds for the special case of a Gaussian disturbance. Given a parameter (probability) p such that $0 < p < 1$, we will define n parameters $\tilde{p}_i$ such that

$$0 < \tilde{p}_i < 1,\ i = 1, \ldots, n; \qquad \sum_{i=1}^{n} \tilde{p}_i = 1 - p \tag{6}$$

Also, for a vector x, x_i denotes its ith component, and for a square matrix Σ, the notation $[\Sigma]_{i,i}$ indicates its ith diagonal element.

3.1 General Distribution

Theorem 15. **(PUB Computation – General Case)** Consider the system (1). Assume that $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix and suppose that $w(t)$ is a stochastic process whose increments are uncorrelated with zero mean values and with incremental covariance matrix $\Sigma_w \mathrm{d}t$. Let $0 < p < 1$ and $\tilde{p}_i$, $i = 1, \ldots, n$, be defined as in (6). Then, for any $\varepsilon > 0$, the set $S = \{x : |x_i| \le b_i + \varepsilon;\ i = 1, \ldots, n\}$ is a PUB for the system with probability p, where

$$b_i \triangleq \sqrt{\frac{[\Sigma_x]_{i,i}}{\tilde{p}_i}};\quad i = 1, \ldots, n$$

and Σ_x is the solution of the Lyapunov equation (5).

The proof of Theorem 15 is identical to that of Theorem 12 in Kofman et al. (2012) for discrete-time systems.

3.2 Gaussian Distribution

The following theorem, valid for the special case of a Gaussian disturbance, provides tighter bounds than those of Theorem 15.

Theorem 16. **(PUB Computation – Gaussian Noise)** Consider the system (1). Assume $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix and suppose that $w(t)$ is a Wiener process with incremental covariance matrix $\Sigma_w \mathrm{d}t$. Let $0 < p < 1$ and $\tilde{p}_i$, $i = 1, \ldots, n$, be defined as in (6). Then, for any $\varepsilon > 0$, the set $S = \{x : |x_i| \le b_i + \varepsilon;\ i = 1, \ldots, n\}$ is a probabilistic ultimate bound for the system with probability p, where

$$b_i \triangleq \sqrt{2[\Sigma_x]_{i,i}}\mathrm{erf}^{-1}(1 - \tilde{p}_i);\quad i = 1, \ldots, n \tag{7}$$

and where Σ_x is the solution of the Lyapunov Equation (5) and erf is the *error function*: $\mathrm{erf}(z) \triangleq \frac{2}{\sqrt{\pi}} \int_0^z e^{-\zeta^2} d\zeta$.

The proof of Theorem 16 is identical to that of Theorem 1 in Kofman et al. (2011) for discrete-time systems.

4. PIS COMPUTATION

Here, again, we first propose a method to compute probabilistic invariant sets for (1) that can be used for stochastic processes $w(t)$ with uncorrelated increments and arbitrary distributions, and then we provide a method for the particular case of Gaussian noises. We will assume the matrix A in (1) to be diagonalisable. The symbol $\preceq$ will denote the elementwise inequality between two vectors, i.e., for $\alpha, \beta \in \mathbb{R}^n$, $\alpha \preceq \beta$ if and only if $\alpha_i \le \beta_i$, $i = 1, \ldots, n$. For a matrix M with complex entries, M^* will denote the conjugate transpose of M.

4.1 General Distribution

Theorem 17. **(γ–PIS Computation – General Case)** Consider the system (1), where matrix A is assumed to be Hurwitz and diagonalisable. Suppose that $w(t)$ is a stochastic process whose increments are uncorrelated with zero mean values and incremental covariance matrix $\Sigma_w \mathrm{d}t$. Let $0 < p < 1$ and $\tilde{p}_i$, $i = 1, \ldots, n$, be defined as in (6). Then, the set $S = \{x : |V^{-1}x| \preceq b\}$ is a γ–PIS for the system with probability p, where V is a similarity transformation such that $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = V^{-1}AV$ is the Jordan diagonal decomposition of matrix A, and the components of $b = [b_1 \ \ldots \ b_n]^T$ are computed according to

$$b_i \triangleq \sqrt{\frac{[\Sigma_v]_{i,i}}{2|\mathbb{R}\mathrm{e}(\lambda_i)|(1-\gamma^2)\tilde{p}_i}};\quad i = 1, \ldots, n \tag{8}$$

with $\Sigma_v = V^{-1}\Sigma_w (V^{-1})^*$.

Proof. With the linear transformation $x(t) = Vz(t)$, system (1) becomes

$$\mathrm{d}z(t) = \Lambda z(t)\mathrm{d}t + V^{-1}\mathrm{d}w(t) \tag{9}$$

with $z \in \mathbb{C}^n$, $w(t) \in \mathbb{R}^n$, $V^{-1} \in \mathbb{C}^{n\times n}$, and $\Lambda \in \mathbb{C}^{n\times n}$ being a diagonal matrix. Defining $v(t) \triangleq V^{-1}w(t)$, the incremental covariance of $v(t)$ results $\Sigma_v \mathrm{d}t = V^{-1}\Sigma_w(V^{-1})^*\mathrm{d}t$, and the ith component of (9) is

$$\mathrm{d}z_i(t) = \lambda_i z_i(t)\mathrm{d}t + \mathrm{d}v_i(t) \tag{10}$$

The expected value of $z_i(t)$ then verifies $\mathrm{E}[z_i(t)] = e^{\lambda_i(t-t_0)}z_i(t_0)$, since we assume that $z_i(t_0)$ is known. The variance of $z_i(t)$ can be computed from (10) as

$$\mathrm{var}[z_i(t)] = \frac{1 - e^{2\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}}{2|\mathbb{R}\mathrm{e}(\lambda_i)|}[\Sigma_v]_{i,i}$$

Suppose that $x(t_0) \in \gamma S$. Thus, $|z(t_0)| = |V^{-1}x(t_0)| \preceq \gamma b$ and $|z_i(t_0)| \le \gamma b_i$. Then, for all $t > t_0$ it results that

$$|\mathrm{E}[z_i(t)]| = |e^{\lambda_i(t-t_0)}z_i(t_0)| \le e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}\gamma b_i \tag{11}$$

From Inequality (11), it follows that

$$\begin{aligned}
&\Pr[|z_i(t)| \ge b_i] \\
&= \Pr[|z_i(t)| - e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}\gamma b_i \ge b_i(1 - \gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})] \\
&\le \Pr[|z_i(t)| - |\mathrm{E}[z_i(t)]| \ge b_i(1 - \gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})] \\
&\le \Pr[|z_i(t) - \mathrm{E}[z_i(t)]| \ge b_i(1 - \gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})]
\end{aligned}$$

Chebyshev's inequality establishes that

$$\begin{aligned}
&\Pr\left[|z_i(t) - \mathrm{E}[z_i(t)]| \ge b_i(1 - \gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})\right] \\
&\le \frac{\mathrm{var}[z_i(t)]}{b_i^2(1 - \gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2}
\end{aligned} \tag{12}$$

and then it results that

$$\Pr[|z_i(t)| \geq b_i] \leq \frac{\mathrm{var}[z_i(t)]}{b_i^2(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2} = \frac{1-e^{2\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}}{2|\mathbb{R}\mathrm{e}(\lambda_i)|b_i^2(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2}[\Sigma_v]_{i,i}$$

The expression

$$\frac{1-e^{2\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}}{(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2} \tag{13}$$

is maximized when $e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)} = \gamma$. Then, it results that $\Pr[|z_i(t)| > b_i] \leq \Pr[|z_i(t)| \geq b_i] \leq \frac{[\Sigma_v]_{i,i}}{2|\mathbb{R}\mathrm{e}(\lambda_i)|b_i^2(1-\gamma^2)} = \tilde{p}_i$ for all $t > t_0$. Thus, the probability

$$\Pr[|z(t)| \not\preceq b] \leq \sum_{i=1}^{n} \Pr[|z_i(t)| > b_i] \leq \sum_{i=1}^{n} \tilde{p}_i = 1-p$$

for all $t > t_0$, and then,

$$\Pr[|z(t)| \preceq b] = \Pr[|V^{-1}x(t)| \preceq b] = \Pr[x(t) \in S] \geq p$$

which proves that S is a γ–PIS with probability p. □

Remark 18. Notice that b_i in Eq.(8) goes to infinity as γ goes to one. This is consistent with the observation made in Remark 3 above, that a PIS cannot be defined without using a factor γ less than one to restrict the initial states due to the infinite-bandwidth nature of the continuous-time white noise disturbance (see, e.g., the discussions in Åström (1970) about the latter fact).

4.2 Gaussian Distribution

Here we obtain tighter bounds for γ-PIS for the case of Gaussian noises by replacing the use of Chebyshev's inequality with specific properties of Gaussian distributions.

Theorem 19. **(γ-PIS Computation – Gaussian Noise)** Consider the system (1), where matrix A is assumed to be Hurwitz and diagonalisable. Suppose that $w(t)$ is a Wiener process with incremental covariance matrix $\Sigma_w \mathrm{d}t$. Let $0 < p < 1$ and $\tilde{p}_i$, $i = 1, \ldots, n$, be defined as in (6) with the restriction that for each pair of complex conjugate eigenvalues λ_i, $\lambda_j = \bar{\lambda}_i$, we take $\tilde{p}_i = \tilde{p}_j$. Then, the set $S = \{x : |V^{-1}x| \preceq b\}$ is a γ–PIS for the system with probability p, where V is a similarity transformation such that $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = V^{-1}AV$ is the Jordan diagonal decomposition of matrix A, and the components of $b = [b_1 \ \ldots \ b_n]^T$ are computed as

$$b_i \triangleq \sqrt{\frac{[\Sigma_v]_{i,i}}{|\mathbb{R}\mathrm{e}(\lambda_i)|(1-\gamma^2)}}\mathrm{erf}^{-1}(1-\tilde{p}_i); \quad i = 1, \ldots, n$$

with $\Sigma_v = V^{-1}\Sigma_w(V^{-1})^*$.

Proof. When λ_i is real, the proof is almost identical to that of Theorem 17 above. We replace here Chebyshev's inequality of Eq.(12) by the following expression valid for Gaussian distributions

$$\Pr\left[|z_i(t) - \mathrm{E}[z_i(t)]| \geq b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})\right] = 1 - \mathrm{erf}\left(\frac{b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})}{\sqrt{2\mathrm{var}[z_i(t)]}}\right)$$

and then we obtain,

$$\begin{aligned}\Pr[|z_i(t)| > b_i] &\leq 1 - \mathrm{erf}\left(b_i\sqrt{\frac{(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2|\mathbb{R}\mathrm{e}(\lambda_i)|}{(1-e^{2\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})[\Sigma_v]_{i,i}}}\right) \\ &\leq 1 - \mathrm{erf}\left(b_i\sqrt{\frac{(1-\gamma^2)|\mathbb{R}\mathrm{e}(\lambda_i)|}{[\Sigma_v]_{i,i}}}\right) = \tilde{p}_i\end{aligned} \tag{14}$$

In the last step we used the fact that $\mathrm{erf}(\cdot)$ is a monotonically increasing function and we maximized the expression of Eq.(13) as in the proof of Theorem 17.

In the case of complex eigenvalues, Eq.(10) can be split into real and imaginary parts $z_i(t) = \mathbb{R}\mathrm{e}[z_i(t)] + j\mathbb{I}\mathrm{m}[z_i(t)]$, where both components are Gaussian processes and the variance can be written as $\mathrm{var}[z_i(t)] = \mathrm{var}[\mathbb{R}\mathrm{e}[z_i(t)]] + \mathrm{var}[\mathbb{I}\mathrm{m}[z_i(t)]]$. Then, the proof follows that of Theorem 2 in Kofman et al. (2011) for discrete-time systems, replacing $t_0 + N$ by t and $b_i(1-|\lambda_i|^N)$ by $b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})$. □

5. CONTROL DESIGN

We consider here the problem of, given a nonnegative vector b and a probability p, find a controller gain K such that the closed loop system

$$\mathrm{d}x(t) = (A+BK)x(t)\mathrm{d}t + H\mathrm{d}v(t) \tag{15}$$

has a PUB $S = \{x : |x| \preceq b\}$ with probability p.

We shall assume that (A, B) is in its controller canonical form and that the system has a single input. Also, we shall assume that the disturbance $v(t)$ is *matched* with the input, i.e., $H = BG$ and that it has a covariance Σ_v.

Theorems 15–16 show that the PUB depends on the diagonal entries of the covariance matrix Σ_x. Thus, this is a problem of *covariance assignment* similar to the one treated in Sreeram et al. (1996).

When matrix A is in its controller canonical form, the covariance matrix that solves Eq.(5) has a *Xiao structure* (Xiao et al., 1992).

Definition 20. (Xiao matrix). Given a vector $0 \preceq z \in \mathbb{R}^k$, we define the Xiao matrix $\mathcal{X}(z)$ as

$$\mathcal{X}(z) = \begin{bmatrix} z_1 & 0 & -z_2 & 0 & z_3 & \cdots & \cdot \\ 0 & z_2 & 0 & -z_3 & 0 & \cdots & \cdot \\ -z_2 & 0 & z_3 & 0 & -z_4 & \cdots & \cdot \\ 0 & -z_3 & 0 & z_4 & 0 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & z_n \end{bmatrix} \tag{16}$$

Before presenting the main result of this section, we introduce the following lemma.

Lemma 21. Let $g : (0, 1) \to \mathbb{R}^+$ be a strictly monotonically decreasing function with $\mathrm{Im}(g) = \mathbb{R}^+$, let $b \succeq 0$ be a vector in $\mathbb{R}^n$ and let $0 < p < 1$. Then, there exist n constants $0 < \tilde{p}_i < 1$ for $i = 1, \cdots, n$ such that $\sum_{i=1}^{n} \tilde{p}_i = 1-p$ and the Xiao matrix

$$\Sigma_x = \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1)^2} \ \frac{(b_2)^2}{g(\tilde{p}_2)^2} \ \cdots \ \frac{(b_n)^2}{g(\tilde{p}_n)^2}\right]^T\right)$$

is positive definite.

Proof. Let us suppose that there exist $0 < \tilde{p}_i^{(k)} < 1$ for $i = 1, \cdots, n$ with $\sum_{i=1}^{n} \tilde{p}_i^{(k)} = 1-p$ such that the matrix

$$\Sigma_k \triangleq \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1^{(k)})^2}\ \frac{(b_2)^2}{g(\tilde{p}_2^{(k)})^2}\ \cdots\ \frac{(b_k)^2}{g(\tilde{p}_k^{(k)})^2}\right]^T\right) \tag{17}$$

is positive definite. We shall prove that there exist $0 < \tilde{p}_i^{(k+1)} < 1$ for $i = 1, \cdots, n$ with $\sum_{i=1}^n \tilde{p}_i^{(k+1)} = 1 - p$ such that the matrix

$$\Sigma_{k+1} \triangleq \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1^{(k+1)})^2}\ \frac{(b_2)^2}{g(\tilde{p}_2^{(k+1)})^2}\ \cdots\ \frac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k+1)})^2}\right]^T\right) \tag{18}$$

is also positive definite. We first form the matrix

$$\tilde{\Sigma}_{k+1} = \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1^{(k)})^2}\ \frac{(b_2)^2}{g(\tilde{p}_2^{(k)})^2}\ \cdots\ \frac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k)})^2}\right]^T\right)$$
$$= \begin{bmatrix} \Sigma_k & c_k \\ c_k^T & \dfrac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k)})^2} \end{bmatrix} = \begin{bmatrix} \Sigma_k & c_k \\ c_k^T & \tilde{d}_{k+1} \end{bmatrix} \tag{19}$$

If the product $c_k^T \cdot (\Sigma_k)^{-1} c_k < \tilde{d}_{k+1}$ then $\tilde{\Sigma}_{k+1} > 0$ and we can choose $\tilde{p}_i^{(k+1)} = \tilde{p}_i^{(k)}$ and the matrix Σ_{k+1} defined as in Eq.(18) is positive definite.

Otherwise, if $c_k^T \cdot (\Sigma_k)^{-1} c_k \geq \tilde{d}_{k+1}$, we first compute

$$r_{k+1} = \frac{c_k^T \cdot (\Sigma_k)^{-1} c_k}{\tilde{d}_{k+1}} \tag{20}$$

and choose a constant $\alpha > 1$ to calculate

$$[\Sigma_{k+1}]_{i,i} = \frac{[\Sigma_k]_{i,i}}{\alpha r_{k+1}} \quad \text{for } 1 \leq i \leq k \tag{21}$$

Then we take

$$\tilde{p}_i^{(k+1)} = \begin{cases} g^{-1}\left(\dfrac{b_i}{\sqrt{[\Sigma_{k+1}]_{i,i}}}\right) & \text{for } 1 \leq i \leq k \\ \tilde{p}_i^{(k)} \dfrac{1 - p - \sum_{j=1}^k \tilde{p}_j^{(k+1)}}{1 - p - \sum_{j=1}^k \tilde{p}_j^{(k)}} & \text{for } i > k \end{cases} \tag{22}$$

That way, we ensure that $\sum_{i=1}^n \tilde{p}_i^{(k+1)} = 1 - p$, and, taking into account that $g(\tilde{p}_i)$ monotonically decreases and $[\Sigma_{k+1}]_{i,i} < [\Sigma_k]_{i,i}$ for $i \leq k$, it results that

$$\tilde{p}_i^{(k+1)} < \tilde{p}_i^{(k)} \text{for } i \leq k \quad \text{and} \quad \tilde{p}_i^{(k+1)} > \tilde{p}_i^{(k)} \text{for } i \geq k+1$$

Then, we have

$$\Sigma_{k+1} = \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1^{(k+1)})^2}\ \frac{(b_2)^2}{g(\tilde{p}_2^{(k+1)})^2}\ \cdots\ \frac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k+1)})^2}\right]^T\right)$$
$$= \begin{bmatrix} \dfrac{\Sigma_k}{\alpha r_{k+1}} & \dfrac{c_k}{\alpha r_{k+1}} \\ \dfrac{c_k^T}{\alpha r_{k+1}} & \dfrac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k+1)})^2} \end{bmatrix} = \begin{bmatrix} \dfrac{\Sigma_k}{\alpha r_{k+1}} & \dfrac{c_k}{\alpha r_{k+1}} \\ \dfrac{c_k^T}{\alpha r_{k+1}} & d_{k+1} \end{bmatrix}$$

with $d_{k+1} > \tilde{d}_{k+1}$. Thus, it results that

$$\frac{c_k^T}{\alpha r_{k+1}} \cdot \left(\frac{\Sigma_k}{\alpha r_{k+1}}\right)^{-1} \cdot \frac{c_k}{\alpha r_{k+1}} = \frac{c_k^T \cdot (\Sigma_k)^{-1} c_k}{\alpha r_{k+1}}$$
$$= \frac{\tilde{d}_{k+1}}{\alpha} < d_{k+1}$$

and then Σ_{k+1} is positive definite.

The proof by induction then concludes by observing that Σ_1 is positive definite for any choice of the parameters $\tilde{p}_i^{(1)} > 0$ such that $\sum_{i=1}^n \tilde{p}_i^{(1)} = 1 - p$. We can initially take, in particular

$$\tilde{p}_i^{(1)} = \frac{1-p}{n} \tag{23}$$

□

Based on this result, the following Theorem establishes that an arbitrary PUB can be assigned to the system of Eq.(15) with a proper choice of the feedback gain K.

Theorem 22. (PUB Assignment). Given a system

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + Bu(t)\mathrm{d}t + H\mathrm{d}v(t) \tag{24}$$

with (A, B) in controller canonical form, $H = B \cdot G$ where the disturbance vector $v(t)$ is a stochastic process with uncorrelated increments and zero mean, and given a vector $b \succeq 0$ and a probability $0 < p < 1$, there exist a control law $u(t) = K \cdot x$ such that $S = \{x : |x| \preceq b + \varepsilon\}$, for any $\varepsilon > 0$, is a PUB with probability p of the closed loop system.

Proof. Let Σ_v be the covariance matrix of v. Defining $w(t) \triangleq Hv(t)$, the covariance of $w(t)$ results

$$\Sigma_w = H\Sigma_v H^T = BG\Sigma_v G^T B^T = B\Sigma B^T$$

where $\Sigma = G\Sigma_v G^T$ is the covariance of $Gv(t)$.

Thus, according to Theorems 15 and 16, provided that $A + BK$ is Hurwitz, the closed loop system of Eq.(15) has a PUB $S = \{x : |x| \preceq b + \varepsilon\}$ with probability p where

$$b_i = \sqrt{[\Sigma_x]_{i,i} g(\tilde{p}_i)}$$

with $\sum_{i=1}^n \tilde{p}_i = 1 - p$ and

$$g(\tilde{p}_i) = \begin{cases} 1/\sqrt{\tilde{p}_i} & \text{for a general distribution} \\ \sqrt{2}\mathrm{erf}^{-1}(1 - \tilde{p}_i) & \text{for a Gaussian distribution} \end{cases}$$

and where Σ_x is the solution of the Lyapunov equation

$$(A + BK)\Sigma_x + \Sigma_x(A + BK)^T = -B\Sigma B^T \tag{25}$$

Notice that in both cases (general and Gaussian distribution), the function g verifies the hypothesis of Lemma 21. Thus, the constants $\tilde{p}_i$ can be chosen such that the matrix

$$\Sigma_x = \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1)^2}\ \frac{(b_2)^2}{g(\tilde{p}_2)^2}\ \cdots\ \frac{(b_n)^2}{g(\tilde{p}_n)^2}\right]^T\right)$$

is positive definite.

According to Lemmas 3.1 and 3.2 in Sreeram and Agathoklis (1992), this positive definite Xiao matrix Σ_x is an assignable covariance matrix of the controller-form pair (A, B). That is, there exists K such that Σ_x is the solution of the Lyapunov equation (25). Moreover, it results that

$$A + BK = \bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & \bar{A}_n & & \end{bmatrix}$$

is a Hurwitz matrix with

$$\bar{A}_n = -\left[\Sigma_x^{-1} h\right]^T \tag{26}$$

and

$$h = [[\Sigma_x]_{2,n}\ [\Sigma_x]_{3,n}\ \cdots\ [\Sigma_x]_{n,n}\ \Sigma/2]^T \tag{27}$$

from where it results that

$$K = \bar{A}_n - A_n \tag{28}$$

A_n being the last row of matrix A (recall that in the controller canonical form $B = [0 \ldots 0\ 1]^T$).

Then, taking K from Eq.(28) the closed loop system has the desired PUB, which concludes the proof. $\square$

From Theorem 22 and Lemma 21 the following algorithm can be devised to find the control law $u(t) = K \cdot x(t)$ such that system (24) has a PUB of size b with probability p:

(1) Obtain the covariance matrix $\Sigma_x > 0$:
  - (a) Take $k = 1$ and $\tilde{p}_i^{(1)}$ as in Eq.(23).
  - (b) Form Σ_k from Eq. (17) and $\tilde{\Sigma}_{k+1}$ from Eq.(19).
  - (c) If $\tilde{\Sigma}_{k+1} > 0$, take $\tilde{p}_i^{(k+1)} = \tilde{p}_i^{(k)}$ and go to step (1e).
  - (d) Otherwise, choose $\alpha > 1$ and take $\tilde{p}_i^{(k+1)}$ from Eqs.(20)–(22).
  - (e) Let $k := k + 1$. If $k < n$ go back to step (1b).

(2) Calculate $\Sigma = G\Sigma_v G^T$ and compute K from Eqs.(26)–(28).

6. EXAMPLE

We consider a system described by Eq.(24) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad H = B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \tag{29}$$

where $v(t)$ is a Wiener process with incremental covariance $\Sigma_v \mathrm{d}t = 0.01\mathrm{d}t$.

We want this system to have a PUB

$$S = \{x : |x| < b + \varepsilon\} \quad \text{where } b = [0.1\ 0.1\ 0.1]^T$$

for all $\varepsilon > 0$ with probability $p = 0.9$.

The algorithm derived above with the choice $\alpha = 2$ provides the values $\tilde{p}_1 = \tilde{p}_2 = 0.002617$, $\tilde{p}_3 = 0.09477$, and $\Sigma_x = \mathcal{X}([0.0011\ \ 0.0011\ \ 0.00358]^T)$. Then, the resulting control gain is

$$K = [-2.0176\ \ -3.2446\ \ -2.0176]$$

In order to verify the result, we performed 10000 simulations of the system from the initial state $x(t_0) = 10 \cdot b$ (outside S) and for each instant of time $t_k = 0.1k$, with $k = 0, \cdots 1000$ we evaluated the *exit ratio* e as the number of times $x(t_k)$ lies outside the PUB divided by 10000. We found that for any $t_k > 12$, between 8.3% and 11.3% of the simulations lie outside the calculated PUB, which is close to the maximum theoretical probability $(1 - p)$ of 10%.

We also computed a γ–PIS with $\gamma = 0.9$ and probability $p = 0.9$ for the system. For that goal, we set $\tilde{p}_1 = \tilde{p}_2 = \tilde{p}_3 = 0.1/3$ obtaining a set $S = \{x : |V^{-1}x| \preceq b_p\}$ with

$$V = \begin{bmatrix} -0.636 & 0.251 - 0.241i & 0.251 + 0.241i \\ 0.573 & 0.194 + 0.483i & 0.194 - 0.483i \\ -0.517 & -0.779 & -0.779 \end{bmatrix}$$

and $b_p = [0.2796\ 0.3343\ 0.3343]^T$.

As before, we run 10000 simulations from an initial state $x_0 = [-0.145,\ \ -0.00278,\ \ 0.000173]^T$ located on the border of the set of initial conditions γS of the γ–PIS S. We then computed the exit ratio as a function of time, obtaining the results shown in Figure 1. It can be seen that, near the beginning of the simulation, about 1.7% of the simulated trajectories abandon the γ–PIS. This set was obtained so as to ensure that the probability of abandoning the set is less than 10%; so, in spite of some conservatism, the numerical result is in the order of magnitude of the theoretical bound.

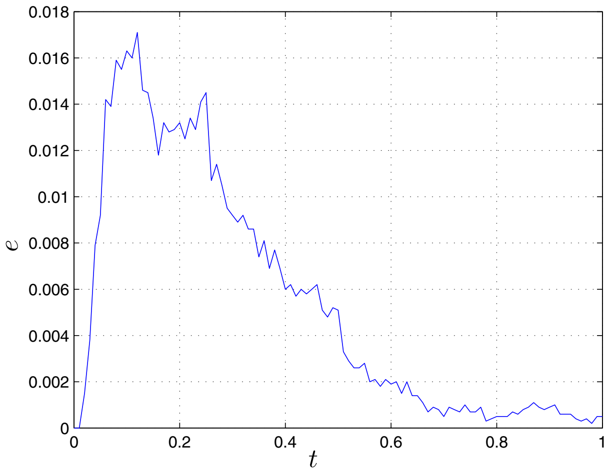

Fig. 1. Exit ratio vs. t for the γ–PIS

7. CONCLUSIONS

We have extended the notions of PUB and PIS to the continuous-time domain, deriving their main properties and providing formulas for their calculation. In the case of PIS, a redefinition was required to take into account the fundamental limitations imposed by the infinite-bandwidth nature of continuous-time white noise. Then, a controller design technique was presented to assign a predetermined PUB having a given probability p for a system given in controller canonical form. The results were illustrated with a numerical example. Future work will include the extension of the results to more general classes of linear systems.

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