

# Discrete-Time Robust Steady-State Control of Nonlinear Multivariable Systems: A Unified Approach

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**Abstract:** The steady-state control of multivariable nonlinear discrete-time, time-invariant systems in the presence of arbitrary unmeasurable but bounded disturbances is addressed in this paper. The pseudoinverse model approach as a unified concept to cope with possible noninvertibility and to achieve a desired behavior of a wide class of both linear and of nonlinear multi-input multi-output square and also nonsquare systems is proposed. It is assumed that the number of the system outputs is not less than the number of its control inputs. Some results regarding equilibrium states are given. In particular, it is shown that the equilibrium state may not exist, in general. A simple linear pseudoinverse-based controller of the integral action is designed for regulating these nonlinear multivariable systems. The properties of robust stability and the boundedness of all signals caused by this controller are derived. Numerical examples are given to support the theoretical investigations.

**Keywords:** Discrete time, feedback control methods, model-based control, multi-input/multi-output systems, nonlinearity, robustness, stability analysis

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## 1. INTRODUCTION

The problem of controlling multivariable systems subjected to arbitrary unmeasurable disturbances stated several decades ago remains actual up to now (Liu and Peng, 2002; Lyubchik, 2011). It is important problem from both theoretical and practical point of view (Freudenberg and Middleton, 1999; Glad and Ljung, 2000; Skogestad and Postlethwaite, 1996).

Since the seventies, the internal model method becomes popular among other methods dealing with an improvement of the control system by exploiting the different types of plant and disturbances models. Based on this principle, multivariable regulator problem was first approached by Francis and Wonham (1976).

A perspective modification of the internal model control principle is the so-called model inverse approach. The perfect output control performance is an important multivariable control problem closely related to inverse systems. The problem of inversion of linear time-invariant multivariable systems has attracted the attention of several researches (Lovass-Nagy *et al.*, 1976; Seraji, 1989). During last years, a significant progress in this research area has been achieved by Liu and Peng (2002), Marro *et al.* (2002) and Lyubchik (2011). Most of their works except (Lyubchik, 2011; Marro *et al.*, 2002) dealt with continuous-time multivariable systems.

To the best of author's knowledge, an inverse model approach to ensuring perfect steady-state regulation and disturbance rejection in linear discrete-time multivariable sys-

tems was first advanced in Lee *et al.* (1968, chapt. 8). Similar discrete-time process control system containing the inverse model-based controller was developed by Skurikhin *et al.* (1990) to deal with steady-state control of this system in the presence of model/plant mismatch. The steady-state control of linear multivariable systems discussed in Seraji (1989, p. 2094) in the framework of the problem of minimal inversion has also been studied in Katkovnik and Pervozvansky (1973) who derived the robust stability conditions of nonlinear discrete-time multivariable control systems with a linear model-based feedback. Meanwhile, general results related to the feedback design for the robust global stability of continuous-time multi-input multi-output systems are established in Isidori (1999, sect. 11.5).

Unfortunately, the inverse model approach is quite unacceptable if the systems to be controlled are square but singular or if they are nonsquare. Several researches including Skogestad and his colleagues whose works are cited in (Skogestad *et al.*, 1988) observed that the inverse model-based controller may be also not admissible for designing some process control systems containing ill-conditioned plants because they may become (almost) noninvertible in the presence of an uncertainty.

It turned out that the so-called generalized inverse (pseudoinverse) model approach first proposed in Lovass-Nagy *et al.* (1976) can be exploited to cope with the noninvertibility of nonsquare systems. In this paper, their approach is extended to controlling a wide class of discrete-time multivariable systems.

The basic contribution of this paper is the utilization of the pseudoinverse model concept as a tool for dealing with the steady-state control of both linear and nonlinear multivariable systems in the presence of arbitrary bounded disturbances. The main effort is focused on deriving robustness and boundedness results.

## 2. THE DESCRIPTION OF CONTROL SYSTEM AND PROBLEM FORMULATION

The plant to be controlled is a nonlinear multivariable time-invariant system whose static characteristic is

$$y = \varphi(u), \quad (1)$$

where  $y = [y^{(1)}, \dots, y^{(m)}]^T$  denotes the  $m$ -dimensional output vector,  $u = [u^{(1)}, \dots, u^{(r)}]^T$  denotes the  $r$ -dimensional input (control) vector, and  $\varphi: \mathbf{R}^r \rightarrow \mathbf{R}^m$  represents some unknown nonlinear vector-valued function given by

$$\varphi(u) = [\varphi^{(1)}(u), \dots, \varphi^{(m)}(u)]^T. \quad (2)$$

Suppose that the number of inputs does not exceed the number of outputs:

$$r \leq m. \quad (3)$$

The following basic assumption with respect to the nonlinearity  $\varphi(u)$  will be required.

*Assumption 1.* The components  $\varphi^{(1)}(u), \dots, \varphi^{(m)}(u)$  of  $\varphi(u)$  in (2) are all the continuously differentiable functions of the variables  $u^{(1)}, \dots, u^{(r)}$  such that each partial derivative does not change its sign and remains uniformly bounded for all  $u$  from  $\mathbf{R}^r$  according to

$$b_{\min}^{(ij)} \leq \partial \varphi^{(i)}(u) / \partial u^{(j)} \leq b_{\max}^{(ij)}, \quad 0 < b_{\min}^{(ij)} b_{\max}^{(ij)} < \infty, \quad (i=1, \dots, m; j=1, \dots, r), \quad (4)$$

where  $b_{\min}^{(ij)}, b_{\max}^{(ij)}$  are assumed to be known.

In order to implement the discrete-time control, the signals  $y^{(1)}(t), \dots, y^{(m)}(t)$  given in the continuous time  $t$  need to be sampled with a sampling period  $T_0$  to yield the sequences  $\{y^{(i)}(nT_0)\}$ , whereas the control signals are of zero-order sample-hold type, i.e.,

$$u^{(i)}(t) = u^{(i)}(nT_0) \text{ for } nT_0 \leq t < (n+1)T_0, \quad i=1, \dots, r.$$

As in (Katkovnik and Pervozvansky, 1973; Lee *et al.*, 1968), suppose that the sampling period  $T_0$  is large enough so that the transient stage caused by stepwise changes of inputs  $u^{(1)}(t), \dots, u^{(r)}(t)$  at each  $(n-1)$ th time instant  $t = (n-1)T_0$  may practically be completed during the time interval  $[(n-1)T_0, nT_0)$ . In view of (1), this narrative description of the discrete-time steady-state control gives that the steady state of this multivariable system can be mathematically modelled by the first-order nonlinear difference equation

$$y_n = \varphi(u_{n-1}) \quad (5)$$

similar to that in Katkovnik and Pervozvansky (1973), if any disturbances are absent. In this equation, the notations  $y_n := y(nT_0)$  and  $u_n := u(nT_0)$  are introduced (for the simplicity of exposition).

In practical applications, the outputs  $y^{(1)}(t), \dots, y^{(m)}(t)$  are usually influenced by certain classes of persistent external disturbances  $d^{(1)}(t), \dots, d^{(m)}(t)$ , respectively. Then, instead of (5), another equation

$$y_n = \varphi(u_{n-1}) + d_n \quad (6)$$

with the disturbance vector  $d_n := [d_n^{(1)}, \dots, d_n^{(m)}]^T$  as a steady-state model of system will be further considered.

Now, the following assumption about  $\{d_n\}$  is introduced.

*Assumption 2.* The components of  $d_n$  are upper bounded in modulus:

$$|d_n^{(i)}| \leq \varepsilon_i \quad (i=1, \dots, m). \quad (7)$$

Let  $y^* := [y^{*(1)}, \dots, y^{*(m)}]^T$  ( $y^{*(i)} \equiv \text{const}$ ) be some nonzero vector defining the desired output vector (a given set-point).

The following assumption with respect to this vector is made.

*Assumption 3.*  $y^*$  is not the  $m$ -dimensional zero-vector  $0_m = [0, \dots, 0]^T$  implying that

$$|y^{*(1)}| + \dots + |y^{*(m)}| \neq 0. \quad (8)$$

Similarly to Katkovnik and Pervozvansky (1973), the control law will be chosen of the following form

$$u_n = u_{n-1} + A e_n, \quad (9)$$

where  $A$  is a fixed  $r \times m$  matrix chosen by the designer, and  $e_n$  represent the output error vector at  $n$ th time instant  $t = nT_0$  specified as

$$e_n = y^* - y_n, \quad (10)$$

The equations (9), (10) describe the simple linear controller of the integral action.

The problem is to derive conditions under which the closed-loop nonlinear control system given by (6), (9), (10) provided that  $d_n \equiv 0$  will be robust stable for all the family of nonlinearities  $\varphi(u)$  satisfying (4), and also will remain BIBS (bounded-input bounded-state) stable with an arbitrary disturbance vector  $d_n \neq 0$  whose components satisfy (7).

## 3. PRELIMINARIES

Denoting  $b^{(ij)}(u) := \partial \varphi^{(i)}(u) / \partial u^{(j)}$ , introduce the matrix

$$B(u) = \begin{pmatrix} b^{(11)}(u) & \dots & b^{(1r)}(u) \\ \vdots & & \vdots \\ b^{(m1)}(u) & \dots & b^{(mr)}(u) \end{pmatrix} \quad (11)$$

which represents the  $m \times r$  Jacobian matrix whose elements play a role of some “dynamical” gains from the  $j$ th input  $u^{(j)}$  to the  $i$ th output  $y^{(i)}$  for each fixed  $u \in \mathbf{R}^r$ . In view of (4), the rank of  $B(u)$  given by (11) satisfies

$$1 \leq \text{rank } B(u) \leq r, \quad (12)$$

where (3) together with the well-known property of the rank of any matrix are utilized.

Let  $u^e$  and  $y^e = \varphi(u^e)$  define an equilibrium state  $\{u^e, y^e\}$  of the feedback control system (6), (9), (10) with no disturbance. It can be clarified that the vector  $u^e$  is a solution  $u = u^e$  of the equation

$$A(y^* - \varphi(u)) = 0. \quad (13)$$

In the linear case, where  $\varphi(u)$  is defined as

$$\varphi(u) = Bu$$

with some numerical  $m \times r$  matrix  $B = (b^{(ij)})$  whose rank satisfies (12), the equation (13) becomes

$$A(y^* - Bu) = 0. \quad (14)$$

It turns out that  $u^e$  may not exist, in general, even in the linear case if  $B$  is not the square non-singular matrix. In this case, the answer to the question related to the existence of the equilibrium state  $\{u^e, y^e\}$  will be given below.

*Lemma 1.* Subject to Assumption 3, the equilibrium state of the feedback control system, consisting of the controller given by (9), (10) and of the plant

$$y_n = Bu_{n-1}, \quad (15)$$

exists iff

$$\text{rank}(AB) = \text{rank}(AB : Ay^*). \quad (16)$$

*Proof.* Immediate from (14) after rewriting this equation as

$$ABu = Ay^* \quad (17)$$

and applying the well-known Kronecker–Capelli theorem given in Marcus and Minc (1964, item 3.1.2) to (17).  $\square$

*Comment 1.* Suppose that, in addition to (8),  $y^*$  does not lie on the range of  $B$  denoted as  $\mathfrak{R}(B)$ :

$$y^* \notin \mathfrak{R}(B).$$

(The definition of  $\mathfrak{R}(B)$  may be found, in particular, in Albert (1972, p. 10).) By the definition of the set  $\mathfrak{R}(B)$  to which  $0_m$  belongs, it directly follows that  $(-Bu) \in \mathfrak{R}(B)$ . This gives that, for any  $u \in \mathbf{R}^r$ , the vector  $y^* - Bu$  must belong to the sum of  $\{y^*\}$ , consisting of the unique  $y^* \neq 0_m$ , and of  $\mathfrak{R}(B)$  which is some linear subspace whose dimension is  $\dim \mathfrak{R}(B) = \text{rank } B$ ; see Albert (1972, p. 52). Obviously,  $\{y^*\} + \mathfrak{R}(B)$  does not contain  $0_m$  and is the

manifold in  $\mathbf{R}^m$  parallel to  $\mathfrak{R}(B)$ . On the other hand, by the definition of the so-called null space of  $A$  denoted as  $\mathfrak{N}(A)$  and given in Albert (1972, p. 10), it can be concluded from (3) that the vector  $y^* - Bu$  must also belong to  $\mathfrak{N}(A)$ . This fact shows that equilibrium points  $u^e$  exist if and only if the intersection of  $\mathfrak{N}(A)$  and of  $\{y^*\} + \mathfrak{R}(B)$  is a non-empty set:

$$\mathfrak{N}(A) \cap (\{y^*\} + \mathfrak{R}(B)) \neq \emptyset. \quad (18)$$

Therefore, the requirement (18) gives the necessary and sufficient conditions for the existence of the equilibrium point  $y^e = Bu^e$ . The geometric interpretation of these conditions is presented in Fig. 1.

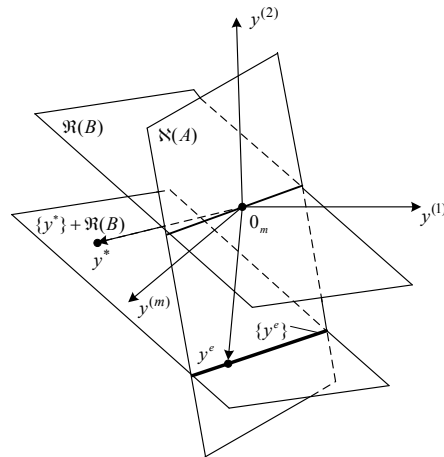


Fig. 1. The meaning of the expression (18)

To demonstrate the fact that the equilibrium state of the control system (15), (9), (10) for  $d_n \equiv 0$  may not exist, an example is given below.

*Example 1:* Let

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}, \quad A = \begin{pmatrix} 0.05 & 1 & -0.1 \\ -1 & 0.05 & 0.05 \end{pmatrix}$$

( $\text{rank } B = 1, \text{rank } A = 2$ ). Put  $y^* = [5, 13, 7]^T$ . In this case, the equation

$$\begin{pmatrix} 2.15 & 4.3 \\ -0.95 & -1.9 \end{pmatrix} u = \begin{pmatrix} 12.55 \\ -4 \end{pmatrix}$$

corresponding to (17) has no solution. With this  $y^*$  and initial  $u_0 = [1, 2]^T$ , the behavior of this control system is shown in Fig. 2. It is observed that the norm of the control input  $\|u_n\|$  unboundedly increases whereas the norm of the output  $y_n$  goes to a constant as  $n \rightarrow \infty$ .

*Lemma 2.* Let an equilibrium state,  $\{u^e, y^e\}$ , of the nonlinear control system (6), (9), (10) exist. If Assumption 1 takes place, then the sufficient condition under which this system

will be stable in the sense that  $u_n \xrightarrow{n \rightarrow \infty} u^e$  meaning  $y_n \xrightarrow{n \rightarrow \infty} y^e$  is

$$\sup_{u \in \mathbf{R}^r} \|I_r - AB(u)\| < 1 \quad (19)$$

for any norm  $\|\cdot\|$ , where  $I_r$  denotes the  $r \times r$  identity matrix.

*Proof.* See the proof of Theorem 1 in Katkovnik and Pervozvansky (1973). The proof of Lemma 1 is based on using the so-called contraction mapping principle after exploiting the fact that if  $\varphi(u)$  is differentiable then

$$\varphi(u_n) - \varphi(u^e) = \int_0^1 B(u^e + \tau(u_n - u^e))(u_n - u^e) d\tau; \quad (20)$$

see Polyak (2010, p. 4). □

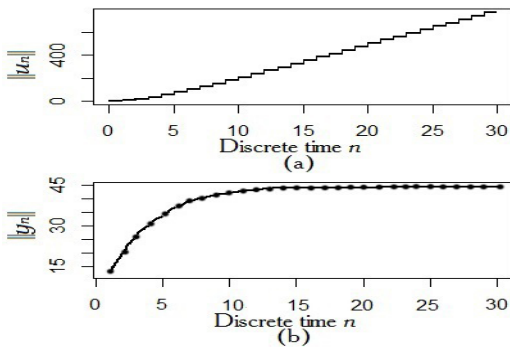


Fig. 2. (a) the norm of control input; (b) the norm of output in Example 1

*Comment 2.* Note that if (19) is satisfied then the feedback control system described by (6), (9), (10) is stable but not *vice versa*. Namely, if  $r = m$  and  $B(u)$  is a singular  $r \times r$  matrix for some  $u \in \mathbf{R}^r$ , then at least one of the eigenvalues,  $\lambda_i$ , of the matrix  $Q_u(u) := I_r - AB(u)$  is 1. In fact, if  $\det B(u) = 0$  then  $\lambda_i(AB(u)) = 0$  which yields

$$\lambda_i(Q_u(u)) = 1 - \lambda_i(AB(u)) = 1 \quad (21)$$

for some  $i$ , since the property

$$\lambda_i(I - C) = 1 - \lambda_i(C)$$

is valid for any matrix  $C$ . See Marcus and Minc (1972, item 2.15.3). On the other hand, due to the Brauer theorem given in Marcus and Minc (1972, p. 145), it can be written

$$\min\{\|Q_u(u)\|_1, \|Q_u(u)\|_\infty\} \geq |\lambda_i(Q_u(u))| \quad (22)$$

using the notations of the corresponding norms

$$\|P\|_1 := \max_{1 \leq i \leq r} \sum_{j=1}^r |p^{(ij)}|, \quad \|P\|_\infty := \max_{1 \leq j \leq r} \sum_{i=1}^r |p^{(ij)}| \quad (23)$$

of some  $r \times r$  matrix  $P = (p^{(ij)})$  taken from Polyak and Shcherbakov (2002, p. 259). Further, by the Browne theorem which can be found in Marcus and Minc (1972, p. 144) one has

$$\|Q_u(u)\|_2 \geq |\lambda_i(Q_u(u))|, \quad (24)$$

where  $\|Q\|_2 := \max_{1 \leq i \leq r} \lambda_i^{1/2}(Q^T Q)$  denotes the spectral norm of  $Q$  used in Polyak and Shcherbakov (2002, p. 259).

By virtue of (21), the inequalities (22) and (24) do not guarantee that (19) will be satisfied. It turns out that at least in a linear case, the stability condition given by (19) can be relaxed. This fact is established in the theorem below.

*Theorem 1.* Consider the linear feedback control system (15), (9), (10) with  $d_n \equiv 0$ , subject to Assumption 3. Suppose the condition (16) is satisfied. Then this system will be stable if and only if the matrix  $Q_u := I_r - AB$  has no eigenvalues outside the unit circle and if its eigenvalues on the unit circle correspond to Jordan blocks of order 1.

*Proof.* Immediate from Lemma 1 and utilizing the equation

$$\tilde{u}_n = Q_u \tilde{u}_{n-1} \quad \text{with } \tilde{u}_n = u_n - u^e$$

produced by (9), (10), (14), (15). See Willems (1970, p. 49). □

*Comment 3.* Note that if  $B$  is not a matrix of the full rank implying  $1 \leq \text{rank } B < r$ , then the conditions of Lemma 2 can be satisfied whereas  $\|Q_u\| = 1$ . Moreover, an interesting observation is here. Namely, since (9), (10), (15) yield

$$e_n = Q_y e_{n-1},$$

where  $Q_y := I_m - BA$ , it can be seen that there exists always a finite  $\lim_{n \rightarrow \infty} e_n = e_\infty$  if all  $\lambda_i(Q_y)$ s satisfy the same condition as  $\lambda_i(Q_u)$ s, whereas a finite  $\lim_{n \rightarrow \infty} u_n$  does not exist if there is no an equilibrium state. See Fig. 2.

## 4. MAIN RESULTS

### 4.1. Control strategy

The so-called generalized inverse model approach is proposed as a unified approach to the steady-state control of both linear and of nonlinear multivariable systems. According to this approach, a fixed matrix  $B_0 = (b_0^{(ij)})$  with the elements satisfying  $b_{\min}^{(ij)} \leq b_0^{(ij)} \leq b_{\max}^{(ij)}$  is taken. Next, the generalized (pseudoinverse)  $r \times m$  matrix  $B_0^+ = (\beta_0^{(ij)})$  defined by

$$B_0^+ = \lim_{\delta \rightarrow 0} B_0^T (B_0^T B_0 + \delta^2 I_r)^{-1} \quad (25)$$

is specified. (The definition (25) of  $B_0^+$  is given in Albert (1972, p. 19).) By choosing the matrix  $A$  as

$$A = B_0^+, \quad (26)$$

the equation (9) produces the linear pseudoinverse model-based control

$$u_n = u_{n-1} + B_0^+ e_n. \quad (27)$$

Note that if  $r = m$  and  $B_0$  is a square non-singular matrix, then (26) transforms to

$$A = B_0^{-1},$$

since  $B^+$  becomes the usual inverse matrix,  $B^+ = B^{-1}$ ; see Albert (1972, p. 20).

To implement the control law (27), one needs the discrete integrator whose output is

$$u_n = \sum_{k=1}^n \Delta u_k, \quad (28)$$

where

$$\Delta u_n = B_0^+ e_n. \quad (29)$$

Due to (29), (28), this controller plays the role of an I-type multivariable controller with a matrix gain  $B_0^+$  (see Fig. 3).

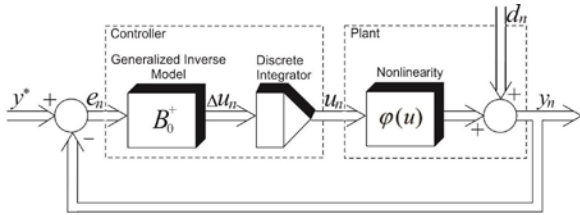


Fig. 3. Configuration of the control system (6), (10), (28), (29)

#### 4.2. Robustness and boundedness properties

The robust stability property of the closed-loop system containing the linear controller described in (27) together with (10) and applied to controlling the nonlinear plant (5) is the basic result presented in the next theorem.

**Theorem 2.** Consider the feedback control system described by (5), (10), (27), subject to Assumptions 1 and 3. Let  $B_0$  be the fixed matrix of the full rank, i.e.,

$$\text{rank } B_0 = r. \quad (30)$$

Then this system will be robust stable for all the family of nonlinearities  $\varphi(u)$  satisfying (4) if

$$q < 1 \quad (31)$$

$$\text{with } q := \max_{1 \leq k \leq r} \sum_{i=1}^r \sum_{j=1}^m \max_{\underline{\delta}^{(ij)} \leq \delta^{(ij)} \leq \bar{\delta}^{(ij)}} |\beta_0^{(kj)} \delta^{(ji)}|,$$

where

$$\underline{\delta}^{(ij)} = b_{\min}^{(ij)} - b_0^{(ij)}, \quad \bar{\delta}^{(ij)} = b_{\max}^{(ij)} - b_0^{(ij)}. \quad (32)$$

*Proof.* Exploits the one fact given in the following lemma.

**Lemma 3.** If  $B_0^+$  is the  $r \times m$  matrix of full rank then

$$B_0^+ B_0 = I_r. \quad (33)$$

*Proof of Lemma 3.* In view of Binet-Cauchy formula given in Marcus and Minc (1964, p. 14) it can be concluded from (30) that  $\det(B_0^T B_0) \neq 0$ . In this case, the inversion  $(B_0^T B_0)^{-1}$  exists and  $B_0^+$  is determined as  $B_0^+ = (B_0^T B_0)^{-1} B_0^T$ . See Al-

bert (1972, p. 21). This allows to establish the validity of (33).  $\square$

Now, defining the  $m \times r$  matrix  $\Delta(u) = (\delta^{(ij)}(u))$  as

$$\Delta(u) = B(u) - B_0, \quad (34)$$

it can be observed that, due to (4) together with (32), its elements lie within the intervals

$$\underline{\delta}^{(ij)} \leq \delta^{(ij)}(u) \leq \bar{\delta}^{(ij)}. \quad (35)$$

Further, utilizing (33) together with (32), (34) and (35), the condition (19) of Lemma 2 can be replaced by

$$\max_{\Delta(u): \delta^{(ij)}(u) \in [\underline{\delta}^{(ij)}, \bar{\delta}^{(ij)}]} \|B_0^+ \Delta(u)\| < 1, \quad (36)$$

where (26) has been used. Next, employing the definitions of the norm  $\|\cdot\|_1$  given in first expression of (23) and the matrix product, from (36) the validity of (31) with the notation of  $q$  follows. This completes the proof.  $\square$

**Corollary.** Under the conditions of Theorem 2, the feedback control system (6), (10), (28) will be robust stable if

$$\max_{\Delta(u): \delta^{(ij)}(u) \in [\underline{\delta}^{(ij)}, \bar{\delta}^{(ij)}]} \|\Delta(u)\|_1 < 1 / \|B_0^+\|_1. \quad (37)$$

*Proof.* Result follows from the definition of  $\|\cdot\|_1$  after employing the fact that the inequality  $\|B_0^+\| \|\Delta\| < 1$  guarantees that (36) will be satisfied.  $\square$

**Comment 4.** The conditions (31), (37) are the sufficient conditions for the robust stability of the nonlinear closed-loop system (5), (10), (27). The verification of (31) can be reduced to the linear programming problem similar to that in Polyak and Shcherbakov (2002, Theorem 4.15). Note that in the case of square system ( $r = m$ ), the set of all  $B(u)$ s needs to be the set of nonsingular matrices in order to satisfy (37). This fact follows from Lemma 7.2 of Polyak and Shcherbakov (2002, p. 201).

The boundedness property is derived in the theorem below.

**Theorem 3.** Let in addition to the conditions of Theorem 2,  $d_n \neq 0$  and Assumption 2 be satisfied. If  $q < 1$  then

$$\lim_{n \rightarrow \infty} \sup \|u_n - u^e\| \leq q \|u_0 - u^e\| + \|B_0^+\| \varepsilon (1 - q)^{-1}.$$

*Proof.* Proceeds along the lines of the proof of Theorem 1 in Katkovnik and Pervozvansky (1973) by utilizing (20) *mutatis mutandis*. Due to space limitation, details are omitted.  $\square$

**Comment 5.** It can be observed that the condition (31) is also sufficient to achieve the robust stability of the linear closed-loop system described in Theorem 1 for the interval matrices  $\bar{B} = (\bar{b}^{(ij)})$  whose elements satisfy  $b_{\min}^{(ij)} \leq \bar{b}^{(ij)} \leq b_{\max}^{(ij)}$ . The conditions of this theorem in which  $\lambda_i(Q_u)$ s are replaced by corresponding  $\lambda_i(\bar{Q}_u)$ s with  $\bar{Q}_u := I_r - B_0^+ \bar{B}$  give the necessary and sufficient conditions guaranteeing its robustness properties.

4.3. Simulation

To illustrate the robustness and boundedness properties established above, results of two simulation examples called Example 2 and Example 3 are presented. The nonlinear system

$$\left. \begin{aligned} y_n^{(1)} &= 5u_{n-1}^{(1)} + 4u_{n-1}^{(2)} + 12 \tanh(u_{n-1}^{(1)}) + 18 \tanh(u_{n-1}^{(2)}) + d_n^{(1)} \\ y_n^{(2)} &= 2u_{n-1}^{(1)} + 4u_{n-1}^{(2)} + 8 \tanh(u_{n-1}^{(1)}) + 14 \tanh(u_{n-1}^{(2)}) + d_n^{(2)} \\ y_n^{(3)} &= 8u_{n-1}^{(1)} + u_{n-1}^{(2)} + 10 \tanh(u_{n-1}^{(1)}) + 20 \tanh(u_{n-1}^{(2)}) + d_n^{(3)} \end{aligned} \right\}$$

was considered. In both experiments,  $B_0$  was chosen as

$$B_0 = \begin{pmatrix} 11 & 6 & 13 \\ 13 & 11 & 11 \end{pmatrix}^T \text{ giving } B_0^+ = \alpha \begin{pmatrix} -55 & -1406 & 1471 \\ 366 & 1474 & -990 \end{pmatrix}$$

and satisfying  $b_0^{(ij)} \in [b_{\min}^{(ij)}, b_{\max}^{(ij)}]$ , where  $\alpha = 1/10082$ .

*Example 2.* In this example, the disturbance  $d_n$  was absent.

*Example 3.* In this example,  $d_n$  was simulated as a pseudo-random variable within  $[-1, 1]$ .

Results of the simulation experiments with  $y^* = [5, 13, 7]^T$  and initial  $u_0 = [1, 2]^T$  showing the robust stability and the boundedness of all signals are depicted in Fig. 4 and 5.

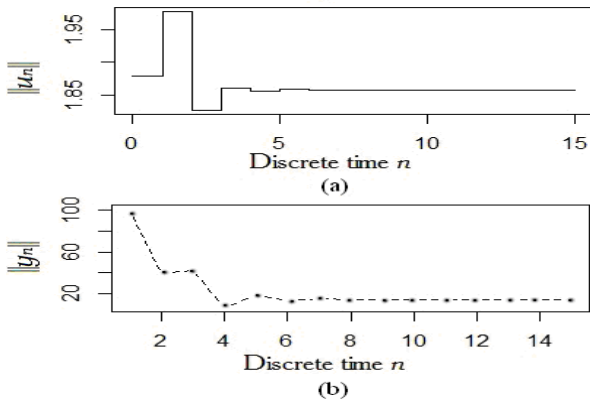


Fig. 4. (a) the norm of control input; (b) the norm of output in Example 2

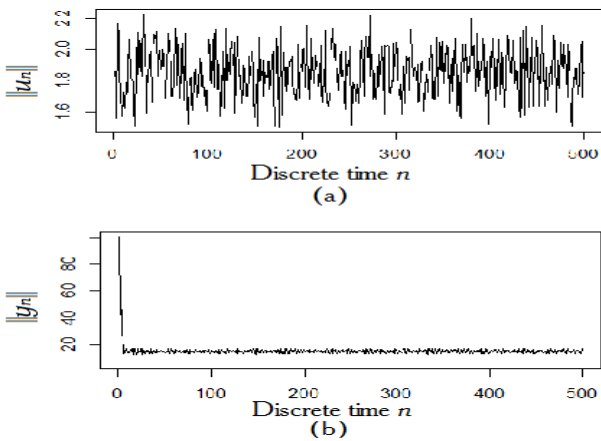


Fig. 5. (a) the norm of control input; (b) the norm of output in Example 3

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