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Event-based Stabilization of Nonlinear Time-Delay Systems

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Abstract: In this paper, a universal formula is proposed for event-based stabilization of nonlinear time-delay systems affine in the control. The feedback is derived from the seminal law proposed by E. Sontag (1989) and then extended to event-based control of nonlinear undelayed systems. Under the assumption of the existence of a control Lyapunov-Krasovsky functional (CLKF), it enables smooth (except at the origin) asymptotic stabilization while ensuring that the sampling intervals do not contract to zero. Global asymptotic stability is obtained under the small control property assumption. Moreover, the control can be proved to be smooth anywhere under certain conditions. Simulation results highlight the ability of the proposals.

Keywords: Event-based control, Nonlinear systems, Time delay, Stabilization, CLKF.

INTRODUCTION

The classical way to address a discrete-time feedback for nonlinear systems is i) to implement a (periodic) continuous-time control algorithm with a sufficiently small sampling period (this procedure is denoted as *emulation*). However, the hardware used to sample and hold the plant measurements or compute the feedback control action may make impossible the reduce of the sampling period to a level that guarantees acceptable closed-loop performance, as demonstrated in Hsu and Sastry (1987). Furthermore, although periodicity simplifies the design and analysis, it results in a conservative usage of resources since the control law is computed and updated at the same rate regardless it is really required or not. Other ways are ii) the application of sampled-data control algorithms based on an approximated discrete-time model of the process, like in Nešić and Teel (2004), or iii) the modification of a continuous-time stabilizing control using a general formula to obtain a redesigned control suitable for sampled-data implementation, as done in Nešić and Grüne (2005). Finally, iv) event-triggered approaches have also been suggested as a solution, where the control law is event-driven. These techniques are resource-aware implementations, they overcome drawbacks of emulation, redesigned control and complexity of the underlying nonlinear sampled-data models.

Although event-based control is well-motivated, only few works report theoretical results about the stability, convergence and performance, see Anta and Tabuada (2010); Marchand et al. (2013) and the references therein. On the other hand, only few works deal with time-delay systems (which are of interest here), like in Lehmann and Lunze (2011, 2012); Guinaldo et al. (2012); Durand (2013) for linear systems. Moreover, in the best knowledge of the authors, this is the first time an event-based control strategy is proposed for general nonlinear time-delay systems.

The work in Marchand et al. (2013) is based on the universal formula of Sontag (1989). An event-based stabilization of general (undelayed) nonlinear systems affine in the input is proposed, where the control updates ensure the strict decrease of a *control Lyapunov function* (CLF) and so is asymptotically stable the closed-loop system. The concept of CLF, which is a useful tool for designing robust control laws for nonlinear systems, has been extended to time-delay systems in the form of *control Lyapunov-Razumikhin functions* (CLRF) and *control Lyapunov-Krasovsky functionals* (CLKF), see Jankovic (1999, 2000, 2003). The latter form is more flexible and easier to construct than CLRFs. Moreover, if a CLKF is known for a nonlinear time-delay system, several stabilizing control laws can be constructed using one of the universal formulas derived for CLFs (such as the Sontag's formula for instance) to achieve global asymptotic stability of the closed-loop system.

In the present paper, the universal event-based formula of Marchand et al. (2013) is extended for the stabilization of affine in the control nonlinear *time-delay* systems. The class of time-delay systems under consideration is restricted here to depend on some discrete delays and a distributed delay. Note also that only state delays are considered whereas delays in the control signal (input delays) are not considered. The rest of the document is organized as follows. In section 1, definitions are introduced and the problem is stated. The main contribution is then presented in section 2. The smooth control particular case is also concerned and an example is depicted. An analysis finally concludes the paper.

1. PRELIMINARIES

1.1 Event-triggered stabilization of nonlinear systems

Let consider the general nonlinear dynamical system

$$\dot{x}(t) = f\big(x(t), u(t)\big) \quad (1)$$
$$\text{with} \quad x(0) := x_0$$

with $x(t) \in \mathcal{X} \subset \mathbb{R}^p$, $u(t) \in \mathcal{U} \subset \mathbb{R}^q$ and f is a Lipschitz function vanishing at the origin. Note that only null stabilization is considered in this paper and the dependence on t can be omitted in the sequel for the sake of simplicity. Also, let define $\mathcal{X}^* := \mathcal{X}\backslash\{0\}$ hereafter.

Definition 1.1. (Event-based feedback).
By *event-based feedback* we mean a set of two functions, that are ***i)*** *an event function* $\epsilon : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ that indicates if one needs (when $\epsilon \leq 0$) or not (when $\epsilon > 0$) to recompute the control law and ***ii)*** *a feedback function* $\upsilon : \mathcal{X} \to \mathcal{U}$.

The solution of (1) with event-based feedback (ϵ, υ) starting in x_0 at $t = 0$ is then defined as the solution of the differential system

$$\dot{x}(t) = f\big(x(t), \upsilon(x_i)\big) \quad \forall t \in [t_i, t_{i+1}[\quad (2)$$
$$\text{with} \quad x_i := x(t_i) \quad (3)$$

where the time instants t_i, with $i \in \mathbb{N}$, are considered as *events* (they are determined when the event function ϵ vanishes and denote the sampling time instants) and x_i is the memory of the state value at the last event. With this formalization, the control value is updated each time ϵ becomes negative. Usually, one tries to design an event-based feedback so that ϵ cannot remain negative (and so is updated the control only punctually). In addition, one also wants that two events are separated with a non vanishing time interval avoiding the *Zeno* phenomenon. All these properties are encompassed with the *Minimal inter-Sampling Interval* (MSI) property introduced in Marchand et al. (2013). In particular:

Property 1.2. (Semi-uniformly MSI).
An event-triggered feedback is said to be *semi-uniformly MSI* if and only if the inter-execution times can be below bounded by some non zero minimal sampling interval $\underline{\tau}(\delta) > 0$ for any $\delta > 0$ and any initial condition x_0 in the ball $\mathcal{B}(\delta)$ centered at the origin and of radius δ.

Remark 1.3. A semi-uniformly MSI event-driven control is a piecewise constant control with non zero sampling intervals (useful for implementation purpose).

A particular event-based feedback has been proposed in Marchand et al. (2013), based on the universal formula of Sontag (1989). In order to then understand how was built this strategy, we first recall some seminal results for the stabilization of continuous-time systems. Let consider the affine in the control nonlinear dynamical system

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t) \quad (4)$$
$$\text{with} \quad x(0) := x_0$$

where f and g are smooth functions with f vanishing at the origin.

Definition 1.4. (Control Lyapunov function).
A smooth and positive definite functional $V : \mathcal{X} \to \mathbb{R}$ is a *control Lyapunov function (CLF)* for system (4) if for each $x \neq 0$ there is some $u \in \mathcal{U}$ such that

$$\alpha(x) + \beta(x)u < 0 \quad (5)$$
$$\text{with} \quad \left| \begin{array}{l} \alpha(x) := L_f V(x) = \dfrac{\partial V}{\partial x} f(x) \\ \beta(x) := L_g V(x) = \dfrac{\partial V}{\partial x} g(x) \end{array} \right.$$

where $L_f V$ and $L_g V$ are the Lie derivatives of f and g functions respectively.

Property 1.5. (Small control property).
If for any $\mu > 0$, $\varepsilon > 0$ and x in the ball $\mathcal{B}(\mu)\backslash\{0\}$, there is some u with $\|u\| \leq \varepsilon$ such that inequality (5) holds, then it is possible to design a feedback control that asymptotically stabilizes the system (Sontag (1989)).

Theorem 1.6. (Sontag's universal formula).
Assume that system (4) admits V as a CLF. For any real analytic function $q : \mathbb{R} \to \mathbb{R}$ such that $q(0) = 0$ and $bq(b) > 0$ for $b \neq 0$, let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\phi(a, b) := \begin{cases} \dfrac{a + \sqrt{a^2 + bq(b)}}{b} & \text{if} \quad b \neq 0 \\ 0 & \text{if} \quad b = 0 \end{cases} \quad (6)$$

Then, the feedback $\upsilon : \mathcal{X} \to \mathcal{U}$, smooth on $\mathcal{X}^*$, defined by

$$\upsilon(x) := -\beta(x)\, \phi\Big(\alpha(x), \|\beta(x)\|^2\Big) \quad (7)$$

with $\alpha(x)$ and $\beta(x)$ defined in (5), is such that (5) is satisfied for all $x \in \mathcal{X}^*$.

Property 1.7. If the CLF V in Theorem 1.6 satisfies the *small control property*, then taking $q(b) = b$ in ϕ in (6), the control is continuous at the origin and so is globally asymptotically stable the closed-loop system.

The event-based feedback in Marchand et al. (2013) is based on such an approach, where the control law υ is similar to the one in (7) (but with a lightly different function ϕ) and event function ϵ is related to the time derivative of the CLF in order to ensure a (global) asymptotic stability of the closed-loop system. In the present paper, this event-based feedback is extended for the stabilization of nonlinear time-delay systems. Actually, the construction is quite similar, this is why the event-based feedback for nonlinear undelayed systems is not detailed here.

1.2 Stabilization of (time-triggered) time-delay systems

Hereafter, the state of a time-delay system is described by $x_d : [-r, 0] \to \mathcal{X}$ defined by $x_d(t)(\theta) = x(t + \theta)$. This notation, used in Jankovic (2000) in particular, seems more convenient than the more conventional $x_t(\theta)$. Note that the dependence on t and θ can be omitted in the sequel for the sake of simplicity, writing $x_d(\theta)$ – or only x_d – instead of $x_d(t)(\theta)$ for instance. Let consider the affine in the control nonlinear dynamical time-delay system

$$\dot{x} = f(x_d) + g(x_d)u \quad (8)$$
$$\text{with} \quad x_d(0)(\theta) := \chi_0(\theta)$$

where f, g are smooth functions and $\chi_0 : [-r, 0] \to \mathcal{X}$ is a given initial condition. Note that the class of time-delay system under consideration has been restricted to depend on l discrete delays and a distributed delay in the form

$$\begin{aligned}
&\dot{x} = \Phi(x_\tau) + g(x_\tau)u && (9)\\
&\text{with} \quad \Phi(x_\tau) := f_0(x_\tau) + \int_{-r}^{0} \Gamma(\theta) F\big(x_\tau, x(t+\theta)\big) d\theta\\
&\text{and} \quad x_\tau := \big(x, x(t-\tau_1), \ldots, x(t-\tau_l)\big)
\end{aligned}$$

where f_0, g and $F : \mathbb{R}^{(l+2)p} \to \mathbb{R}^\Gamma$ are smooth functions of their arguments. Without loss of generality, it is assumed that $F(x_\tau, 0) = 0$ and the matrix $\Gamma : [-r, 0] \to \mathbb{R}^{p\times\Gamma}$ is piecewise continuous (hence, integrable) and bounded.

Remark 1.8. The restriction (9) on this class of delay systems is needed to avoid the problems that arise due to non-compactness of closed bounded sets in the space $\big(C([-r,0],\mathcal{X}), \|\cdot\|\big)$, where $C([-r,0],\mathcal{X})$ denotes the space of continuous functions from $[-r, 0]$ into $\mathcal{X}$. This is discussed in Jankovic (1999, 2000).

Remark 1.9. Input delays of the form $u(t-\tau)$ are not considered in this paper. However, the control law is computed using the state x_d of the time-delay system.

Definition 1.10. (Control Lyapunov-Krasovsky functional). A smooth functional $V : \mathcal{X} \to \mathbb{R}$ of the form

$$V(x_d) = V_1(x) + V_2(x_d) + V_3(x_d) \qquad (10)$$

$$\text{with} \quad \left| \begin{aligned} &V_2(x_d) = \sum_{j=1}^{l} \int_{-\tau_j}^{0} S_j(x(t-\varsigma)) d\varsigma \\ &V_3(x_d) = \int_{-r}^{0}\int_{t+\theta}^{t} L(\theta, x(\varsigma)) d\varsigma d\theta \end{aligned}\right.$$

where V_1 is a smooth, positive definite, radially unbounded function of the current state x, V_2 and V_3 are non-negative functionals respectively due to the discrete delays and the distributed delay in (9), $S_j : \mathcal{X} \to \mathbb{R}$ and $L : \mathbb{R}^+ \times \mathcal{X} \to \mathbb{R}$ are non-negative integrable functions, smooth in the x-argument, is a *control Lyapunov-Krasovsky functional (CLKF)* for system (9) if there exists a function λ, with $\lambda(s) > 0$ for $s > 0$, and two class $\mathcal{K}_\infty$ functions κ_1 and κ_2 such that

$$\kappa_1(|\chi_0|) \leq V(\chi_d) \leq \kappa_2(\|\chi_d\|)$$

and

$$\beta_d(\chi_d) = 0 \Rightarrow \alpha_d(\chi_d) \leq -\lambda(|\chi_0|) \qquad (11)$$

$$\text{with} \quad \left| \begin{aligned} &\alpha_d(x_d) := L_f^* V(x_d) \\ &\beta_d(x_d) := L_g V_1(x_d) \end{aligned}\right.$$

for all piecewise continuous functions $\chi_d : [-r, 0] \to \mathcal{X}$, where χ_0 is defined in (8).

Remark 1.11. Whereas the classical Lie derivative notation is used in $L_g V_1(x) = \frac{\partial V_1}{\partial x} g(x)$ for the CLKF part V_1 which is function of the current state x, an extended Lie derivative is required for functionals of the form (10). $L_f^* V$, initially defined in Jankovic (2000), comes from the time derivative of the CLKF V in (10) along trajectories of the system (9), that is

$$\dot{V} = L_f^* V(x_d) + L_g V_1(x_d) u = \alpha_d(x_d) + \beta_d(x_d) u \qquad (12)$$

$$\begin{aligned}\text{with} \quad L_f^* V(x_d) := \frac{\partial V_1}{\partial x}\Phi + \sum_{j=1}^{l}\Big(S_j(x) - S_j(x(t-\tau_j))\Big)\\ + \int_{-r}^{0}\Big(L(\theta, x) - L(\theta, x(t+\theta))\Big) d\theta\end{aligned}$$

where Φ is defined in (9).

The Sontag's universal formula (Theorem 1.6) has been extended in Jankovic (2000) for the stabilization of nonlinear time-delay systems (9) with a CLKF of the form (10). This can be summarized as follows:

Theorem 1.12. (Sontag's universal formula with CLKF). Assume that system (9) admits a CLKF of the form (10). For any real analytic function $q : \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$, both defined in Theorem 1.6, let the feedback $\upsilon : \mathcal{X} \to \mathcal{U}$, smooth on $\mathcal{X}^*$, defined by

$$\upsilon(x_d) := -\beta_d(x_\tau)\,\phi\Big(\alpha_d(x_d), \|\beta_d(x_d)\|^2\Big) \qquad (13)$$

with x_τ and α_d, β_d defined in (9) and (11) respectively. Then υ is such that (11) is satisfied for all non zero piecewise continuous functions $\chi_d : [-r, 0] \to \mathcal{X}$.

Property 1.13. If the CLKF V in Theorem 1.12 satisfies the *small control property*, then taking $q(b) = b$ in ϕ in (6), the control is continuous at the origin and so is globally asymptotically stable the closed-loop system.

1.3 Contribution of the paper

In the present paper, the event-based approach previously developed in Marchand et al. (2013) is extended for nonlinear time-delay systems admitting a CLKF.

In the sequel, let

$$x_{di} := x_d(t_i) \qquad (14)$$

be the memory of the delayed state value at the last event, by analogy with (3).

2. EVENT-BASED STABILIZATION OF NONLINEAR TIME-DELAY SYSTEMS

It is possible to design an event-based feedback control that asymptotically stabilizes time-delay systems (9) with a CLKF of the form (10):

Theorem 2.1. (Event-based universal formula with CLKF). If there exists a CLKF V of the form (10) for system (9), then the event-based feedback (ϵ, υ) – see Definition 1.1 – defined by

$$\upsilon(x_d) = -\beta_d(x_\tau)\Delta(x_\tau)\gamma(x_d) \qquad (15)$$

$$\begin{aligned}\epsilon(x_d, x_{di}) = -\alpha_d(x_d) - \beta_d(x_d)\upsilon(x_{di})\\ -\sigma\sqrt{\alpha_d(x_d)^2 + \theta(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\end{aligned} \qquad (16)$$

with

- α_d and β_d as defined in (11) ;
- $\Delta : \mathcal{X}^* \to \mathbb{R}^{q\times q}$ (a tunable parameter) and $\theta : \mathcal{X} \to \mathbb{R}$ are smooth positive definite functions ;
- $\gamma : \mathcal{X} \to \mathbb{R}$ is defined by

$$\gamma(x_d) := \begin{cases} \dfrac{\alpha_d(x_d) + \sqrt{\alpha_d(x_d)^2 + \theta(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T} \\ \quad \text{if} \quad x_d \in \mathcal{S}_d \\ 0 \quad \text{if} \quad x_d \notin \mathcal{S}_d \end{cases} \tag{17}$$

- $\mathcal{S}_d := \{x_d \in \mathcal{X} \mid \|\beta_d(x_d)\| \neq 0\}$;
- $\sigma \in [0, 1[$ is a tunable parameter ;

where x_{di} and x_τ are defined in (14) and (9) respectively, is semi-uniformly MSI, smooth on $\mathcal{X}^*$ and such that the time derivative of V satisfies (11) $\forall x \in \mathcal{X}^*$.

Remark 2.2. The simplification made with respect to the original result in Marchand et al. (2013) (for the stabilization of nonlinear undelayed systems) resides in the assumptions made for the functions θ and Δ, that are more restrictive here whereas they are assumed to be definite only on the set $\mathcal{S}_d$ in the original work.

Remark 2.3. The idea behind the construction of the event-based feedback (15)-(16) is to compare the time derivative of the CLKF V i) in the event-based case, that is applying $\upsilon(x_{di})$, and ii) in the classical case, that is applying $\upsilon(x_d)$ instead of $\upsilon(x_{di})$. The event function is the weighted difference between both, where σ is the weighted value. By construction, an event is enforced when the event function ϵ vanishes to zero, that is hence when the stability of the event-based scheme does not behave as the one in the classical case. Also, the convergence will be faster with higher σ but with more frequent events in return. $\sigma = 0$ means updating the control when $\dot{V} = 0$.

Also, properties inherited from Marchand et al. (2013) complete Theorem 2.1. In particular:

Property 2.4. (Global asymptotic stability).
If the CLKF V in Theorem 2.1 satisfies the *small control property*, then the event-based feedback (15)-(16) is continuous at the origin and so is globally asymptotically stable the closed-loop system.

Property 2.5. (Smooth control).
If there exists some smooth function $\omega : \mathcal{X} \to \mathbb{R}^+$ such that on $\mathcal{S}_d^* := \mathcal{S}_d \backslash \{0\}$

$$\omega(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T - \alpha_d(x_d) > 0$$

then the control is smooth on $\mathcal{X}$ as soon as $\theta(x_d)\|\Delta(x_d)\|$ vanishes at the origin with

$$\theta(x_d) := \omega(x_d)^2\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T - 2\alpha_d(x_d)\omega(x_d) \tag{18}$$

2.1 Proofs

Proof of Theorem 2.1: The proof follows the one developed in Marchand et al. (2013) for event-based control of systems without delays (4). First, let define hereafter

$$\psi(x) := \sqrt{\alpha_d(x)^2 + \theta(x)\beta_d(x)\Delta(x)\beta_d(x)^T} \tag{19}$$

Let begin establishing γ is smooth on $\mathcal{X}^*$. For this, consider the algebraic equation

$$\begin{aligned} P(x_d, \zeta) := \beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T\zeta^2 \\ - 2\alpha_d(x_d)\zeta - \theta(x_d) = 0 \end{aligned} \tag{20}$$

Note first that $\zeta = \gamma(x)$ is a solution of (20) for all $x_d \in \mathcal{X}$. It is easy to prove that the partial derivative of P with respect to ζ is always strictly positive on $\mathcal{X}^*$

$$\frac{\partial P}{\partial \zeta} := 2\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T\zeta - 2\alpha_d(x_d) \tag{21}$$

Indeed, when $\|\beta_d(x_d)\| = 0$, (11) gives $\frac{\partial P}{\partial \zeta} = -2\alpha_d(x_d) \geq 2\lambda(|\chi_0|) > 0$ and when $\|\beta_d(x_d)\| \neq 0$, (17) gives $\frac{\partial P}{\partial \zeta} = 2\sqrt{\alpha_d(x_d)^2 + \theta(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T} > 0$ replacing ζ in (21) by the expression of γ (since $\zeta = \gamma(x)$ is a solution of (20)). Therefore $\frac{\partial P}{\partial \zeta}$ never vanishes at each point of the form $\{(x_d, \gamma(x_d)) | x_d \in \mathcal{X}^*\}$. Furthermore, P is smooth w.r.t. x_d and ζ since so are α_d, β_d, θ and Δ. Hence, using the implicit function theorem, γ is smooth on $\mathcal{X}^*$.

The decrease of the CLKF of the form (10) when applying the event-based feedback (15)-(16) is easy to prove. For this, let consider the time interval $[t_i, t_{i+1}]$, that is the interval separating two successive events. Recall that x_{di} denotes the value of the state when the i^{th} event occurs and t_i the corresponding time instant, as defined in (14). At time t_i, when the event occurs, the time derivative of the CLKF, i.e. (12), after the update of the control is

$$\frac{dV}{dt}(x_{di}) = \alpha_d(x_{di}) + \beta_d(x_{di})\upsilon(x_{di}) = -\psi(x_{di}) < 0$$

when substituting (17) in (15), where ψ is defined in (19). More precisely, defining a compact set not containing the origin, that is $\Omega = \{x_d \in CP([-r, 0], \mathcal{X}) : d \leq \|x_d\| \leq D\}$, where $CP([-r, 0], \mathcal{X})$ denotes the space of piecewise continuous functions from $[-r, 0]$ into $\mathcal{X}$, d and D are some constant in $\mathbb{R}^+$. If V is a CLKF for the system of the form (9) then for all $0 < \delta < D$ there exists $\varepsilon > 0$ such that $\alpha_d(\chi_d) \geq -\frac{1}{2}\lambda(|\chi_0|) \Rightarrow |\beta_d(\chi_d)| \geq \varepsilon$ for $\chi_d \in \Omega$. This gives

$$\dot{V} \leq -\lambda(|x|)$$

One can refer to Lemma 1 in Jankovic (2000), and Jankovic (1999), for further details. With this updated control, the event function (16) hence becomes strictly positive

$$\epsilon(x_{di}, x_{di}) = (1 - \sigma)\psi(x_{di}) > 0$$

since $\sigma \in [0, 1[$, where ψ is defined in (19). Furthermore, the event-function necessarily remains positive before the next event by continuity, because an event will occur when $\epsilon(x_d, x_{di}) = 0$ (see Definition 1.1). Therefore, on the interval $[t_i, t_{i+1}]$, one has

$$\begin{aligned} \epsilon(x_d, x_{di}) &= -\alpha_d(x_d) - \beta_d(x_d)\upsilon(x_{di}) - \sigma\psi(x_d) \\ &= -\frac{dV}{dt}(x_d) - \sigma\psi(x_d) \geq 0 \end{aligned}$$

which ensures the decrease of the CLKF on the interval since $\sigma\psi(x_d) \geq 0$, where ψ is defined in (19). Moreover, t_{i+1} is necessarily bounded since, if not, V should converge to a constant value where $\frac{dV}{dt} = 0$, which is impossible thanks to the inequality above. The event function precisely prevents this phenomena detecting when $\frac{dV}{dt}$ is *close to vanish* and updates the control if it happens, where σ is a tunable parameter fixing how "close to vanish" has to be the time derivative of V.

To prove that the event-based control is MSI, one has to prove that for any initial condition in an a priori given set, the sampling intervals are below bounded. First of all, notice that events only occur when ϵ becomes negative (with $x_d \neq 0$). Therefore, using the fact that when $\beta_d(x_d) = 0$, $\alpha_d(x_d) < -\lambda(|\chi_0|)$ (because V is a CLKF as defined in Definition 1.10), it follows from (16), on $\{x_d \in \mathcal{X}^* \,|\, \|\beta_d(x_d)\| = 0\}$, that

$$\epsilon(x_d, x_{di}) = -\alpha_d(x_d) - \sigma|\alpha_d(x_d)| = (1-\sigma)\lambda(|\chi_0|) > 0$$

because $\sigma \in [0,1[$ and $\lambda(s) > 0$ for $s > 0$. Therefore, there is no event on the set $\{x_d \in \mathcal{X} | \|\beta_d(x_d)\| = 0\} \cup \{0\}$. The study is then restricted to the set $\mathcal{S}_d^* = \{x_d \in \mathcal{X}^* | \|\beta_d(x_d)\| \neq 0\}$, where θ and Δ are strictly positive by assumption. Rewriting the time derivative of the CLKF along the trajectories yields

$$\begin{aligned}\frac{dV}{dt}(x_d) &= \alpha_d(x_d) + \beta_d(x_d)\upsilon(x_{di}) \\ &= -\psi(x_d) + \beta_d(x_d)\big(\upsilon(x_{di}) - \upsilon(x_d)\big) \end{aligned} \quad (22)$$

when using the definition of $\upsilon(x_d)$ in (15) and (17), where ψ is defined in (19). Let define for $x_{di} \in \mathcal{S}_d$, the level $\vartheta_i := V(x_{di})$ and the set $\mathcal{V}_{\vartheta_i} := \{x_d \in \mathcal{X} | V(x_d) \leq \vartheta_i\}$. From the choice of the event function, it follows from (22) that x_d belongs to $\mathcal{V}_\vartheta \subset \mathcal{V}_{\vartheta_i}$. Note that if x_{di} belongs to $\mathcal{S}_d$, this is not necessarily the case for x_d that can escape from this set. First see that, since i) $\theta(x_d)$ is such that $\alpha_d(x_d)^2 + \theta(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T > 0$ for all $x_d \in \mathcal{S}_d^*$, and ii) $\alpha_d(x_d)$ is necessarily non-zero on the frontier of $\mathcal{S}_d$ (except possibly at the origin)

$$\begin{aligned}\frac{dV}{dt}(x_{di}) &= -\psi(x_{di}) \\ &\leq - \inf_{\substack{x_{di}\in\mathcal{S}_d \\ \text{s.t. } V(x_{di})=\vartheta_i}} \psi(x_{di}) =: -\varphi(\vartheta_i) < 0 \end{aligned} \quad (23)$$

Considering now the second time derivative of the CLKF

$$\ddot{V}(x_d) = \left(\frac{\partial \alpha_d}{\partial x_d}(x_d) + \upsilon(x_{di})^T \frac{\partial \beta_d^T}{\partial x_d}(x_d)\right)\Theta(x_d, x_{di}) \quad (24)$$
$$\text{with} \quad \Theta(x_d, x_{di}) := \Phi(x_\tau) + g(x_\tau)\upsilon(x_{di})$$

where Φ is defined in (9). By continuity of all the involved functions (except for Γ in Φ which is piecewise continuous but bounded by assumption), both terms can be bounded for all $x_d \in \mathcal{V}_{\vartheta_i}$ by the following upper bounds $\varrho_1(\vartheta_i)$ and $\varrho_2(\vartheta_i)$ such that

$$\varrho_1(\vartheta_i) := \sup_{\substack{x_{di}\in\mathcal{S}_d \text{ s.t. } V(x_{di})=\vartheta_i \\ x_d \in \mathcal{V}_{\vartheta_i}}} \left\| \frac{\partial \alpha_d}{\partial x_d}(x_d) + \upsilon(x_{di})^T \frac{\partial \beta_d^T}{\partial x_d}(x_d) \right\|$$

$$\varrho_2(\vartheta_i) := \sup_{\substack{x_{di}\in\mathcal{S}_d \text{ s.t. } V(x_{di})=\vartheta_i \\ x_d \in \mathcal{V}_{\vartheta_i}}} \|\Theta(x_d, x_{di})\|$$

where Θ is defined in (24). Therefore, $\dot{V}$ is strictly negative at any event instant t_i and cannot vanish until a certain time $\underline{\tau}(\vartheta_i)$ is elapsed (because its slope is positive). This minimal sampling interval is only depending on the level ϑ_i. A bound on $\underline{\tau}(\vartheta_i)$ is given by the inequality

$$\frac{dV}{dt}(x_d) \leq \frac{dV}{dt}(x_{di}) + \rho_1\rho_2(t - t_i) \qquad x \in \mathcal{V}_{\vartheta_i}$$

that yields

$$\underline{\tau}(\vartheta_i) \geq \frac{\varphi(\vartheta_i)}{\varrho_1(\vartheta_i)\varrho_2(\vartheta_i)} > 0$$

where φ is defined in (23). As a consequence, the event-based feedback (15)-(16) is semi-uniformly MSI.

Proof of Property 2.4: To prove the continuity of υ at the origin, one only needs to consider the points in $\mathcal{S}$ since $\upsilon(x_d) = 0$ if $\|\beta_d(x_d)\| = 0$. Then (15) gives

$$\begin{aligned}\|\upsilon(x_d)\| &\leq \frac{|\alpha_d(x_d)|}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\|\Delta(x_d)\beta_d(x_d)^T\| \\ &\quad + \frac{\psi(x_d)}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\|\Delta(x_d)\beta_d(x_d)^T\| \\ &\leq \frac{2|\alpha_d(x_d)|}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\|\Delta(x_d)\beta_d(x_d)^T\| \\ &\quad + \sqrt{\theta(x_d)\|\Delta(x_d)\|} \end{aligned} \quad (25)$$

With the small control property (see Property 1.5), for any $\varepsilon > 0$, there is $\mu > 0$ such that for any $x_d \in \mathcal{B}(\mu)\backslash\{0\}$, there exists some u with $\|u\| \leq \varepsilon$ such that $L_f^* V(x_d) + [L_g V_1(x_d)]^T u = \alpha_d(x_d) + \beta_d(x_d)u < 0$ and therefore $|\alpha_d(x_d)| < \|\beta_d(x_d)\|\varepsilon$. It follows

$$\|\upsilon(x_d)\| \leq \frac{2\varepsilon\|\beta_d(x_d)\|\|\Delta(x_d)\beta_d(x_d)^T\|}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T} + \sqrt{\theta(x_d)\|\Delta(x_d)\|}$$

Since the function $(v_1, v_2) \to \frac{\|v_1\|\|v_2\|}{v_1^T v_2}$ is continuous w.r.t. its two variables at the origin where it equals 1, since θ and Δ are also continuous, since $\theta(x_d)\|\Delta(x_d)\|$ vanishes at the origin, for any ε', there is some μ' such that $\forall x_d \in \mathcal{B}(\mu')\backslash\{0\}$, $\|\upsilon(x_d)\| \leq \varepsilon'$ which ends the proof of continuity.

Proof of Property 2.5: With θ as in (18), the control in (15), (17) becomes $\upsilon(x_d) = -\beta_d(x_d)\Delta(x_d)\omega(x_d)$ which is obviously smooth on $\mathcal{X}$.

2.2 Example

Consider the nonlinear time-delay system

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_2 + x_{2d} + x_1^3 + u \\ \text{with} \quad & x_{2d} := x_2(t - \tau) \end{aligned} \quad (26)$$

that admits a CLKF (proposed in Jankovic (2000))

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}\int_{-\tau}^{0} x_2^2(\theta)d\theta \quad (27)$$
$$\text{with} \quad \left| \begin{aligned} \alpha_d &= x_2(-x_2 + x_{2d} + x_1^3) + \frac{1}{2}(x_2^2 - x_{2d}^2) \\ \beta_d &= x_1 + x_2 \end{aligned} \right.$$

Indeed, setting, $\lambda(|x|) = \frac{1}{4}|x|^4$, one obtains

$$\begin{aligned} &\beta_d = 0 \Rightarrow x_1 = -x_2 \\ &\Rightarrow \alpha_d = -\frac{1}{2}(x_2 - x_{2d})^2 - x_2^4 \leq -x_2^4 \leq -\lambda(|x|) \end{aligned}$$

which proves that (27) is a CLKF for (26) using Definition 1.10.

The time evolution of x, $\upsilon(x)$ and the event function $\epsilon(x, x_i)$ is depicted in Fig. 1, for $\Delta = I_p$ (the identity matrix), $\theta(x)$ is as defined in (18) (for smooth control everywhere), with $\omega = 0.1$, $\sigma = 0.1$, $x_0 = (1\ -2)^T$ and a time delay $\tau = 2\,s$. One could remark that only 5 events occurs in the $20\,s$ simulation time (including the first event at $t = 0$) when applying the proposed event-based approach (15)-(16). Furthermore, x_1 and x_2 slowly converge to 0, as one can see in the $200\,s$ simulation time in Fig. 2.

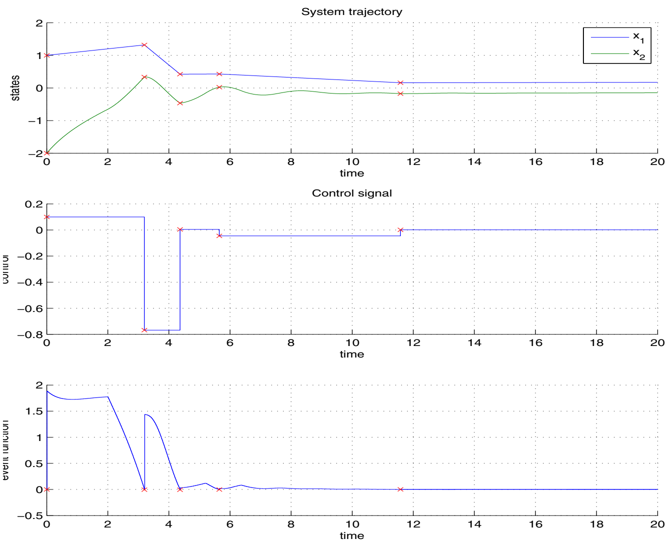

Fig. 1. Simulation results of system (26) with CLKF as in (27) and event-based feedback (15)-(16).

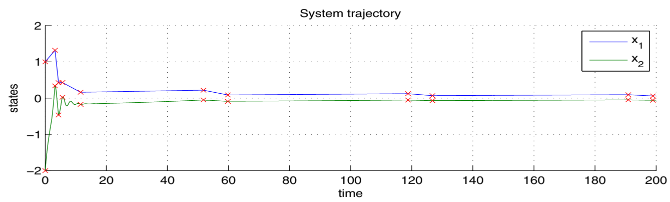

Fig. 2. Convergence of x_1 and x_2 to 0.

CONCLUSION

In this paper, an extension of the *Sontag*'s universal formula was proposed for event-based stabilization of nonlinear *time-delay* systems. Whereas the original work deals with control Lyapunov functions, some *control Lyapunov-Krosovsky functionals* (CLKF) are now required for a global (except at the origin) asymptotic stabilization of systems with state delays. The sampling intervals do not contract to zero. Moreover, the control is continuous at the origin if the CLKF fulfills the small control property. With additional assumption, the control can be proved to be smooth everywhere. Some simulation results were provided, they notably highlighted the low frequency of events of the proposal.

Next step is to also consider input delays. Another way of investigation could be to develop event-based strategies for nonlinear systems based on other universal formulas, like the formula of Freeman and Kokotovic (1996) or the domination redesign formula of Sepulchre et al. (1997), using CLRF and CLKF in the spirit of Jankovic (2000) (for the time-triggered case).

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REFERENCES

Anta, A. and Tabuada, P. (2010). To sample or not to sample: Self-triggered control for nonlinear systems. *IEEE Transactions on Automatic Control*, 55, 2030–2042.

Durand, S. (2013). Event-based stabilization of linear system with communication delays in the measurements. Proceedings of the American Control Conference.

Freeman, R. and Kokotovic, P. (1996). *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques.* Birkhäuser Basel.

Guinaldo, M., Lehmann, D., Sánchez, J., Dormido, S., and Johansson, K.H. (2012). Distributed event-triggered control with network delays and packet losses. In *Proceedings of the 51st IEEE Conference on Decision and Control.*

Hsu, P. and Sastry, S. (1987). The effect of discretized feedback in a closed loop system. In *Proceedings of the 26th IEEE Conference on Decision and Control.*

Jankovic, M. (1999). Control Lyapunov-Razumikhin functions to time-delay systems. In *Proceedings of the 38th IEEE Conference on Decision and Control.*

Jankovic, M. (2000). Extension of control Lyapunov functions to time-delay systems. In *Proceedings of the 39th IEEE Conference on Decision and Control.*

Jankovic, M. (2003). Control of nonlinear systems with time-delay. In *Proceedings of the 42th IEEE Conference on Decision and Control.*

Lehmann, D. and Lunze, J. (2011). Event-based control with communication delays. In *Proceedings of the 18th IFAC world congress.*

Lehmann, D. and Lunze, J. (2012). Event-based control with communication delays and packet losses. *International Journal of Control*, 85, 563–577.

Marchand, N., Durand, S., and Guerrero-Castellanos, J.F. (2013). A general formula for event-based stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 58(5), 1332–1337.

Nešić, D. and Grüne, L. (2005). Lyapunov-based continuous-time nonlinear controller redesign for sampled-data implementation. *Automatica*, 41(7), 1143–1156.

Nešić, D. and Teel, A. (2004). A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Transactions on Automatic Control*, 49(7), 1103–1122.

Sepulchre, R., Jankovic, M., and Kokotovic, P. (1997). *Constructive Nonlinear Control.* Springer-Verlag.

Sontag, E.D. (1989). A "universal" construction of Artstein's theorem on nonlinear stabilization. *Systems & Control Letters*, 13, 117–123.