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Controllability, Observability and Eigenvalue Assignment on Isolated Time Scales

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Abstract: In this paper, some theoretical results on controllability, observability and duality for isolated time scales systems are given. It is also shown that eigenvalue assignment via a state feedback (output injection) controller on isolated time scales yields desired system behavior of a continuous system. Furthermore, a method for designing such a controller is introduced.

Keywords: Time scales systems, Nonuniform Sampling, Controllability, Observability, Duality, State Feedback, Output Injection, Eigenvalue Assignment

1. INTRODUCTION

Time scales is introduced as a PhD thesis by Stefan Hilger in 1988 which unifies continuous and discrete time systems. Recently, time scales has gained attention in control theory (Jackson et al. [2009], Pawluszewicz and Bartosiewicz [2005], Gravagne et al. [2009], Davis et al. [2009b], Davis et al. [2010], Bartosiewicz and Pawiuszewicz [2004], Sevim and Goren-Sumer [2012]).
In computer controlled systems, constant sampling periods may not be achieved due to some practical problems like jitter, computational delays, communication delays, etc. They often result in undesired system behavior including instability. The concept of isolated time scales provides a natural framework and a powerful tool to analyze systems under nonuniform sampling.
We provided the fundamental system theoretical results for dynamical systems defined on isolated time scales, namely controllability, observability and duality. Since controllability (observability) does not imply arbitrary eigenvalue assignment on time scales, a new concept "assignability" is introduced. Also, we gave a design method based on the time scales model of a continuous system for eigenvalue assignment via a time scales state feedback (output injection) controller.
In the second section we gave a summary of the time scales concept and provide the necessary theoretical results. Definitions and results on controllability, observability, duality are given in the third section. Section four is devoted to eigenvalue assignment via state feedback and output injection. Some illustrative examples are given in Section 5. Also, a discussion about open questions are given in Conclusion.

2. TIME SCALES SUMMARY

In this section, elementary definitions and theorems are given. Most of the definitions and results are summarized from Bohner and Peterson [2001], Hilger [1990], Agarwal et al. [2002], Bohner and Lutz [2001] and Gravagne et al. [2007]. The proofs of these results can be found in the corresponding references. Some new results used throughout this paper are also developed in this section. The proofs of these results are also given.

2.1 Time Scales

Time scales is defined as a nonempty subset of real numbers ($\mathbb{T} \neq \emptyset$, $\mathbb{T} \subseteq \mathbb{R}$). Time scales systems become continuous systems for $\mathbb{T} = \mathbb{R}$ and become discrete time systems for $\mathbb{T} = h\mathbb{Z} \triangleq \{hk | k \in \mathbb{Z}\}$ for any $h > 0$. But any subset of real numbers can be arbitrarily selected as time scales.

Definition 1. The functions σ, $\rho : \mathbb{T} \to \mathbb{T}$ are defined as

$$\sigma(t) \triangleq \inf\{s \in \mathbb{T} \mid s > t\} \text{ and } \rho(t) \triangleq \sup\{s \in \mathbb{T} \mid s < t\}$$

and called as *forward jump operator* and *backward jump operator* respectively. Also, $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$ are assumed.

Definition 2. Any selected point $t \in \mathbb{T}$ is called

i) *right-dense* if $\sigma(t) = t$
ii) *right-scattered* if $\sigma(t) > t$
iii) *left-dense* if $\rho(t) = t$
iv) *left-scattered* if $\rho(t) < t$
v) *isolated* if $\rho(t) < t < \sigma(t)$
vi) *dense* if $\rho(t) = t = \sigma(t)$

Definition 3. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined as $\mu(t) \triangleq \sigma(t) - t$.

Definition 4. A time scales with all points are isolated, is called *isolated time scales* and denoted as $\mathbb{T}_+$. In another way,

$$\mathbb{T}_+ \triangleq \{t_k | k \in \mathbb{Z}\}, t_k \in \mathbb{R} \text{ and } t_k < t_{k+1}, \forall k \in \mathbb{Z}.$$

Then, we can find graininess function of $\mathbb{T}_+$ as follows:

$$\mu_k \triangleq \mu(t_k) = \sigma(t_k) - t_k = t_{k+1} - t_k, \forall k \in \mathbb{Z}.$$

Definition 5. The set of all possible graininess values of a time scales $\mathbb{T}$ is called *graininess set of* $\mathbb{T}$ and denoted as $M(\mathbb{T})$.

A few examples of graininess sets can be given as follows:

i) $M(\mathbb{R}) = \{0\}$.
ii) $M(h\mathbb{Z}) = \{h\}$ for some $h > 0$.
iii) $M(\mathbb{T}_+) = \{\mu_k = t_{k+1} - t_k \mid k \in \mathbb{Z}, t_k \in \mathbb{T}_+\}$ for any isolated time scales $\mathbb{T}_+$.

Note that $M(\mathbb{T})$ is a countable set for any time scales $\mathbb{T}$.

2.2 Delta Derivative

Definition 6. We define the set $\mathbb{T}^\kappa$ as follows:

$$\mathbb{T}^\kappa \triangleq \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if } \sup\mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup\mathbb{T} = \infty \end{cases}$$

Definition 7. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. f is called *delta differentiable* at t, if there exists an α such that, for any given $\varepsilon > 0$ a neighborhood of t ($\mathcal{N} = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) exists with the property

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leqslant \varepsilon\, |\sigma(t) - s|\,, \forall s \in \mathcal{N}.$$

In this case, α is called as *delta derivative of f at t* and denoted as $f^\Delta(t)$. If f is delta differentiable for all $t \in \mathbb{T}^\kappa$, then f is called *delta differentiable (or simply differentiable) on* $\mathbb{T}$ and $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ becomes a new function.
If f is differentiable on $\mathbb{T}$, delta derivative of f becomes as follows:

$$f^\Delta(t) = \begin{cases} \lim\limits_{s \to t, s \in \mathbb{T}} \dfrac{f(t) - f(s)}{t - s} & , \text{if } \mu(t) = 0 \\ \dfrac{f(\sigma(t)) - f(t)}{\mu(t)} & , \text{if } \mu(t) > 0 \end{cases}$$

Note that delta derivative becomes classical derivative for $\mathbb{T} = \mathbb{R}$, and forward difference for $\mathbb{T} = \mathbb{Z}$.

The notation $f^\sigma(t) = f(\sigma(t))$ will be used in the rest of the paper.

Definition 8. $F : \mathbb{T} \to \mathbb{R}$ is called *antiderivative* of $f : \mathbb{T}^\kappa \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. In this case, the definite antiderivative is denoted as

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \tag{1}$$

for $s, t \in \mathbb{T}$.

Note that antiderivative becomes classical integral for $\mathbb{T} = \mathbb{R}$ and sum for $\mathbb{T} = \mathbb{Z}$. On an isolated time scale,

$$\int_{t_0}^{t_k} f(\tau)\Delta\tau = \sum_{i=0}^{k-1} \mu_i f(t_i) \tag{2}$$

can be written where $t_i \in \mathbb{T}_+$ $(i = 1, 2, 3, \dots)$.

Definition 9. A function f defined on $\mathbb{T}$ is called *rd-continuous* if it is continuous at all right-dense points and has a left limit on all left-dense points.

Theorem 10. (Hilger [1990]). Every rd-continuous function has an antiderivative.

2.3 Linear Dynamic Equations

Definition 11. Let $A(t)$ be a $n \times m$ matrix valued function defined on $\mathbb{T}$.

i) $A(t)$ is called *rd-continuous* if all elements of $A(t)$ is rd-continuous.
ii) $A(t)$ is called *differentiable* if all elements of $A(t)$ is differentiable. In this case, the delta derivative of $A(t)$ is defined as

$$A^\Delta(t) = (a_{ij}^\Delta(t))\ ,\ \ (i = 1, 2, ..., n)(j = 1, 2, ..., m)$$

where $A(t) = (a_{ij}(t))$.

Definition 12. Let $A(t)$ be a $n \times n$ matrix valued function defined on $\mathbb{T}$. $A(t)$ is called *regressive* if the matrix

$$I + \mu(t)A(t)$$

has an inverse for all $t \in \mathbb{T}^\kappa$.

Definition 13. Let $A(t)$ be a regressive $n \times n$ matrix valued function defined on $\mathbb{T}$. Then the unary operator $\ominus$ can be defined as

$$\ominus A(t) \triangleq -A(t)[I + \mu(t)A(t)]^{-1}.$$

Theorem 14. Let $A(t)$ be a regressive $n \times n$ matrix valued function defined on $\mathbb{T}$. Then the following properties hold for all $t \in \mathbb{T}$:

i) $\ominus A(t)$ is regressive.
ii) $\ominus(\ominus A(t)) = A(t)$
iii) $\ominus(A^T(t)) = (\ominus A(t))^T$
iv) $-A(t)[I + \mu(t)A(t)]^{-1} = -[I + \mu(t)A(t)]^{-1}A(t)$

Proof. The results follow by just writing the definitions. The only trick used is writing

$$I = [I + \mu(t)A(t)][I + \mu(t)A(t)]^{-1}.$$

Theorem 15. (Bohner and Lutz [2001]). Let A be a $n \times n$ regressive, matrix valued function defined on $\mathbb{T}$. Suppose $f : \mathbb{T} \to \mathbb{R}^n$ is a rd-continuous function, $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then, the initial value problem

$$x^\Delta = A(t)x + f(t), \quad x(t_0) = x_0 \tag{3}$$

has one and only one solution.

Definition 16. Let $t_0 \in \mathbb{T}$ and A be a $n \times n$ regressive, matrix valued function defined on $\mathbb{T}$. Then, the unique solution of the initial value problem

$$X^\Delta = A(t)X, \quad X(t_0) = I \tag{4}$$

is called *the matrix exponential function* and denoted as $e_A(\cdot, t_0)$.

Theorem 17. (Bohner and Peterson [2001]). Let $t, s, r \in \mathbb{T}$ and A be a $n \times n$ regressive, matrix valued function defined on $\mathbb{T}$. Then,

i) $e_0(t, s) = I$ and $e_A(t, t) = I$
ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$
iii) $e_A(t, s)$ has an inverse.
iv) $e_A^{-1}(t, s) = e_{\ominus A^T}^T(t, s)$
v) $e_A(t, s) = e_A^{-1}(s, t)$
vi) $e_A(t, s)e_A(s, r) = e_A(t, r)$

Theorem 18. (Bohner and Peterson [2001]). Let A be a $n \times n$ regressive, matrix valued function defined on $\mathbb{T}$. Suppose $f : \mathbb{T} \to \mathbb{R}^n$ is a rd-continuous function, $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then, the unique solution of the initial value problem (3) can be given as

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^t e_A(t, \sigma(\tau))f(\tau)\Delta\tau. \tag{5}$$

2.4 Time Scales Models of Continuous Systems

In order to derive a time scales model of a continuous time system, the following definitions and theorems are given.

Definition 19. Let X be a real square matrix. Then, the expc function is defined as follows:

$$\text{expc}(X) = I + \frac{X}{2!} + \frac{X^2}{3!} + ... + \frac{X^{n-1}}{n!} + ... \quad (6)$$

Theorem 20. Let X be a real square matrix and $t, s \in \mathbb{R}$ are constants. Then, the expc function satisfies the following equations:

i) $e^X = \text{expc}(X)X + I$
ii) $\text{expc}(X)X = X\text{expc}(X)$
iii) $t\text{expc}(tX) = \int_0^t e^{\tau X} d\tau$
iv) $(t+s)\text{expc}((t+s)X) = t\text{expc}(tX) + se^{tX}\text{expc}(sX)$

Proof. Only the sketch of the proofs are given.

i) It easily follows from the series expansion of e^X and the definition of expc.
ii) It follows from the definition.
iii) Expand the $e^{\tau X}$ to infinite series and take integral of each component to obtain the result.
iv) Use the previous relation to obtain

$$(t+s)\text{expc}((t+s)X) = \int_0^t e^{\tau X} d\tau + \int_t^{t+s} e^{\tau X} d\tau.$$

Then the result follows by changing the boundaries of the second integral.

Theorem 21. (Gravagne et al. [2007]). Let A, B, C, D are constant real matrices with the sizes $n \times n, n \times m, r \times n$ and $r \times m$, respectively. The point-to-point time scales model of the continuous time linear dynamic system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \;\; x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (7)$$

can be given as

$$\begin{aligned} x^\Delta(t) &= F(\mu(t))x(t) + G(\mu(t))u(t), \;\; x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (8)$$

where
$F(\mu(t)) \triangleq \text{expc}[A\mu(t)]A$ and $G(\mu(t)) \triangleq \text{expc}[A\mu(t)]B$.

Proposition 22. The state transition matrices of (7) and (8) are exactly same, meaning

$$e^{A(t-t_0)} = e_F(t, t_0) \quad (9)$$

for all $t_0, t \in \mathbb{T}$, where $\mathbb{T}$ is any selected time scales.

Proof. It easily follows from the e_A formula for isolated time scales and definition of $F(\mu(t))$.

Proposition 23. $F(\mu(t))$ in (8) is always regressive.

Proof. From the definition of $F(\mu)$ and properties of expc, we can write

$$(I + \mu(t)F(\mu(t))) = I + \mu(t)\frac{e^{A\mu(t)} - I}{\mu(t)} = e^{A\mu(t)}$$

which is always an invertible matrix.

Remark 24. F and G depend on the value of $\mu(t)$ at a point $t \in \mathbb{T}$ rather than the t itself, ie. $F : M(\mathbb{T}) \to \mathbb{R}^{n\times n}$ and $G : M(\mathbb{T}) \to \mathbb{R}^{n\times m}$. Hence, we will drop t and write $F(\mu)$ and $G(\mu)$, but keeping in mind that actual values of μ depend on t, while μ is an element of the set $M(\mathbb{T})$. With this description, it is possible to regard an isolated time scales system model as some collection of time invariant system models at each time $t \in \mathbb{T}_+$.

3. CONTROLLABILITY AND OBSERVABILITY

In this section definitions of controllability and observability and related conditions available in the literature are given. Also some new results developed for special cases in this study, are provided. Furthermore, dual system is defined for a system on time scales in the general case.

3.1 Controllability

Definition 25. Let $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ be rd-continuous matrix valued functions on $\mathbb{T}$ with $m \leqslant n$. The linear regressive system

$$x^\Delta(t) = A(t)x(t) + B(t)u(t), \; x(t_0) = x_0 \quad (10)$$

is *controllable* on $[t_0, t_f]$ if for any given x_0, there exists a rd-continuous control signal $u(t)$ such that $x(t_f) = x_f$.

Theorem 26. Let the linear regressive system (10) be defined on an isolated time scales $\mathbb{T}_+$ and $t_0, t_f \in \mathbb{T}_+$. Then, the system is controllable on the interval $[t_0, t_f]$ if and only if

$$\text{rank}\,[P_0 \;\, P_1 \;\, \dots \;\, P_{f-1}] = n$$

where $P_i = e_A(t_f, t_{i+1})B(t_i)$.

Proof. The solution of (10) given as in (5) becomes

$$\begin{aligned} x(t_f) &= e_A(t_f, t_0)x_0 + \sum_{i=0}^{f-1} \mu_i e_A(t_f, t_{i+1})B(t_i)u(t_i) \\ &= e_A(t_f, t_0)x_0 + \sum_{i=0}^{f-1} \mu_i P_i u(t_i) \end{aligned}$$

on an isolated time scale, where $t_i \in \mathbb{T}_+$ $(i = 0, 1, 2, \dots)$. By defining $x_f = x(t_f)$ and $u_i = u(t_i)$ we can write

$$x_f - e_A(t_f, t_0)x_0 = [\mu_0 P_0 \;\, \mu_1 P_1 \;\, \dots \;\, \mu_{f-1}P_{f-1}] \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{f-1} \end{bmatrix}.$$

It is well-known that a solution exists to this equation for all $x_0, x_f \in \mathbb{R}^n$ if and only if

$$\text{rank}\,[\mu_0 P_0 \;\, \mu_1 P_1 \;\, \dots \;\, \mu_{f-1}P_{f-1}] = n.$$

Since μ_i are positive scalars, their multiplication does not affect the rank of the matrix. Hence, the result follows by the definition of controllability.

Theorem 27. Let (8) be the time scales model of the linear continuous system (7) on an isolated time scales $\mathbb{T}_+$ and $t_0, t_f \in \mathbb{T}_+$. Then, the system (8) is controllable on the interval $[t_0, t_f]$ if and only if

$$\text{rank}\,[R_0 \;\, R_1 \;\, \dots \;\, R_{f-1}] = n$$

where $R_i = \text{expc}(A(t_f - t_i))B$.

Proof. By using Theorem 26 the system (8) is controllable on the interval $[t_0, t_f]$ if and only if

$$\text{rank}\,[P_0 \;\, P_1 \;\, \dots \;\, P_{f-1}] = n$$

where $P_i = e_F(t_f, t_{i+1})G(\mu_i)$. By using (9) and definition of $G(\mu)$, we can write

$$P_i = e^{A(t_f - t_{i+1})}\text{expc}(A\mu_i)B.$$

Now define

$$S_i \triangleq S_{i+1} + (t_{i+1} - t_i)P_i$$

where $S_f \triangleq 0$ by definition. We are going to show that

$$S_i = (t_f - t_i)\text{expc}(A(t_f - t_i))B$$

by using backwards induction, starting from $f-1$ to 0. For $i=f-1$ the claim is true by definition. Now suppose the claim is true for $i+1$. Hence we can write

$$\begin{aligned} S_i =&(t_f - t_{i+1})\text{expc}(A(t_f - t_{i+1}))B \\ &+ (t_{i+1} - t_i)e^{A(t_f - t_{i+1})}\text{expc}(A(t_{i+1} - t_i))B. \end{aligned}$$

By using Theorem 20-*iv* we can obtain the desired result. Note that S_i is constructed from P_i with scalar multiplication and addition. This can be interpreted as the replacement of P_i with S_i by linear column operations in the matrix. The rank of a matrix is invariant under these operations. Also, multiplying columns of a matrix with scalars does not change the rank. Since, $S_i = (t_f - t_i)R_i$, it follows that

$$\text{rank}\,[R_0 \; R_1 \; \dots \; R_{f-1}] = \text{rank}\,[P_0 \; P_1 \; \dots \; P_{f-1}]$$

and the proof is complete.

3.2 Observability

Definition 28. Let $A(t) \in \mathbb{R}^{n\times n}$ and $C(t) \in \mathbb{R}^{r\times n}$ be rd-continuous matrix valued functions on $\mathbb{T}$ with $r \leqslant n$. The linear regressive system

$$\begin{aligned} x^{\Delta}(t) &= A(t)x(t),\, x(t_0) = x_0 \\ y(t) &= C(t)x(t) \end{aligned} \tag{11}$$

is *observable* on $[t_0, t_f]$ if any initial state x_0 can be uniquely determined from the output signal $y(t)$ for $t \in [t_0, t_f)$.

Theorem 29. Let the linear regressive system (11) be defined on an isolated time scales $\mathbb{T}_+$ and $t_0, t_f \in \mathbb{T}_+$. Then, the system is observable on the interval $[t_0, t_f]$ if and only if

$$\text{rank}\begin{bmatrix} O_0 \\ O_1 \\ \vdots \\ O_{f-1} \end{bmatrix} = n$$

where $O_i = C(t_i)e_A(t_i, t_0)$.

Proof. The solution of (11) given as in (5) becomes

$$\begin{aligned} y(t_i) &= C(t_i)x(t_i) \\ &= C(t_i)e_A(t_i, t_0)x_0 \\ &= O_i x_0 \end{aligned}$$

on an isolated time scale, where $t_i \in \mathbb{T}_+$ $(i = 0, 1, 2, \dots)$. By using the above solution, we can write

$$\begin{bmatrix} y(t_0) \\ y(t_1) \\ \vdots \\ y(t_{f-1}) \end{bmatrix} = \begin{bmatrix} O_0 \\ O_1 \\ \vdots \\ O_{f-1} \end{bmatrix} x_0.$$

It is well-known that any x_0 can be uniquely determined if and only if

$$\text{rank}\begin{bmatrix} O_0 \\ O_1 \\ \vdots \\ O_{f-1} \end{bmatrix} = n.$$

Hence, the result follows by the definition of observability.

Theorem 30. Let (8) be the time scales model of the linear continuous system (7) on an isolated time scales $\mathbb{T}_+$ and $t_0, t_f \in \mathbb{T}_+$. Then, the system (8) is observable on the interval $[t_0, t_f]$ if and only if

$$\text{rank}\begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{f-1} \end{bmatrix} = n$$

where $Q_i = Ce^{A(t_i - t_0)}$.

Proof. It easily follows from the Theorem 29 and (9).

3.3 Duality

Theorem 31. Let $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ and $C(t) \in \mathbb{R}^{r\times n}$ be rd-continuous matrix valued functions on $\mathbb{T}$ with $m, r \leqslant n$. The linear regressive system

$$\begin{aligned} x^{\Delta}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{12}$$

is controllable on $[t_0, t_f]$ if and only if the dual system

$$\begin{aligned} x^{\Delta}(t) &= \ominus A^T(t)x(t) + [I + \mu(t)A^T(t)]^{-1}C^T(t)u(t) \\ y(t) &= B^T(t)[I + \mu(t)A^T(t)]^{-1}x(t) \end{aligned} \tag{13}$$

is observable on $[t_0, t_f]$ where $x(t_0) = x_0 \in \mathbb{R}^n$ and $t_0, t_f \in \mathbb{T}$. Also (12) is observable on $[t_0, t_f]$ if and only if (13) is controllable on $[t_0, t_f]$.

Proof. For the first part of the proof, it is sufficient to show the following equality:

$$e_A(t_0, \sigma(t))B(t) = e^T_{\ominus A^T}(t, t_0)[I + \mu(t)A(t)]^{-1}B(t)$$

for all $t \in \mathbb{T}$. But it easily follows from the properties in Theorem 17.
Similarly, to prove the second part we need to show that

$$e^T_A(t, t_0)C^T(t) = e_{\ominus A^T}(t_0, \sigma(t))[I + \mu(t)A^T(t)]^{-1}C^T(t)$$

for all $t \in \mathbb{T}$. It also easily follows from the properties in Theorem 17.

4. EIGENVALUE ASSIGNMENT VIA STATE FEEDBACK AND OUTPUT INJECTION

In this section, we provide a new definition "Assignability" and give a method for eigenvalue assignment via state feedback and output injection for systems on isolated time scales.

4.1 Assignability

It is well-known that controllability (observability) implies arbitrary eigenvalue assignment using static state feedback (output injection) in linear time invariant systems. However, this is not true in general in time scales systems. Therefore, we give a new definition "Assignability" and provide a sufficient condition for assignability for systems on time scales.

Definition 32. Let $\mathbb{T}$ be a time scales, $F : M(\mathbb{T}) \to \mathbb{R}^{n\times n}$ and $G : M(\mathbb{T}) \to \mathbb{R}^{n\times m}$ be matrix valued functions. The pair $(F(\mu), G(\mu))$ is called *assignable* if there exists a matrix valued function $K : M(\mathbb{T}) \to \mathbb{R}^{m\times n}$ such that the characteristic polynomial of the matrix $F(\mu) - G(\mu)K(\mu)$ is equal to some nth order polynomial

$$p_c(s, \mu) = s^n + a_{n-1}(\mu)s^{n-1} + \dots + a_0(\mu)$$

with arbitrarily selected coefficient functions $a_i : M(\mathbb{T}) \to \mathbb{R}$ $(i = 0, 1, 2, \dots, n-1)$ for all $\mu \in M(\mathbb{T})$.

Definition 33. Let $\mathbb{T}$ be a time scales, $F : M(\mathbb{T}) \to \mathbb{R}^{n\times n}$ and $G : M(\mathbb{T}) \to \mathbb{R}^{n\times m}$ be matrix valued functions. The *assignability matrix* of F and G is defined as

$$\mathcal{A}(F,G)(\mu) \triangleq \left[G(\mu)\ F(\mu)G(\mu)\ \dots\ F^{n-1}(\mu)G(\mu)\right].$$

Theorem 34. Let $\mathbb{T}$ be a time scales, $F : M(\mathbb{T}) \to \mathbb{R}^{n\times n}$ and $G : M(\mathbb{T}) \to \mathbb{R}^{n\times m}$ be matrix valued functions. The pair $(F(\mu), G(\mu))$ is assignable if

$$\text{rank}\mathcal{A}(F,G)(\mu) = n$$

for all $\mu \in M(\mathbb{T})$.

Proof. First let $g : M(\mathbb{T}) \to \mathbb{R}^n$ is a vector valued function and assume that $\text{rank}\mathcal{A}(F,g)(\mu) = n$, ie. $\mathcal{A}(F,g)(\mu)$ is an invertible square matrix for all $\mu \in M(\mathbb{T})$. Since $M(\mathbb{T})$ is a countable set we can use the well-known Ackermann's formula to obtain $k : M(\mathbb{T}) \to \mathbb{R}^n$ as

$$k^T(\mu) = [0\ 0\ \dots\ 1]\,\mathcal{A}^{-1}(F,g)(\mu)p_c(F(\mu),\mu) \tag{14}$$

such that the characteristic polynomial of the matrix $F(\mu) - g(\mu)k^T(\mu)$ is equal to $p_c(s,\mu)$ for all $\mu \in M(\mathbb{T})$. Now to expand this result to the matrix case, it is easy to show that there exists a vector $f : M(\mathbb{T}) \to \mathbb{R}^m$ such that $\text{rank}\mathcal{A}(F,Gf)(\mu) = n$. Then $K(\mu) = f(\mu)k^T(\mu)$ can be written where $k^T(\mu)$ can be found as in above result.

Criterion 35. Let $\mathbb{T}$ be a time scales and $A \in \mathbb{R}^{n\times n}$ be a constant matrix. $(A,\mathbb{T})$ satisfies the *KHN (Kalman-Ho-Narendra) criterion* if

$$\mu(\lambda - \gamma) \neq 2k\pi j \tag{15}$$

for any nonzero integer k and for all $\mu \in M(\mathbb{T})$ where λ and γ are any pair of eigenvalues of A and j is the pure imaginary number.

Theorem 36. Let (8) be the time scales model of the linear continuous system (7) on an isolated time scales $\mathbb{T}_+$. The pair $(F(\mu), G(\mu))$ is assignable if (A,B) is a controllable pair and $(A,\mathbb{T}_+)$ satisfies the KHN criterion. Furthermore, the pair $(F^T(\mu), C^T)$ is assignable if (A,C) is an observable pair and $(A,\mathbb{T}_+)$ satisfies the KHN criterion.

Proof. Write the definitions of $F(\mu)$ and $(G(\mu))$ in the expression of the rank condition given in Theorem 34 and follow the steps given in Chen [1970] Appendix D which are developed for discrete time systems.

4.2 Eigenvalue Assignment

Note that the eigenvalues of the time scales model of a system changes with the values of $\mu(t)$. A natural question follows that how to select these varying eigenvalues to achieve desired system characteristics. To do this, we select the desired eigenvalues for continuous time systems and convert these eigenvalues to time scales with the map

$$\phi(X,\mu) \triangleq \frac{e^{X\mu} - I}{\mu} \tag{16}$$

where X is any square matrix and $\mu \in M(\mathbb{T}_+)$ for any selected isolated time scales $\mathbb{T}_+$. Then time scales controller can be found as in the proof of Theorem 34, provided that the time scales system is assignable. This method can only be used if the following conjecture is true.

Conjecture 37. Let $A \in \mathbb{R}^{n\times n}$ be a constant matrix and $b \in \mathbb{R}^n$ be a constant vector such that (A,b) is a controllable pair. Let $\mathbb{T}_+$ be an isolated time scales and define

$$F(\mu) \triangleq \text{expc}(A\mu)A = \phi(A,\mu) = \frac{e^{A\mu} - I}{\mu}$$

$$g(\mu) \triangleq \text{expc}(A\mu)b$$

for all $\mu \in M(\mathbb{T}_+)$ (see Theorem 21). Let $(A,\mathbb{T}_+)$ satisfies KHN criterion, so that $(F(\mu), g(\mu))$ is an assignable pair (Theorem 36). Let $k \in \mathbb{R}^n$ such that

$$\text{eig}(A - bk^T) = \{\lambda_1, \dots, \lambda_n\}$$

where $\lambda_i \in \mathbb{C}$ $(i = 1, 2, \dots, n)$. Assignability assumption implies that there exists a $k : M(\mathbb{T}_+) \to \mathbb{R}^n$ such that

$$\text{eig}(F(\mu) - g(\mu)k^T(\mu)) = \{\phi(\lambda_1,\mu), \dots, \phi(\lambda_n,\mu)\}$$

for all $\mu \in M(\mathbb{T}_+)$. Define $F_c(\mu) \triangleq F(\mu) - g(\mu)k^T(\mu)$ and $A_c = A - bk^T$. Then there exists a $\delta = \delta(\mu_{\max}) \geqslant 0$ such that

$$\left\| e_{F_c}(t,t_0) - e^{A_c(t-t_0)} \right\| \leqslant \delta \tag{17}$$

for all $t \in \mathbb{T}_+$ where $t_0 \in \mathbb{T}_+$ and $\mu_{\max} = \max(M(\mathbb{T}_+))$.

The proof of this conjecture is an ongoing work and we could not find a counter example so far. Also, MIMO case of this conjecture will be considered.

Remark 38. This method can also be used for output injection if $(F^T(\mu), C^T)$ is assignable.

5. EXAMPLE

Example 39. Consider the following unstable continuous time system

$$\dot{x}(t) = \begin{bmatrix} 0.4 & 1 & 0 \\ -0.4 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1\ 0\ 0]\,x(t)$$

Let the desired eigenvalues of the closed loop system be $\{-2,-2,-5\}$. Then the following figures are obtained with $x(t_0) = x_0 = [0\ 1\ 1]^T$, by using the time scales state feedback controller on a randomly selected isolated time scales $\mathbb{T}_+$ such that $0 < \mu < 0.2$ for all $\mu \in M(\mathbb{T}_+)$.

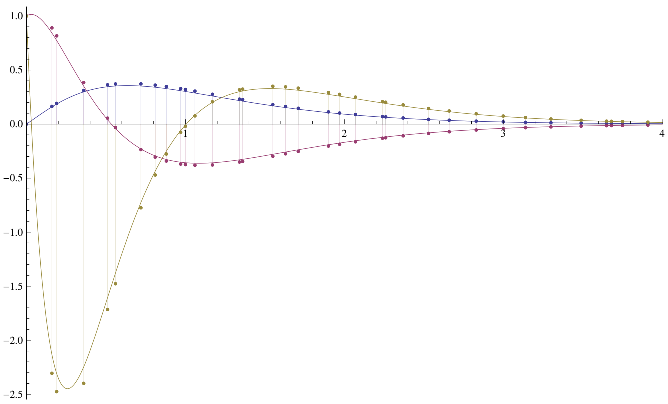

Fig. 1. States of the time scales closed loop system with time scales controller and states of the closed loop system with continuous controller.

6. CONCLUSION AND FUTURE WORK

In this paper, we provided a method for controlling a continuous time systems with an isolated time scales state

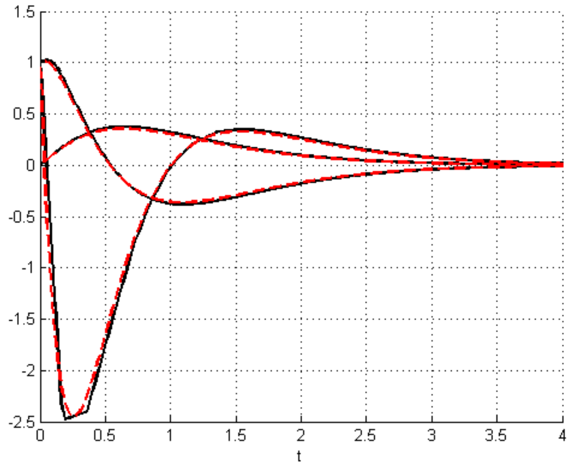

Fig. 2. States of the continuous time closed loop system with time scales controller (solid line) and with continuous controller (dotted line).

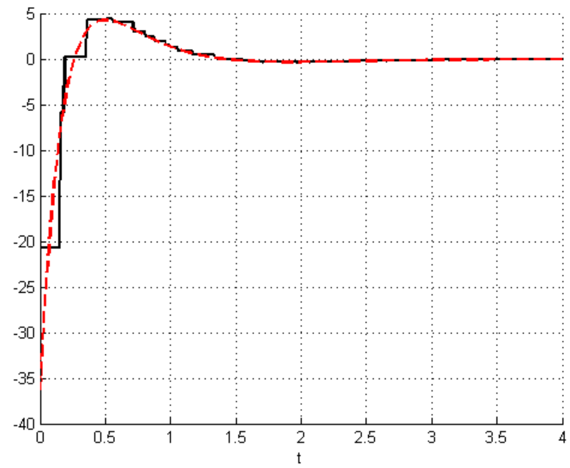

Fig. 3. Time scales (solid line) and continuous time (dotted line) state feedback control signal.

feedback (output injection) controller via eigenvalue assignment. This means, we can use classical control methods for systems with nonuniform sampling.

We gave the necessary and sufficient conditions for controllability, observability and duality on isolated time scales respectively with theorems 26-27, 29-30 and 31. Furthermore, sufficient conditions on assignability are provided with theorems 34 and 36, which are used to design static state feedback (output injection) controller on isolated time scales.

Some natural questions arise which we have not covered in this paper, because they are not solved yet. One of the open questions is that with which conditions the controllability and observability of a continuous system is preserved under nonuniform sampling. Unfortunately, the answer to this question is not trivial and left as future work.

The relation between assignability and controllability (observability) can be considered. Note that it is easy to find a controllable but not assignable time scales system, however the converse is not known.

Lastly, we need to prove that Conjecture 37 is true at least under some conditions to guarantee that our method works.

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