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Discontinuous Control of Nonlinear Systems with convex input constraint via Locally Semiconcave Control Lyapunov Functions

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Abstract: Recently, locally semiconcave control Lyapunov functions (LS-CLFs) play important roles in nonlinear control theory. Many LS-CLF based stabilizing controllers are proposed.
In this paper, we consider the locally asymptotic stabilization problem of the input affine nonlinear systems with convex input constraints. To design a stabilizing state feedback under the input constraints, we employ the LS-CLF and convex optimization theory.
Due to nonsmoothness of LS-CLF, the proposed state feedback is discontinuous on the state space. Therefore, we consider sample and hold solutions and guarantee the asymptotic stability of the closed loop system in the sense of sample stability. The effectiveness of the proposed method is confirmed by the numerical example.

Keywords: nonlinear control, discontinuous control, control Lyapunov function, sample stability, input constraint, convex optimization

1. INTRODUCTION

Control Lyapunov functions (CLFs) play an important role in recent nonlinear control theory. In particular, semiconcave CLF based discontinuous controller design attracts much attention (Rifford (2002); Clarke (2010); Nakamura et al. (2013)). For input affine nonlinear systems defined on Euclidian spaces, Rifford proposed an asymptotic stabilizing controller based on a semiconcave CLF(Rifford (2002)). Nakamura et al introduced the concept of the locally semiconcave control Lyapunov function (LS-CLF) that an assumption of semiconcavity was relaxed to a local one (Nakamura et al. (2013)). By using LS-CLFs, they proposed an asymptotic stabilizing controller for nonlinear systems defined on noncontractible manifolds. Although (locally) semiconcave CLFs are nonsmooth, asymptotic stabilization is attained by an input that minimizes a generalized derivative of a (locally) semiconcave CLF along the trajectory of solutions (Clarke (2010)).

However, previously proposed methods do not consider input constraints that often exist in actual control systems. For example, the controller proposed in Rifford (2002) is not guaranteed to satisfy the input constraint despite the control input is assumed to be restricted to a compact convex set. To apply semiconcave CLF based controllers to actual control systems, we need to solve the important problem.

In this paper, we design a discontinuous locally asymptotic stabilizing state feedback controller for nonlinear systems having a convex input constraint; the control input is restricted to a compact convex set (Suárez et al. (2001)). Convex optimization based continuous stabilizing state feedback have been already proposed for the class of systems (Satoh et al. (2008)). To extend the method to discontinuous state feedback design, we employ disassembled differential of semiconcave functions (Nakamura et al. (2013)). We show that the proposed controller asymptotically stabilizes the desired equilibrium under the input constraint. Moreover, we discuss the continuity of the proposed controller at the desired equilibrium. The effectiveness of the proposed method is confirmed by a numerical example of position control of a two-wheeled mobile robot.

2. PRELIMINARIES

In this section, we introduce basic definitions of mathematical terms and their fundamental properties.

2.1 Sample-and-hold Solution and Sample Stability

In this paper, we consider the following input affine nonlinear system:

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \qquad (1)$$

where $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, and U is a compact set. We assume that every $f, g_i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in \{1, \ldots, m\}$ is locally Lipschitz continuous with respect to x, and $f(0) = 0$.

To deal with discontinuous state feedback control for (1), we consider the following sample-and-hold solution as the closed loop solution (Clarke (2010); Cortés (2008)).

Definition 1. (Partition). Any infinite sequence $\pi = \{t_i \in \mathbb{R}_{\geq 0}\}_{i \in \mathbb{Z}_{\geq 0}}$ consisting of numbers $0 = t_0 < t_1 < t_2 < \cdots$ with $\lim_{i\to\infty} t_i = +\infty$ is called a partition, and the number $d(\pi) := \sup_{i\in\mathbb{Z}_{\geq 0}}(t_{i+1} - t_i)$ is called its diameter.

Definition 2. (Sample-and-hold solution). Let $k : \mathbb{R}^n \to U; x \mapsto k(x)$ be a given state feedback, π a partition, and $x_0 \in \mathbb{R}^n$ an initial state. The sample-and-hold solution $\psi(t, x_0, k(x)) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times U \to \mathbb{R}^n$ for (1) is defined as the mapping such that $\psi(t, x_0, k(x)) = x(t)$, where $x(t)$ is a continuous mapping obtained by recursive solving

$$\dot{x} = f(x(t)) + g(x(t))u(x(t_i)), \tag{2}$$

from the initial time t_i to the maximal time

$$s_i := \max\{t_i, s \in [t_i, t_{i+1}] | x(\cdot) \text{is defined on} [t_i, s)\}, \tag{3}$$

with $x(0) = x_0$.

With sample-and-hold solution $x(t)$, we define local sample stability for (1) as follows (Nakamura et al. (2013)):

Definition 3. (Local sample stability). Consider system (1). Let $\bar{\mathcal{R}} \subset \mathbb{R}^n$ be a bounded set containing the origin, and $\mathfrak{P}$ denotes the set of all open subset of $\bar{\mathcal{R}}$ containing the origin.

A feedback $k : \bar{\mathcal{R}} \to U$ is said to sample stabilize the origin of the system (1) if the following holds for arbitrary sets $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{P}$ such that $\mathcal{R}_1 \subset \mathcal{R}_2$.

(1) There exists a set $\mathcal{M} \subset \bar{\mathcal{R}}$ depending only upon $\mathcal{R}_2$ and two positive numbers $\Omega, T > 0$ depending on $\mathcal{R}_1$ and $\mathcal{R}_2$ such that, for any initial condition $x_0 \in \mathcal{R}_2$, for any partition π of the diameter less than Ω, the corresponding sample-and-hold solution $\psi(t, x_0, k(x))$ satisfies the following conditions:
  (a) $\psi(t, x_0, k(x)) \in \mathcal{R}_1, \ \forall t \geq T$
  (b) $\psi(t, x_0, k(x)) \in \mathcal{M}, \ \forall t \geq 0$

(2) For each $\mathcal{E} \in \mathfrak{P}$, there exists a set $\mathcal{P} \in \mathfrak{P}$ such that if $\mathcal{R}_2 \subset \mathcal{P}$, $\mathcal{M}$ in (1) can be chosen satisfying $\mathcal{M} \subset \mathcal{E}$.

2.2 Locally Semiconcave Control Lyapunov Function and Disassembled differential

In this subsection, we introduce the definition of the the locally semiconcave function and its disassembled differential. Then we define locally semiconcave control Lyapunov function (LS-CLF).

Definition 4. (Locally semiconcave function). A continuous function $V : X \subset \mathbb{R}^n \to \mathbb{R}$ said to be locally semiconcave on X, if for any compact set $\mathcal{M} \subset X$ there exists $E > 0$ such that

$$V(x) + V(y) - 2V\left(\frac{x+y}{2}\right) \leq E\|x - y\|^2 \tag{4}$$

for all $x, y \in \mathcal{M}$ satisfying $(x + y)/2 \in \mathcal{M}$.

Although locally semiconcave functions are nonsmooth, the following good properties hold (Cannarsa and Sinestrari (2004)):

Lemma 1. Let $V : X \subset \mathbb{R}^n \to \mathbb{R}$ be a locally semiconcave function. Then, V is a Lipschitz continuous.

Theorem 2. Let $V : X \subset \mathbb{R}^n \to \mathbb{R}$ be a locally semiconcave function. Then, V can be locally written as the minimum of class C^2 functions. More precisely, for any compact set $\mathcal{M} \subset X$, there exist a compact set $\mathcal{S} \subset \mathbb{R}^{2n}$ and a family of functions $\{\tilde{V}_s\}_{s\in\mathcal{S}}$ such that each $\tilde{V}_s : \mathcal{M} \to \mathbb{R}$ is C^2 with respect to x and

$$V(x) = \min_{s\in\mathcal{S}} \tilde{V}_s(x), \ \forall x \in \mathcal{M}. \tag{5}$$

We define disassembled differential of a locally semiconcave function as follows (Nakamura et al. (2013)):

Definition 5. (Disassembled differential). Let $V : X \subset \mathbb{R}^n \to \mathbb{R}$ be a locally semiconcave function. Then, the following set-valued map $\tilde{D}V : X \to 2^{\mathbb{R}^n}$ is said to be a disassembled differential of V:

$$\tilde{D}V(x) = \left\{ \frac{\partial \tilde{V}_s}{\partial x}(x) \,\Big|\, s \in \{s \in \mathcal{S} | V(x) = \tilde{V}_s(x)\} \right\}. \tag{6}$$

Locally semiconcave control Lyapunov function (LS-CLF) for system (1) is defined as follows (Nakamura et al. (2013)):

Definition 6. (LS-CLF). Let $X \subset \mathbb{R}^n$ be a neighborhood of the origin of system (1). A locally semiconcave control Lyapunov function (LS-CLF) for system (1) is a locally semiconcave function $V : X \to \mathbb{R}$ such that following properties hold.

(A1) V is proper; that is, the set $\{x \in X | V(x) \leq L\}$ is compact for every $L > 0$.

(A2) V is positive definite; that is, $V(0) = 0$ and $V(x) > 0$ for all $x \in X\backslash\{0\}$

(A3) There exists a control u admissible for x, a continuous positive definite function $Q : X \to \mathbb{R}_{\geq 0}$ such that

$$D_V(x; (f(x) + g(x)u)) \leq -Q(x), \forall x \in X\backslash\{0\}, \tag{7}$$

where the directional derivative $D_V(x; v)$ of semiconcave function V is defined as follows:

$$D_V(x; v) := \lim_{t\downarrow 0} \frac{V(x + tv) - V(x)}{t}. \tag{8}$$

The following lemma clarifies that the relation between the directional subderivatibe $D_v(x; v)$ and the disassembled differential $\tilde{D}V(x)$ (Nakamura et al. (2013)):

Lemma 3. Let $V : X \subset \mathbb{R}^n \to \mathbb{R}$ be a locally semiconcave function, and $\tilde{D}V$ denotes its disassembled differential. Then, the following holds:

$$DV(x; v) = \min_{p\in \mathrm{co}\tilde{D}V(x)} \langle p, v\rangle, \tag{9}$$

where co denotes the convex hull.

According to Lemma 3, the following theorem holds (Nakamura et al. (2013)):

Theorem 4. Consider the control system (1) and a locally semiconcave function $V : X \subset \mathbb{R}^n \to \mathbb{R}$ satisfying (A1) and (A2). Then, the function V is an LS-CLF if and only if the following condition (A3') is satisfied:

(A3') For arbitrary $R_2, R_1 \in \mathbb{R}_{>0}$ such that $R_2 > R_1 > 0$, there exist a positive real constant $Q > 0$, and a mapping $p : X \to \mathbb{R}^n$ such that $p(x) \in \tilde{D}V(x)$ such that

$$\min_{u\in U} \langle p(x), f(x) + g(x)u\rangle < -Q, \quad \forall x \in \{x \in X | R_1 \leq V(x) \leq R_2\}. \tag{10}$$

3. PROBLEM STATEMENT

In this section, we describe the problem considered in the present paper.

Firstly, we introduce the concept of convex input constraint for system (1):

Definition 7. (Convex Input Constraint). The control system (1) said to have a convex input constraint $u \in U$ if the following properties hold:

(H1) $U \subset \mathbb{R}^m$ is a compact set.
(H2) U is represented as

$$\begin{aligned} U &= \bigcup_{x \in \mathbb{R}^n} \mathcal{U}_x \\ &= \bigcup_{x \in \mathbb{R}^n} \{u \in \mathbb{R}^m | G_i(x,u) \le 0,\ (i = 1, \ldots, l)\}, \end{aligned} \quad (11)$$

where each $G_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is C^1 with respect to x and u, and a convex function with respect to u for each fixed x.
(H3) $0 \in \mathrm{int} \bigcap_{x \in \mathbb{R}^n} \mathcal{U}_x$.

Remark 1. The above convex input constraint is a general class of input constraints including simple norm input constraints. For example, the convex input constraint occurs when pre-feedback control designs (e.g. gravity compensation) are used.

In this paper, we consider the following problem:

Problem 1. Consider the control system (1) under a convex input constraint. We suppose an LS-CLF $V(x)$ for (1) is obtained.

Then, design an LS-CLF based state feedback controller $u = k(x)$ such that following properties holds:

- the origin of the system (1) is locally sample stabilized.
- given convex input constraint $u \in U$ is satisfied.

4. MAIN THEOREM

The sample stabilizable domain of the origin of system (1) with an LS-CLF$V(x)$ is guaranteed as follows:

Lemma 5. (sample stabilizable domain). Consider control system (1) has a convex input constraint. Let $X \subset \mathbb{R}^n$ be a neighborhood of the origin and $V : X \to \mathbb{R}_{\ge 0}$ an LS-CLF. Moreover, $p : X \to \mathbb{R}^n$ is a mapping such that $p(x) \in \tilde{D}V(x)$, and $R_{max} > 0$ is the maximum constant satisfying

$$\begin{aligned} &\min_{u \in \mathcal{U}_x} \langle p(x), f(x) + g(x)u \rangle < -Q, \\ &\forall x \in \mathcal{W} \backslash \{0\} := \{x \in X | V(x) < \mathcal{R}_{max}\} \backslash \{0\}. \end{aligned} \quad (12)$$

Then, $\mathcal{W}$ is a sample stabilizable domain.

We can prove Lemma 5 by constructing a sample stabilizing controller for (1). The following minimizing input plays a central role in our control design.

Definition 8. (Minimizing Input). Let $\mathcal{W}$ be a sample stabilizable domain with respect to an LS-CLF $V(x)$. Then the minimizing input is the state feedback $\bar{k} : \mathcal{W} \to U$ such that the following conditions hold:

(1) for each fixed $x \in \mathcal{W}$, $u = \bar{k}(x)$ is a solution of the minimization problem

$$\text{Minimize } \langle H(x), u \rangle \quad \text{subject to } u \in \mathcal{U}_x, \quad (13)$$

$$H(x) = [\langle p(x), g_1(x) \rangle, \ldots, \langle p(x), g_m(x) \rangle]^{\mathrm{T}}, \quad (14)$$

(2) $\bar{k}(x) = 0,\ \forall x \in \{x \in \mathcal{W} | H(x) = 0\}$.

Remark 2. Note that the minimizing input $\bar{k}(x)$ is well-defined on $\mathcal{W}$. Moreover, we can easily calculate $\bar{k}(x)$ by convex optimization. The details are provided in subsection 6.1.

Remark 3. In general, $\bar{k}(x)$ is not a unique solution of (13) for each $x \in \mathcal{W}$. However, this fact do not cause any problem for sample stabilization.

Let us state our main result of the present paper.

Theorem 6. Consider control system (1) has a convex input constraint. Let $V : X \to \mathbb{R}_{\ge 0}$ be an LS-CLF, $p : X \to \mathbb{R}^n$ a mapping such that $p(x) \in \tilde{D}V(x)$, and $\bar{k}(x)$ a minimizing input.

Then, the following state feedback $u = k(x)$ locally sample stabilizes the origin of the system (1) in $\mathcal{W}$ and satisfies the convex input constraints $k(x) \in \mathcal{U}_x,\ \forall x \in \mathcal{W}$:

$$k(x) = \begin{cases} \dfrac{P(x)+|P(x)|+C(x)}{2+C(x)} \bar{k}(x) & (H(x) \ne 0) \\ 0 & (H(x) = 0) \end{cases}, \quad (15)$$

$$P(x) = \frac{\langle p(x), f(x) \rangle}{-\langle H(x), \bar{k}(x) \rangle}, \quad (16)$$

where $P : \{x \in \mathcal{W} | H(x) \ne 0\} \to \mathbb{R}$, and $C : \mathcal{W} \to \mathbb{R}_{\ge 0}$ is a function satisfying

$$\begin{aligned} &C(x) \ne 0,\ \forall x \in \{x \in \mathcal{W} | H(x) \ne 0\}, \\ &\lim_{H(x) \to 0} C(x) = 0. \end{aligned} \quad (17)$$

5. PROOF

In this section, we prove Theorem 6 provided in the preceding section. To prove the theorem, we introduce six key lemmas.

Lemma 7. Let $\bar{k}(x)$ be the minimizing input introduced in Theorem 6. Then, the following holds:

$$\langle H(x), \bar{k}(x) \rangle < 0,\ \forall x \in \{x \in \mathcal{W} | H(x) \ne 0\}. \quad (18)$$

Proof. According to (H3), there exists a constant $\alpha_x > 0$ such that

$$\bar{B}(0, \alpha_x) := \{u |\ \|u\|_{\le} \alpha_x\} \subset \mathcal{U}_x. \quad (19)$$

Consider the minimizing input under the norm input constraint $u \in \bar{B}(0, \alpha_x)$, and we can obtain the following inequality:

$$\min_{u \in \bar{B}(0, \alpha_x)} \langle H(x), u \rangle = -\alpha_x \parallel H(x) \parallel < 0. \quad (20)$$

Since $\bar{B}(0, \alpha_x) \subset \mathcal{U}_x$, we can observe that

$$\begin{aligned} \min_{u \in \mathcal{U}_x} \langle H(x), u \rangle &= \langle H(x), \bar{k}(x) \rangle \\ &\le \min_{u \in \bar{B}(0, \alpha_x)} \langle H(x), u \rangle < 0. \end{aligned} \quad (21)$$

□

Lemma 8. Let $P : \{x \in \mathcal{W} | H(x) \ne 0\} \to \mathbb{R}$ be the function defined by (16). Then, the following holds:

$$P(x) < 1,\ \forall x \in \{x \in \mathcal{W} | H(x) \ne 0\} \backslash \{0\}. \quad (22)$$

Proof. According to Lemma 7 and (A3'), we can obtain the following:

$$P(x) = \frac{\langle p(x), f(x)\rangle}{-\langle H(x), \bar{k}(x)\rangle} = 1 - \frac{Q}{\langle H(x), \bar{k}(x)\rangle} < 1. \quad (23)$$

□

Lemma 9. Let $P : \{x \in \mathcal{W} | H(x) \neq 0\} \to \mathbb{R}$ be the function defined by (16). Then, the following holds:

$$\lim_{x\to x^*} P(x) + |P(x)| = 0, \quad \forall x^* \in \{x \in \mathcal{W} | H(x) = 0\}\backslash\{0\}. \quad (24)$$

Proof. Let $\{x_j\}_{j\in\mathbb{N}} \subset \mathcal{W}$ be any sequence converges to x^*. Since $V(x)$ is a LS-CLF, the following holds for each $j \in \mathbb{N}$:

$$\min_{u\in\mathcal{U}_{x_j}} \left[\langle p(x_j), f(x_j)\rangle + \langle H(x_j), u\rangle\right] = \langle p(x_j), f(x_j)\rangle + \langle H(x_j), \bar{k}(x_j)\rangle < -Q. \quad (25)$$

Then there exists $J > 0$ such that

$$\langle p(x_j), f(x_j)\rangle < 0, \ \forall j > J. \quad (26)$$

Since $\langle H(x_j), \bar{k}(x_j)\rangle < 0$ by Lemma 7, we can obtain

$$P(x_j) + |P(x_j)| = 0, \ \forall j > J. \quad (27)$$

□

Lemma 10. Consider control system (1) has a convex input constraint under the assumptions of Theorem 6. Let $k : \mathcal{W} \to U; x \mapsto k(x) \in \mathcal{U}_x$ be a state feedback defined by(15), and R_1, R_2 positive constants such that $0 < R_1 < R_2 < R_{max}$. Then, the following inequality holds:

$$\sum_{i=1}^{m} k_i^2(x) < \infty, \quad \forall x \in \mathcal{R} = \{x \in \mathcal{W} | R_1 \leq V(x) \leq R_2\}. \quad (28)$$

Proof. The result follows from the fact that there exists no sequence $\{x_j\}_{j\in\mathbb{Z}}$ such that $\lim_{j\to\infty} \sum_{i=1}^{m} k_i^2(x_j) = \infty$. For more details, refer to the proof of Lemma 3 in Nakamura et al. (2013). □

Lemma 11. Consider control system (1) has a convex input constraint. Let $k : \mathcal{W} \to U; x \mapsto k(x) \in \mathcal{U}_x$ be a state feedback defined by (15), and R_1, R_2 positive constants such that $0 < R_1 < R_2 < R_{max}$. Then, there exist a constant $G > 0$ and a mapping $p : X \to \mathbb{R}^n$ such that $p(x) \in \tilde{D}V(x)$ such that

$$\langle p(x), f(x) + g(x)k(x)\rangle = \langle p(x), f(x)\rangle + \langle H(x), k(x)\rangle < -G, \quad \forall x \in \mathcal{R} = \{x \in \mathcal{W} | R_1 \leq V(x) \leq R_2\}. \quad (29)$$

Proof. We can prove the lemma in a similar way to the proof of Lemma 4 in Nakamura et al. (2013). Let us consider the following three cases.

(i) $H(x) = 0$:
According to (A3'), we can obtain $\langle p(x), f(x)\rangle < -Q$.

(ii) $H(x) \neq 0$ and $P(x) \leq 0$:
In this case, $\langle p(x), f(x)\rangle \leq 0$ and $P(x) + |P(x)| = 0$. Then by Lemma 7, there exists a constant Q_1 such that

$$\langle p(x), f(x)\rangle + \langle H(x), k(x)\rangle = \langle p(x), f(x)\rangle + \frac{C(x)}{2 + C(x)}\langle H(x), \bar{k}(x)\rangle < -Q_1. \quad (30)$$

(iii) $H(x) \neq 0$ and $P(x) > 0$:
Since $P(x) + |P(x)| = 2P(x)$, $k(x)$ is rewritten to

$$k(x) = \left\{P(x) + \frac{(1 - P(x))\, C(x)}{2 + C(x)}\right\} \bar{k}(x). \quad (31)$$

Note that $(1 - P(x)) > 0$ by Lemma 8. Hence, there exists a constant $Q_2 > 0$ such that

$$\langle p(x), f(x)\rangle + \langle H(x), k(x)\rangle = \frac{(1 - P(x))\, C(x)}{2 + C(x)}\langle H(x), \bar{k}(x)\rangle < -Q_2. \quad (32)$$

Let $G := \min\{Q_1, Q_2\}$, the lemma holds.

□

Lemma 12. Consider control system (1) has a convex input constraint. Let $k : \mathcal{W} \to U; x \mapsto k(x) \in \mathcal{U}_x$ be a state feedback defined by (15). Then, there exists a constant $\Omega > 0$ such that the following inequality holds uniformly on $\mathcal{R}$:

$$V(\psi(t, x_0, k(x))) - V(x_0) \leq -\Omega t, \ \forall t \geq 0 \quad (33)$$

Proof. Since f and g are locally Lipschitz mapping, there exists a constant $L > 0$ such that the following holds for all $x, y \in \mathcal{R}$:

$$\|f(x) + g(x)k(x) - f(y) - g(y)k(x)\| < L\|x - y\|. \quad (34)$$

Moreover, there exist constants $K, M > 0$ such that

$$|V(x) - V(y)| \leq K\|x - y\|, \quad (35)$$

$$|f(x) + g(x)k(x)\| < M, \quad (36)$$

for all $x \in \mathcal{R}$ by Lemmas 1 and 10. According to the semiconcavity of V and Proposition 3.3.1 in Cannarsa and Sinestrari (2004), we can obtain the following inequality:

$$V(\psi(t, x_0, k(x))) - V(x) \leq \langle p(x), \psi(t, x_0, k(x))\rangle + E\|\psi(t, x_0, k(x)) - x\|^2. \quad (37)$$

Then there exists $t^* \in [0, t]$ satisfying

$$V(\psi(t, x_0, k(x))) - V(x) \leq \langle p(x), f(\psi(t^*, x_0, k(x))) + g(\psi(t^*, x_0, k(x)))k(x)\rangle t + E\|\psi(t, x_0, k(x)) - x\|^2, \quad (38)$$

by the mean value inequality (Clarke et al. (1998)). We can obtain the following inequality by the same discussion as Clarke (2010); Nakamura et al. (2013):

$$\begin{aligned} V(\psi(t, x_0, k(x))) - V(x) \leq & \langle p(x), f(x) + g(x)k(x)\rangle t \\ & + \langle p(x), f(\psi(t^*, x_0, k(x))) + g(\psi(t^*, x_0, k(x)))k(x) \\ & - f(x) - g(x)k(x)\rangle t + E\|\psi(t, x_0, k(x)) - x\|^2 \\ \leq & \langle p(x), f(x) + g(x)k(x)\rangle t \\ & + L\|p(x)\| \cdot \|\psi(t^*, x_0, k(x)) - x_0\|t + CM^2 t \\ \leq & - Pt + KLMt^2 + EM^2t^2. \end{aligned} \quad (39)$$

Therefore, for any partitionπ such that

$$d(\pi) \leq P/(2M(KL + EM)), \quad (40)$$

the following holds:

$$V(\psi(t, x_0, k(x))) - V(x_0) \leq -\frac{1}{2}Pt, \ \forall t \geq 0. \quad (41)$$

The lemma holds by choosing $\Omega = P/2$. □

Then, we can prove Theorem 6.

Proof. [Theorem 6]

(i) Local sample stability:
Let $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{P}$ be arbitrary sets satisfying $\mathcal{R}_1 \subset \mathcal{R}_2$. We choose $R_1 > 0$ such that $\{x \in \mathcal{W} | V(x) < R_1\} \subset \mathcal{R}_1$, and $R_2 = \sup_{x \in \mathcal{R}_2} V(x)$. According to Lemmas 11 and 12, the following holds on $\mathcal{R}$ if $t \geq (R_2 - R_1)/\Omega$:

$$V(\psi(t, x_0, k(x))) < R_1. \tag{42}$$

Then, for $x_0 \in \mathcal{R}_2$, we can obtain

$$\begin{aligned} &x(t) \in \mathcal{M} := \{x \in \mathcal{W} | V(x) < R_2\},\ \forall t \geq 0, \\ &x(t) \in \mathcal{R}_1,\ \forall t \geq (R_2 - R_1)/\Omega. \end{aligned} \tag{43}$$

For a given set $\mathcal{E} \in \mathfrak{P}$, we define

$$\overline{\varepsilon} := \sup_{\varepsilon > 0} \{\{x \in \mathcal{W} | V(x) < \varepsilon\} \subset \mathcal{E}\}, \tag{44}$$

and a set $\mathcal{P}$ by

$$\mathcal{P} = \{x \in \mathcal{W} | V(x) < \varepsilon\}. \tag{45}$$

Finally, we can observe that the following holds for $\varepsilon \in (0, \overline{\varepsilon})$:

$$\begin{aligned} &\mathcal{R}_2 \subset \mathcal{P} \Rightarrow R_2 \leq \varepsilon < \overline{\varepsilon} \\ &\Rightarrow \mathcal{M} = \{x \in \mathcal{W} | V(x) < R_2\} \subset \mathcal{E} \end{aligned} \tag{46}$$

(ii) Input constraint $k(x) \in \mathcal{U}_x,\ \forall x \in \mathcal{W}$ is satisfied:
Note that $k(x) = 0$ for all $x \in \{x \in \mathcal{W} | H(x) = 0\}$ because we set $\bar{k}(x) = 0$. According to (H3), it is clear that $k(x) = 0 \in \mathcal{U}_x$.
On the other hand, the following holds for $x \in \{x \in \mathcal{W} | H(x) \neq 0\}$ by Lemma 8:

$$\mu := \frac{P(x) + |P(x)| + C(x)}{2 + C(x)} \in (0, 1]. \tag{47}$$

Recall that $\mathcal{U}_x$ is a convex set and $0, \bar{k}(x) \in \mathcal{U}_x$, we can obtain

$$k(x) = \mu \bar{k}(x) + (1 - \mu) 0 \in \mathcal{U}_x. \tag{48}$$

□

6. SOME REMARKS ON THE PROPOSED CONTROLLER

6.1 Minimizing Input design via Convex Optimization

The existence of the minimizing input $\bar{k}(x)$ is crucial for the proposed controller design. Let us introduce the following lemma to guarantees the existence:

Lemma 13. Consider system (1) has a convex input constraint, and let $V(x)$ be an LS-CLF. Then, there exists a minimizing input $\bar{k}(x)$ on $\mathcal{W}$. In other words, optimization problem (13) has at least a single solution for each fixed $x \in \mathcal{W}$.

Proof. Fix an arbitrary $x \in \mathcal{W}$. According to hypotheses (H1) and (H2), $\mathcal{U}_x$ is a compact set and $\langle H(x), u \rangle$ is a continuous function with respect to u. Then, there exists a global minimum of problem (13) by the extreme value theorem. □

Remark 4. Note that for each $x \in \{x \in \mathcal{W} | H(x) = 0\}$, any $u \in \mathcal{U}_x$ satisfies the condition (49). In Definition 8, we choose $\bar{k}(0) = 0$ for convenience.

Another problem we have to consider is how to construct a minimizing input in general. Since (13) is a convex optimization problem, Karush-Kuhn-Tucker conditions (KKT conditions; see e.g. Boyd and Vandenberghe (2004)) characterize the solutions:

Lemma 14. For each fixed $x \in \mathcal{W}$, $u \in \mathcal{U}_x$ is a solution of optimization problem (13) if and only if the following conditions hold:

$$\begin{aligned} &H(x) + \sum_{i=1}^{l} \lambda_i \frac{\partial G_i(x, u)}{\partial u} = 0, \\ &\lambda_i \geq 0,\ G_i(x, u) \leq 0,\ \lambda_i G_i(x, u) = 0\ (i = 1, \ldots, l), \end{aligned} \tag{49}$$

where $\lambda = [\lambda_1, \ldots, \lambda_l]^{\mathrm{T}} \in \mathbb{R}^l$ is a vector of Lagrange multipliers.

Proof. Fix an arbitrary $x \in \mathbb{R}^n$. Note that $\mathcal{U}_x$ is a convex set and $\langle H(x), u \rangle$ a convex function with respect to u by (H2). Since the problem is a convex optimization, the KKT conditions are sufficient (Boyd and Vandenberghe (2004)). Moreover, the conditions are also necessary for optimality because of the slater constraint qualification is satisfied (Boyd and Vandenberghe (2004)).

Let us introduce the Lagrangian $L : \mathbb{R}^n \times U \times \mathbb{R}^l \to \mathbb{R}$ defined by

$$L(x, u, \lambda) = \langle H(x), u \rangle + \sum_{i=1}^{l} \lambda_i G_i(x, u), \tag{50}$$

and consider the KKT condition

$$\begin{aligned} &\frac{\partial L}{\partial u}(x, u, \lambda) = 0, \\ &\lambda_i \geq 0,\ G_i(x, u) \leq 0,\ \lambda_i G_i(x, u) = 0\ (i = 1, \ldots, l), \end{aligned} \tag{51}$$

we can obtain the condition (49). □

6.2 Continuity at the desired equilibrium

In this subsection, we discuss the continuity of the proposed controller (15) at the desired equilibrium $x = 0$.

In differentiable CLF based controller design, there exists a stabilizing feedback continuous at $x = 0$, if and only if there exists a differentiable CLF which satisfies the following small control property (SCP; Sontag (1989); Bacciotti and Rosier (2005)):

Definition 9. (Small Control Property). Let $V : \mathcal{W} \to \mathbb{R}_{\geq 0}$ be a differentiable control Lyapunov function for system (1).

Then, V said to satisfy SCP if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} &0 \neq \|x\| < \delta\ \Rightarrow\ \exists \|u\| < \varepsilon, \\ &\text{s.t.}\ \left\langle \frac{\partial V}{\partial x}, f(x) + g(x)u \right\rangle < 0. \end{aligned} \tag{52}$$

Recall that LS-CLFs are locally written as the minimum of class C^2 functions (Theorem 2), we can derive the following Theorem:

Theorem 15. Consider system(1) has a convex input constraint. Let V(x) be an LS-CLF and $D \subset X$ any compact set containing $x = 0$. According to Theorem 2, there exist sets S, S_0 such that

$$\begin{aligned} &V(x) = \min_{s \in S} \tilde{V}_s(x),\ \forall x \in D, \\ &\tilde{V}_s(0) = 0,\ \forall s \in S_0 \subset S. \end{aligned} \tag{53}$$

Then, the proposed controller (15) is continuous at $x = 0$ if each $\tilde{V}_s(x),\ s \in S_0$ satisfies the SCP.

Proof. Note that Theorem 15 holds if $P(x)$ in (15) satisfies $\lim_{x\to 0} P(x) = 0$.

Since each $\tilde{V}_s, s \in S_0$ satisfies the SCP, there exist constants $\delta_s, s \in S_0$ such that each $\tilde{V}_s$ satisfies (55). We denote

$$\delta = \min_{s\in S_0} \delta_s. \tag{54}$$

According to the definition of disassembled differential and (54), the following holds for $p(x) \in \tilde{D}V(x)$:

$$\begin{aligned} &0 \neq \|x\| < \delta \ \Rightarrow\ \exists \|u\| < \varepsilon, \\ &\text{s.t. } \langle p(x), f(x) + g(x)u\rangle < 0. \end{aligned} \tag{55}$$

In a small neighborhood of $u = 0$, the following inequality holds for some $\alpha > 0$ by the same manner as in the proof of Lemma 7:

$$-\langle H(x), \bar{k}(x)\rangle \geq \alpha\|H(x)\| > 0. \tag{56}$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| < \delta \ \Rightarrow\ \langle p(x), f(x)\rangle < -\langle H(x), u\rangle < \varepsilon\|H(x)\|. \tag{57}$$

Therefore, we can obtain

$$|P(x)| = \frac{|\langle p(x), f(x)\rangle|}{|-\langle H(x), \bar{k}(x)\rangle|} = \frac{\|\varepsilon H(x)\|}{\|\alpha H(x)\|} = \frac{\varepsilon}{\alpha}, \tag{58}$$

and $\lim_{x\to 0} |P(x)| = 0$ by considering $\varepsilon \to 0$ (i.e., $\delta \to 0$). □

7. NUMERICAL EXAMPLE

In this section, we apply the proposed method to the position control of a two-wheeled mobile robot. The control system of a two-wheeled mobile robot model is given as follows (kimura et al. (2013)):

$$\dot{\hat{x}} = \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} \cos\hat{x}_3/2 & \cos\hat{x}_3/2 \\ \sin\hat{x}_3/2 & \sin\hat{x}_3/2 \\ 1/W & -1/W \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}, \tag{59}$$

where $[\hat{x}_1, \hat{x}_2] \in \mathbb{R}^2$ is the Cartesian coordinate of the center of the robot, and $\hat{x}_3 \in (-\pi/2, \pi/2)$ is the angle between the heading direction and $\hat{x}_1$-axis. The input vector $\hat{u} = [\hat{u}_1, \hat{u}_2]^{\mathrm{T}}$ consists of velocity inputs for respective right and left wheels of the robot. W denotes the distance between two wheels. We consider the following norm input constraint for (59):

$$\hat{u} \in U' := \{\hat{u} \in \mathbb{R}^2 | G(\hat{u}) \leq 0\}, \tag{60}$$

$$G'(\hat{u}) = \sqrt{\hat{u}_1^2 + \hat{u}_2^2} - d^2, \tag{61}$$

where d is a positive constant.

Let us consider the following coordinate and input transformations for (59):

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \hat{x}_1 \\ \tan\hat{x}_3 \\ 2\hat{x}_2 - \hat{x}_1\tan\hat{x}_3 \end{bmatrix}, \tag{62}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (\hat{u}_1 + \hat{u}_2)\cos\hat{x}_3/2 \\ (\hat{u}_1 - \hat{u}_2)\sec^2\hat{x}_3/W \end{bmatrix}. \tag{63}$$

Then, the system is transformed into a well known brockett integrator form:

$$\dot{x} = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ -x_1 \end{bmatrix} u_2 = g_1(x)u_1 + g_2(x)u_2. \tag{64}$$

For the brocket integrator (64), the following LS-CLF is proposed in kimura et al. (2013):

$$V(x) = \left\{ x_1^4 + x_2^4 + \left(\min_{\theta\in[0,2\pi]} F(\theta, x)\right)^2 \right\}^{\frac{1}{2}}, \tag{65}$$

$$F(\theta, x) = \frac{|x_3|^{\frac{3}{2}}}{\left| x_1\cos\theta + x_2\sin\theta + \sqrt{|x_3|}\right|}. \tag{66}$$

Remark 5. The LS-CLF originally introduced in kimura et al. (2013) is the squared of above LS-CLF (65). To achieve fast convergence of the state, We employ its square root function as an LS-CLF.

We design a locally sample stabilizing controller for mobile robot (59) according to the following steps:

(i) Design a locally sample stabilizing controller for brocket integrator (64) based on LS-CLF (65),
(ii) Transform the controller designed in step (i) into a locally sample stabilizing controller for (59) by the inverse transformation of (63).

Note that LS-CLF (65) is differentiable except that $(x_1, x_2) = (0, 0)$. We design a discontinuous mapping p as follows:

$$p(x) = \begin{cases} \dfrac{\partial V}{\partial x} & (x_1, x_2) \neq (0, 0) \\ \dfrac{\partial F}{\partial x}(0, x) & (x_1, x_2) = (0, 0) \end{cases}. \tag{67}$$

According to the input transformation (63), The input constraint (61) for (59) is equivalent to the following input constraint for (64):

$$u \in U := \{u \in \mathbb{R}^2 | G(\hat{u}) \leq 0\}, \tag{68}$$

$$G(x, u) = a_1(x)u_1^2 + a_2(x)u_2^2 - d^2 \leq 0, \tag{69}$$

where $a_1(x) = 2\sec^2\hat{x}_3$ and $a_2(x) = W^2/2\cos^4\hat{x}_3$. Note that the input constraint (68) is not a simple norm constraint, but represented as a convex input constraint.

We can obtain the minimizing input for (64) by using the KKT condition (49):

$$\bar{k}_i(x) = \begin{cases} -\dfrac{d \cdot L_{g_i}V(x)}{2a_i(x)\sqrt{\frac{L_{g_1}V}{a_1(x)} + \frac{L_{g_2}V}{a_2(x)}}} & (x_1, x_2) \neq (0, 0), \\ -\dfrac{d \cdot L_{g_i}F(0, x)}{2a_i(x)\sqrt{\frac{L_{g_1}F(0,x)}{a_1(x)} + \frac{L_{g_2}F(0,x)}{a_2(x)}}} & (x_1, x_2) = (0, 0), \end{cases} \tag{70}$$

where $i = 1, 2$, and $L_{g_i}V(x)$ and $L_{g_i}F(x, 0)$ denote Lie derivatives defined as

$$\begin{aligned} L_{g_i}V(x) &:= \left\langle \frac{\partial V}{\partial x}, g_i(x) \right\rangle, \\ L_{g_i}F(x, 0) &:= \left\langle \frac{\partial F}{\partial x}(x, 0), g_i(x) \right\rangle. \end{aligned} \tag{71}$$

The components of $L_{g_i}V(x)$ and $L_{g_i}F(x, 0)$ are derived from (65) and the calculation of kimura et al. (2013). We can design a local sample stabilizer $u = k(x)$ for (64) by using (15), (16) and (70). Moreover, we can choose $C(x) = \|H(x)\|$ in (15). Finally, we can obtain a local sample stabilizer $\hat{u} = \hat{k}(\hat{x})$ for (59) by applying the inverse transformation of (63) to the obtained controller.

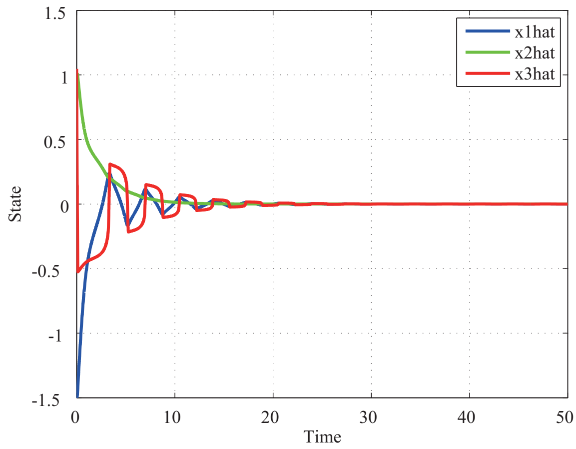

Fig. 1. Simulation: Time response of the state

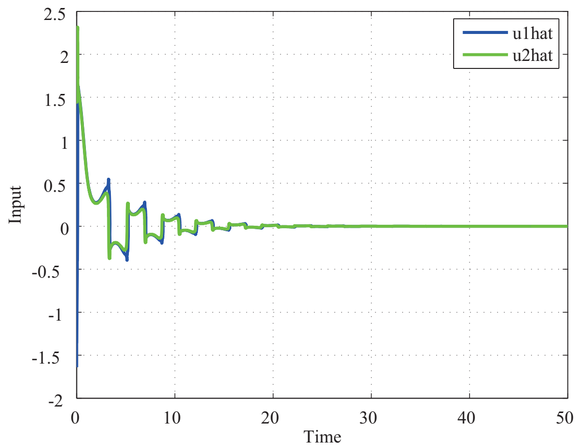

Fig. 2. Simulation: Time response of the input

We show a simulation result in Fig. 1 and 2. The initial value is set at $\hat{x}(0) = [-1.5, 1.0, \pi/3]^T$, and $W = 0.2$, $d = 4$. According to Fig. 1, we can permit the state $\hat{x}$ successfully converges to the origin. Figure. 2 illustrates that the input constraint is satisfied.

8. CONCLUSION

In this paper, we proposed a locally sample stabilizing controller for nonlinear systems having a convex input constraints. The proposed controller is based on a minimizing input associated with disassembled differential of a locally semiconcave control Lyapunov function. To design a minimizing input, we employed convex optimization theory. Moreover, we modified the small control property for locally semiconcave CLFs, and clarified that the proposed controller is continuous at $x = 0$ if the condition (53) is satisfied. The effectiveness of the proposed method was confirmed through a numerical example.

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