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Recovering the Controllability of Complex Networks ⋆

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Abstract: The studies on the controllability of complex networks popularly existing in natural, social and man-made engineered systems have been a critical and attractive subject for both academic and practical communities. To design and maintain a networked system under control, it is vital to explore the mechanism and relationship between the network layout and its controllability. For a fully-controlled complex system, potential malicious attacks and/or random failures will lead to the damage of its internal structure, such as the breakdown of certain control nodes or the loss of the links between state nodes. In this paper, we first introduce the concept of degree of controllability to quantify the control level of the networks. And for the networks whose degree of controllability is not full, we propose two novel optimal recovering strategies, OAN (short for optimal adding-node) strategy and OAE (short for optimal adding-edge) strategy, to repair their controllability. The results of experiments conducted on the various real and model networks demonstrate the effectiveness of these two strategies and the better performance compared to their randomized counterparts, RAN (short for randomized adding-node) strategy and RAE (short for randomized adding-edge) strategy.

Keywords: degree of controllability, optimal adding-node recovering strategy, optimal adding-edge recovering strategy, network controllability, complex networks

1. INTRODUCTION

Complex networked dynamical systems can be seen almost everywhere in our life, from the neural systems, to the social networks, to the large-scale man-made engineered systems (e.g., Internet and power grids). The interaction of distinct units within them naturally gives rise to the complex network structures (Albert and Barabási (2002); Newman (2003); Boccaletti et al. (2006); Barabási (2012)). In recent years, the studies focused on how to develop the capacity to effectively and efficiently control complex networks receive a lot of attention from both fields of network science and control science (Liu et al. (2011); Ding et al. (2013b); Posfai et al. (2013); Wang et al. (2012b); Nepusz and Vicsek (2012); Liu et al. (2012); Wang et al. (2012a); Yan et al. (2012); Cowan et al. (2012); Pu et al. (2012); Nacher and Akutsu (2013); Delpini et al. (2013); Jia et al. (2013); Sorrentino (2007); Wang et al. (2013)).

Consider a directed weighted network $\boldsymbol{G}$ of N state nodes and P control nodes whose time evolution follows the linear time-invariant dynamics (Liu et al. (2011)).

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

where $\mathbf{X}(t) = (x_1(t), x_2(t), ..., x_N(t))^{\mathrm{T}} \in \mathrm{R}^N$ is the state vector of the network at time t; $\mathbf{A} \in \mathrm{R}^{N\times N}$ denotes the state matrix which depicts the linking strength between the state nodes. The input matrix $\mathbf{B} \in \mathrm{R}^{N\times P}$ elucidates the network's control wiring diagram which identifies the N state nodes that are controlled by P-dimensional input vector $\mathbf{u}(t)$ with the independent control signals imposed by P control nodes. $\boldsymbol{G}$ is said to be controllable if and only if with a suitable choice of inputs $\mathbf{u}$, it can be guided from any initial state $\mathbf{X}_o$ to any desired state $\mathbf{X}_f$ within the finite time, which can be numerically judged by the Kalman controllability rank condition,

$$\mathrm{rank}(\mathbf{C} = (\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, ..., \mathbf{A}^{N-1}\mathbf{B})) = N \tag{2}$$

where $\mathbf{C}$ is the controllability matrix (Kalman (1963)).

The property of controllability of networks is the bedrock for various dynamical processes running as desired on them, for example, the traffic flow between routers in Internet (Paxson (1997)) and the power transmission on the power grids (Kundur (1994)). If losing the controllability, by the definition, there exist two separate states of the network can not reach each other within finite time under the properly selected inputs, which means in Internet, the desired traffic distribution on routers may not be achieved and on power grids, we may not be able to configure the voltage and phase at each bus as planned. For a fully-controlled network, potential malicious attacks and/or random failures will lead to the damage of its

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internal structure, such as the breakdown of certain control nodes or the loss of the links between state nodes. This in turn may transform its fully-controllable status to the opposite. In Fig. 1b, the sample network in Fig. 1a loses its controllability after the breakdown of the control node u_1. In Fig. 1c, after the link (x_6, x_5) breaks, the network loses its controllability as well. This naturally gives the spotlight to the research of robustness of controllability of the networks (Liu et al. (2011); Magnien et al. (2011); Albert et al. (2000); Wang et al. (2012b); Pu et al. (2012); Liu et al. (2012); Wang et al. (2013)). In reality, the precise value of system parameters, i.e., the elements of **A** and **B**, are often not known except the zeros that mark the absence of the links between components of the system (Lin (1974)). Hence **A** and **B** are commonly taken as the structural matrices with either fixed zero or free parameters. The structural controllability (Lin (1974)) is used to characterize such systems. In this regard, Liu et al. (Liu et al. (2011)) proposed a graph-theoretic approach to identify the minimum number of state nodes, a.k.a. driver nodes, whose control can offer full capacity to guide the network's dynamics. In addition, they found, to make controllability robust to the attacks on links, it is sufficient to double the critical links in the networks. Pu et al. (Pu et al. (2012)) studied the controllability of directed ER and SF networks under attacks and cascading failures, and found degree-based attacks are more efficient than random attacks. In (Liu et al. (2012)), Liu et al. proposed an efficient random upstream attack strategy against malicious networks. Wang et al. (Wang et al. (2013)) proposed and optimized a control robustness index which can mitigate the destruction of malicious attack through backing up the control routes. These interesting and insightful studies are helpful to deepen our understanding on the robust control of networks.

It is realized that the research activities dedicating on the robustness of controllability of networks so far have mainly been focused on how to better design the sophisticated network structure against malicious attacks (Liu et al. (2011); Wang et al. (2013)) or evaluate the efficiency of various attack strategies targeting the controllability of the networks (Liu et al. (2012); Pu et al. (2012)). There is little effort being devoted to developing effective and efficient avenues to recover the damaged controllability of networks caused by malicious attacks or random failures, e.g., networks in Fig. 1b and Fig. 1c, which is a common scenario in the real world. For example, on power grids, the power electronic devices on the buses have a high probability to fail after a long time running, which means the voltage and the phase on that bus may not be configured desirably, leading to the loss of its controllability. It is for the considerations of efficiency and economics to repair such controllability-damaged dynamical systems rather than redesign a new control scheme for them.

In this paper, we first introduce the concept of degree of controllability to quantify the control level of the networks. It is defined as the ratio of the number of state nodes in a maximum controllable subnetwork to the total number of the state nodes. Clearly it ranges from 0 to 1. If it equals to 1, we call the degree of controllability of the network is full. For the networks whose degree of controllability is not full, which means the networks are not fully controllable (i.e. not all the state nodes under control), we develop two novel optimal recovering strategies, OAN (short for optimal adding-node) strategy, which optimizes the number of control nodes added to the networks for recovering the controllability and OAE (short for optimal adding-edge) strategy, which optimizes the number of edges between the state nodes added to the networks for recovering the controllability. Mathematically they both can be mapped into the constrained combinatorial optimization problems with the binary decision variables and solved by the branch-and-bound techniques (Nemhauser and Wolsey (1988)). The experiments conducted on the various real and model networks demonstrate the effectiveness of both strategies and the better performance compared to their randomized counterparts, RAN (short for randomized adding-node) strategy and RAE (short for randomized adding-edge) strategy.

The main contributions of this paper rest on the following two aspects,

(1) Propose an optimal adding-node recovering strategy for the networks which are not fully controllable;
(2) Propose an optimal adding-edge recovering strategy for the networks which are not fully controllable.

The rest of the paper is organized as follows. Section 2 introduces the concept of degree of controllability of networks. In section 3 and section 4, we propose two recovering strategies, one with optimally adding control nodes and the other with optimally adding links between state nodes. Then we map them into the constrained combinatorial optimization problems. In section 5, experiments on various real and model networks are conducted and detailed discussions are made. Finally, concluding remarks are given.

2. DEGREE OF CONTROLLABILITY OF THE DIRECTED NETWORKS

The computing of rank(**C**) needs the accurate value of network parameters, i.e. the entries in the state matrix **A** and the input matrix **B**. In reality, the elements of **A** and **B** are not often known precisely other than the zeros that depict the absence of connections between nodes of the network. Hence, **A** and **B** are often viewed as structured matrices, i.e. their elements are either fixed or independent free parameters (Lin (1974)). Obviously, the rank of **C** varies as a function of the free parameters of **A** and **B**. However, it reaches the maximal value for all but an exceptional set of values of the free parameters which forms a proper variety with Lebesgue measure zeros in the parameter space (Shields RW (1976); Hosoe (1980)). This maximal value is called generic rank of the controllability matrix **C**, denoted as $\text{rank}_g(\mathbf{C})$, which represents the number of the state nodes in the maximal controllable subnetwork (Hosoe (1980)). When $\text{rank}_g(\mathbf{C}) \equiv N$, the network is said to be structurally controllable, i.e. controllable for almost all sets of values of the free parameters of **A** and **B** other than an exceptional set of values with zero measure (Lin (1974); Hosoe (1980); Poljak (1990)). Lin (Lin (1974)) proposed a graph-theoretical approach to test a given network's structural controllability, depicted in Fig. 2, where the network covered by cacti is structurally controllable.

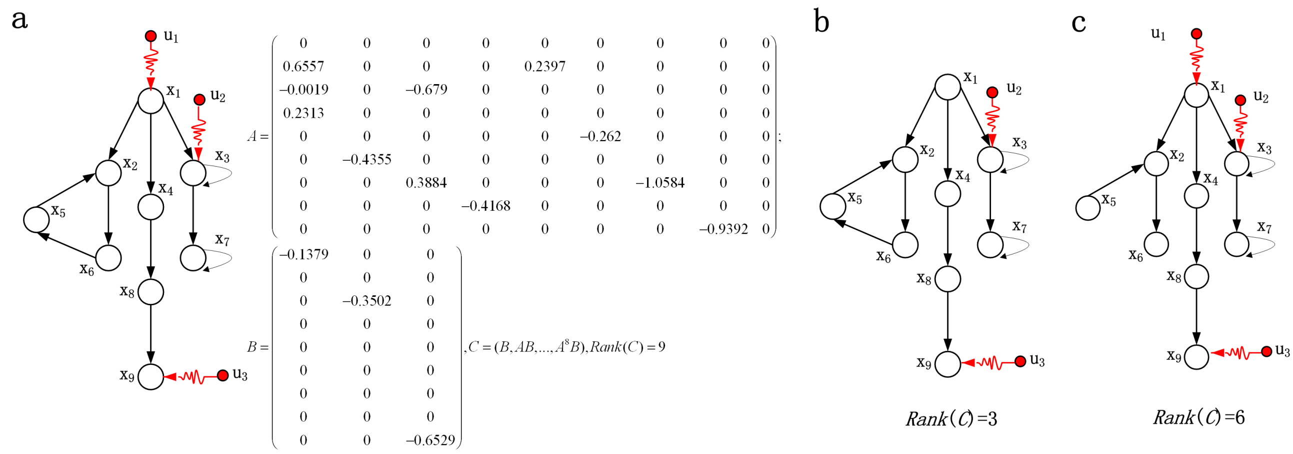

Fig. 1. **An illustrative example on how a network loses its controllability due to malicious attacks or random failures**. **a**, A directed weighted network with nine state nodes and three control nodes whose controllability matrix has the full rank. **b**, When control node u_1 breaks down, the rank of $\mathbf{C}$ becomes 3, indicating the damaged network loses its controllability. **c**, When the link (x_6, x_5) breaks, it leads the rank of $\mathbf{C}$ to 6, losing its controllability.

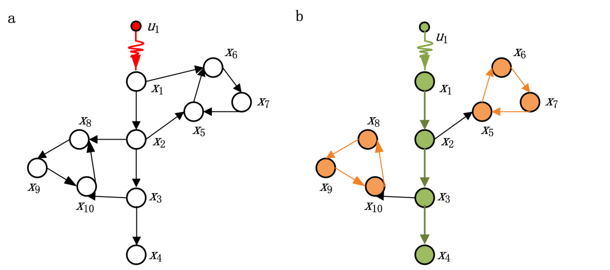

Fig. 2. **Structural controllability of a network**. **a**, A sample network of ten state nodes and one control node. **b**, The network is spanned by cacti which can be decomposed into node-disjoint stems (in green) and cycles (in yellow), indicating it is structurally controllable. For the definitions of cacti, stem and cycle, please refer to (Lin (1974)).

For a directed network $G(\mathbf{A},\mathbf{B})$ of N state nodes and P control nodes, we introduce the concept of degree of controllability to quantify the level of the control of the network, which is defined as the ratio of the number of state nodes of its maximal controllable subnetwork and the total number of the state nodes, N, expressed below,

$$doc = \frac{\text{rank}_g(\mathbf{C})}{N}, \tag{3}$$

It is clear to see doc ranges from 0 to 1. When it equals to 1, the degree of controllability of the network is said to be full. The calculation of doc depends on the calculation of $\text{rank}_g(\mathbf{C})$, which can be mapped into a combinatorial optimization problem over $G(\mathbf{A},\mathbf{B})$ (Hosoe (1980)). According to Hosoe's theorem (Hosoe (1980)), $\text{rank}_g(\mathbf{C})$ is given by

$$\text{rank}_g(\mathbf{C}) = \max_{G_s \in G} |E(G_s)|, \tag{4}$$

where G is the set of all stem-cycle disjoint subnetworks of the accessible part of $G(\mathbf{A},\mathbf{B})$ and $|E(G_s)|$ is the number of the edges in the subnetwork G_s. A state node x_i is called accessible if there exists a path reaching it from one of the control nodes. For example, in Fig. 2a, all state nodes $\{x_1, \ldots, x_{10}\}$ are accessible from the control node u_1. A stem is a directed path starting from a control node and ending in a state node, with no nodes appearing more than once in it, e.g., $u_1 \to x_1 \to x_2 \to x_3 \to x_4$ in Fig. 2a. A stem-cycle disjoint subnetwork G_s consists of stems and cycles only, and the stems and cycles have no node in common, e.g., the subnetwork colored in Fig. 2b. Note that the subnetwork colored in Fig. 2b contains the largest number of edges among all possible stem-cycle disjoint subnetworks, denotes as $G_s^{\max}$. Thus, doc of the network shown in Fig. 2a equals to 1, which means the degree of controllability is full. Note that the advantage of the Equation (4) is that $\text{rank}_g(\mathbf{C})$ can be calculated via linear programming (Poljak (1990)), which gives us an efficient numerical tool to determine the degree of controllability of an arbitrary controlled directed network.

3. PROBLEM FORMULATION OF THE OPTIMAL ADDING-NODE RECOVERING STRATEGY

A fully controlled directed network may lose its controllability due to the potential malicious attacks and/or the random failures (Pu et al. (2012); Liu et al. (2012)), which has been illustrated in Fig. 1. For such damaged networks whose degree of controllability is not full, linking the additional control nodes separately to each of the reasonably identified state nodes or adding links properly between the state nodes can effectively get them recovered (Liu et al. (2011)). Intuitively, the damaged networks can be decomposed into the controllable part and the non-controllable part, and for the non-controllable part, we can identify its minimum number of driver nodes (Liu et al. (2011)) or make the minimum structural perturbation (Wang et al. (2012b)) to transform it into the controllable. But such heuristical recovering approaches can only obtain the suboptimal solutions due to its insufficient ability to tackle the damaged networks as a whole. Here, we are interested in developing the optimal solution framework which adds the minimum number of control nodes or edges into these impaired networks and help them recover.

In this section, we propose an optimal adding-node (OAN for short) recovering strategy. Before we give the procedure of OAN, two definitions are introduced.

Definition 1. (Ding et al. (2013a)) A control scheme of $G(\mathbf{A})$ is defined as a set of the state nodes, which driven by the different input signals imposed from the control nodes can offer the structural controllability for the whole network.

Remark 2. A minimal control scheme is a control scheme from which excluding any state node will shift the network into the structurally uncontrollable, denoted as M, e.g., $\{x_1, x_4, x_5\}$ in Fig. 3a. Note that all the minimal control schemes of a directed network have the same size (Ding et al. (2013a)).

Definition 3. A controlled scheme of $G(\mathbf{A{,}B})$ is defined as a set of the state nodes each of which can find a distinguished driving control node, denoted as F. For example, $\{x_3\}$ or $\{x_4\}$ in Fig. 3a.

Remark 4. $\{x_3, x_4\}$ is not a controlled scheme due to there not existing two different control nodes which are connected to that two state nodes separately.

In general, OAN includes two steps, proceeding as follows,

step 1 Identify a minimal control scheme and a controlled scheme for a given directed network, the pair having the biggest intersection, denoted as M_{big} and F_{big} respectively (Fig. 3a and Fig. 3b).

step 2 For each of the state nodes in $M_{big} - F_{big}$, link an additional control node to it.

Here, we give the proof of the optimality of ONA strategy.

Theorem 5. (optimality of OAN strategy)
$num_{node} = |M_{big} - F_{big}|$, num_{node} denoting the minimum number of the control nodes added for recovering a damaged network's controllability.

Proof. By the definition of the minimal control scheme, to make a directed network structurally controllable, at least we need to link $|M|$ control nodes, each separately to a distinguished state node, these state nodes all appearing in a minimal control scheme. By the definition 3, we know each controlled scheme of a damaged network includes the state nodes each of which can find a distinct control node. Obviously, for recovering the network's controllability, at a minimal, we need to link the additional control nodes to the state nodes appearing in a minimal control scheme, not appearing in a controlled scheme, one on one. Therefore, the minimum number of the control nodes added for recovering a damaged network equals to the size of the difference between a minimal control scheme and a controlled scheme of this network, the pair of which has the biggest intersection.

Fig. 3 shows an illustrative example on this strategy.

A minimal control scheme, M, of $G(\mathbf{A})$ is the set of the unmatched nodes of one of its maximum matchings (Ding et al. (2013a)). The unmatched nodes of a maximum matching of a directed network can be figured out efficiently using its bipartite representation (Hopcroft and Karp (1973)), shown in Fig. 4b. A controlled scheme, F, of $G(\mathbf{A{,}B})$ is formed through the following steps,

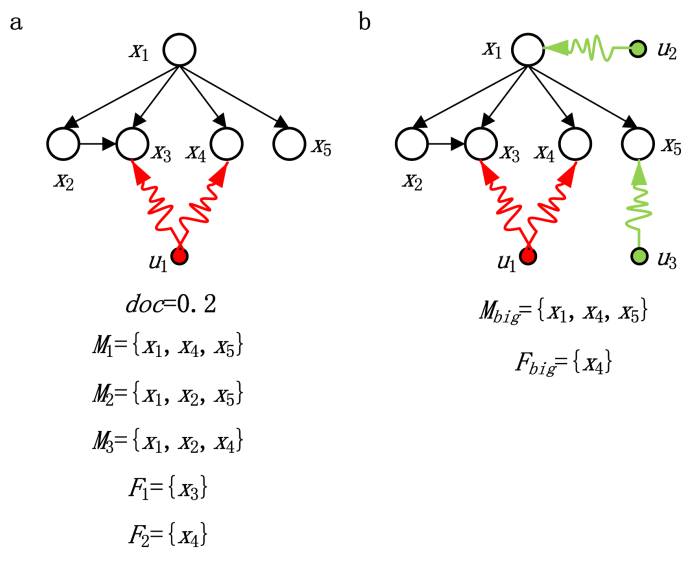

Fig. 3. **An illustrative example of OAN strategy. a**, A sample network of five state nodes and one control node whose degree of controllability is not full. Below the network list its all minimal control schemes and controlled schemes. **b**, F_2 and M_1 have the biggest intersection. So the minimum number of control nodes for recovering this network is 2. Newly added u_2 and u_3 separately connect to the state nodes x_1 and x_5. Note that in general, M_{big} is not unique, e.g., M_3 is also a minimal control scheme which has the biggest intersection with F_2.

step 1 For each control node u_i in $G(\mathbf{A{,}B})$, $i = 1, 2, ..., P$, denote U_i=$\{x_j | b_{ji} \neq 0, j = 1, 2, ..., N\}$ as the set of the controlled state nodes of u_i.

step 2 Draw one node from each U, without duplication, and form a controlled scheme F.

We define q_{ij} as the binary decision variables to indicate whether or not state node x_j is drawn from the set U_i, $i = 1, 2, ..., P, j = 1, 2, ..., N$. At most one state node can be drawn from the set U_i, and the same state node can only be drawn from one set U among the potential multiple sets which include it. These two constraints guarantee a configuration of q_{ij} represents a valid controlled scheme. Note that if x_j doesn't appear in U_i, then the value of q_{ij} always equals to 0. We also define m_{ij} as the binary decision variables to indicate whether or not the *out* node x_i (Fig. 4) can reach or match the *in* node x_j, $i = 1, 2, ..., N, j = 1, 2, ..., N$. One *out* node can match at most one *in* node among its reachable *in* nodes and the same *in* node can only be matched by one *out* node. These two constraints help guarantee a configuration of m_{ij} represents a valid minimal control scheme. Therefore, OAN strategy can be mapped into a constrained combinatorial optimization problem with the binary decision variables. The objective of OAN strategy is to minimize the number of the control nodes added for recovering a damaged network's controllability, or to maximize the size of the intersection between a minimal control scheme and a controlled scheme of the network. The formulation of OAN strategy is given below,

$$\textbf{maximize} \sum_{j=1}^{N}((1-\sum_{i=1}^{N} m_{ij}) * \sum_{i=1}^{P} q_{ij}) \quad (5)$$

s.t.

$$\sum_{i=1}^{N} m_{ij} \leq 1, j = 1,2,...,N \quad (6)$$

$$\sum_{j=1}^{N} m_{ij} \leq 1, i = 1,2,...,N \quad (7)$$

$$\sum_{j=1}^{N}(1-\sum_{i=1}^{N} m_{ij}) = |M|, \quad (8)$$

$$\sum_{i=1}^{P} q_{ij} \leq 1, j = 1,2,...,N \quad (9)$$

$$\sum_{j=1}^{N} q_{ij} \leq 1, i = 1,2,...,P \quad (10)$$

$$m_{ij} \leq s_{ij}, i = 1,2,...,N, j = 1,2,...,N \quad (11)$$

$$q_{ij} \leq c_{ij}, i = 1,2,...,P, j = 1,2,...,N \quad (12)$$

$$m_{ij} \in \{0,1\}, i = 1,2,...,N, j = 1,2,...,N \quad (13)$$

$$q_{ij} \in \{0,1\}, i = 1,2,...,P, j = 1,2,...,N \quad (14)$$

where constraints (6), (7) and (8) guarantee the obtained configuration of m_{ij} is a feasible minimal control scheme; constraints (9) and (10) guarantee the obtained configuration of q_{ij} is a feasible controlled scheme; $|M|$ is the size of any minimal control scheme of $G(\mathbf{A,B})$; s_{ij} is a Boolean constant. $s_{ij} = 1$ indicates there exists a link from state node x_i to x_j in $G(\mathbf{A,B})$, otherwise not; c_{ij} is also a Boolean constant. $c_{ij} = 1$ indicates control node u_i connects to the state node x_j, otherwise not. In section 5, the canonical branch-and-bound technique is employed to provide the optimal solution for this constrained combinatorial optimization problem with the binary variables.

4. PROBLEM FORMULATION OF THE OPTIMAL ADDING-EDGE RECOVERING STRATEGY

OAN strategy recovers a damaged network's controllability via linking additional control nodes separately to each of the properly identified state nodes in the network. While sometimes, in reality, due to some kind of physical restraints, such as the limited installation space, adding control nodes to a network is not that convenient compared to adding links between the state nodes. Taking this into account, in this section, we propose an alternative for the OAN strategy--optimal adding-edge (OAE for short) recovering strategy, which aims to add the minimum number of edges between the state nodes for recovering a damaged network's controllability.

OAE strategy proceeds as follows,

step 1 Add virtual links from state node x_j to x_i if $a_{ij} = 0$ into $G(\mathbf{A,B})$, $i = 1,2,...,N, j = 1,2,...,N$. For example (x_2, x_4) in Fig. 5b. Denote the set of these links as VSL.

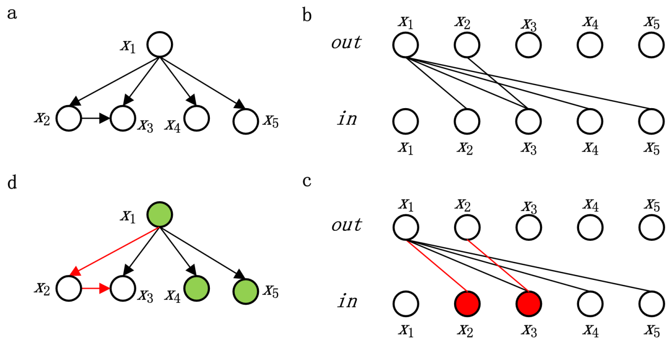

Fig. 4. **Identifying the unmatched nodes of a maximum matching using bipartite representation of a directed network**. **a**, A sample directed network with five state nodes. **b**, The bipartite representation of the directed network in **a**, where nodes are represented as two disjoint sets, *out* and *in*. A directed link from x_1 to x_3 in **a** corresponds to a link from x_1 of *out* set to x_3 of *in* set. **c**, One maximum matching (in red) in bipartite representation where one node can maximumly match another node through one link, leaves node x_2 and x_3 matched (in red). **d**, A maximum matching (in red) in the directed network corresponds to that in **c**. Hence, we get three unmatched nodes (in green), x_1, x_4 and x_5

step 2 Add virtual links from the state nodes to the control nodes into $G(\mathbf{A,B})$, e.g., (x_4, u_1) in Fig. 5. Denote the set of these links as VCL. The newly formed network is denoted as $G'(\mathbf{A,B})$.

step 3 Denote the original edge set as OL. Assign weight $w = 1$ to each edge in VSL and $w = 0$ to each edge in VCL and OL.

step 4 Identify a collection of the node-disjoint cycles which spans $G'(\mathbf{A,B})$ and has the minimum edge weight. For example, $\{(u_1 \to x_1 \to x_3 \to x_4 \to x_2 \to u_1)\}$ in Fig. 5c with the total edge weight equal to 1.

step 5 Turn the virtual edges belonging to VSL of this collection to the real. And these edges are the minimum edges required for recovering the controllability of $G(\mathbf{A,B})$.

Fig. 5 gives an illustrative example on OAE strategy.

Theorem 6. (optimality of OAE strategy)
After performing **step 1** and **step 2**, the edges belonging to VSL of an identified collection of the node-disjoint cycles, which spans $G'(\mathbf{A,B})$ and has the minimum edge weight, are the minimum edges required for recovering the controllability of $G(\mathbf{A,B})$.

Proof. We may assume without loss of generality that all the state nodes in $G(\mathbf{A,B})$ are accessible since it can be fixed by adding links from one control node directly to those state nodes which are not accessible.

After performing **step 1** and **step 2**, it is clear to see in $G'(\mathbf{A,B})$, there are two types of cycles in the network. One is all-state-nodes cycle, the other is control-state-mix cycle. A control-state-mix cycle is formed by adding a link from the top of a stem to the root of the stem. So a collection of the node-disjoint all-state-node and control-state-mix cycles can be viewed as a collection of the node-disjoint stems and elementary cycles by omitting the edges belonging to VCL, which is factorized from a cacti in $G'(\mathbf{A},\mathbf{B})$ (Lin (1974)). When a collection of the node-disjoint cycles spans $G'(\mathbf{A},\mathbf{B})$, e.g., $\{(u_1 \to x_1 \to x_2 \to x_3 \to x_4 \to u_1)\}$, which means the network is covered by a cacti, $G'(\mathbf{A},\mathbf{B})$ is structurally controllable (Lin (1974)). The edges belonging to VSL in this collection are the required edges for recovering the controllability of $G(\mathbf{A},\mathbf{B})$. Due to the weights of the edges belonging to VSL are set to 1 and the weights of the edges belonging to VCL and OL are set to 0, a collection of the node-disjoint cycles which has the minimum edge weight contains the minimum number of edges belonging to VSL which are used to recover the controllability of $G(\mathbf{A},\mathbf{B})$. Thus, the conclusion follows.

We define h_i as the binary decision variables to indicate whether or not the edge e_i from $G'(\mathbf{A},\mathbf{B})$ is picked up to form a collection of the node-disjoint cycles, $i = 1, 2, ..., E$, where E denotes the total number of edges of $G'(\mathbf{A},\mathbf{B})$. Therefore, OAE strategy can be mapped into a constrained binary integer programming problem, whose objective is to find a collection of the node-disjoint cycles which has the minimum number of edges belonging to VSL and spans $G'(\mathbf{A},\mathbf{B})$. The formulation of OAE strategy is given below,

$$\mathbf{minimize}\, num_{edge} = \sum_{i=1}^{E}(w_i h_i) \quad (15)$$

s.t.

$$\sum_{e_i \text{ leaves } v}(h_i) = 1, \text{ for every node v in } G'(\mathbf{A},\mathbf{B}) \quad (16)$$

$$\sum_{e_i \text{ enters } v}(h_i) = 1, \text{ for every node v in } G'(\mathbf{A},\mathbf{B}) \quad (17)$$

$$h_i \in \{0, 1\}, i = 1, 2, ..., E \quad (18)$$

where num_{edge} denotes the minimum number of the edges between state nodes added for recovering $G(\mathbf{A},\mathbf{B})$'s controllability; w_i is the value of the weight assigned to e_i. w_i equals to 1 if $e_i \in VSL$, otherwise 0; constraints (16) and (17) guarantee the picked edges can form a collection of the node-disjoint cycles which spans $G(\mathbf{A},\mathbf{B})$. In section 5, the canonical branch-and-bound technique is employed to provide the optimal solution for this constrained binary integer programming problem.

5. EXPERIMENTAL RESULTS AND DISCUSSIONS

In this section, we first examine the degree of controllability on four real and model networks, including Foodweb (Martinez (1991)), two SF networks both of $N = 100$ and $< k >= 2$, and the network parameter γ set to be 2.5 and 3, respectively (Goh et al. (2001)), and an ER network of $N = 100$ and $< k >= 2$ (Wang and Chen (2003)). Then we add $0.05 * N$ control nodes into each of these four networks, intentionally to emulate the four corresponding controllability-damaged networks. We call them the damaged Foodweb, SF and ER networks. Each added control node is randomly connected to the state nodes. The effectiveness and efficiency of OAN and OAE recovering strategies are subsequently examined on these damaged networks by making a comparison to their counterparts, RAN and RAE strategies, on the recovering speed. The three experiments are implemented using Matlab 2008b and executed on an Intel 2.1Ghz computer.

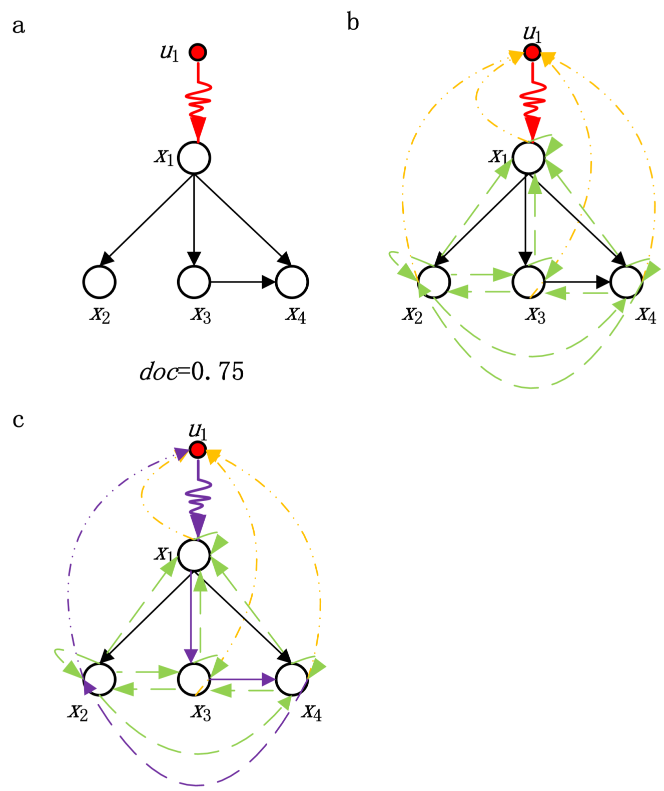

Fig. 5. **An illustrative example of OAE strategy.** **a**, A sample network of four state nodes and one control node whose degree of controllability is not full. **b**, The formed network after performing **step 1** and **step 2**. Edges in green belong to VSL. Edges in yellow belong to VCL. **c**, A collection of the node-disjoint cycles $\{(u_1 \to x_1 \to x_3 \to x_4 \to x_2 \to u_1)\}$ (in purple) has the minimum edge weight equal to 1, which means only one edge (x_4, x_2) is required to recover the controllability of the network in **a**.

Fig. 6 shows the changing curve of *doc* by constantly adding control nodes to the four real and model networks. The added control nodes are randomly linked to the state nodes. From the figure, it is clear to see that at first *doc* goes up dramatically with the increasing of the number of the added control nodes. When the number reaches a critical proportion of the total number of the state nodes, e.g., 0.8 in Foodweb, *doc* turns out to be full, which means the whole network at this point turns structurally controllable. With more control nodes in, *doc* remains full. This kind of changing curve coincides with our intuitive.

Fig. 7 depicts the recovering performance of both OAN strategy and RAN strategy on the four damaged networks by plotting the *doc* as the function of $p_c = \frac{n_c}{N}$, where n_c is the number of added control nodes and N is the total number of the state nodes. In OAN strategy, the linking position of the added control nodes is calculated by Equation (5)-(14), whereas in RAN, the position is randomized. By comparing both strategies' speed to recover the damaged networks, it is clear to see that OAN strategy is much faster to get the controllability-damaged networks recovered than RAN strategy, e.g., in Foodweb,

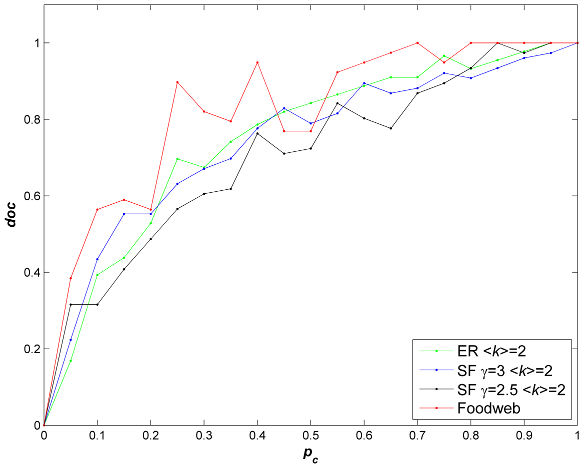

Fig. 6. **The changing curve of** *doc* **by constantly adding the control nodes into the networks**. We plot *doc* as the function of $p_c = \frac{n_c}{N}$, where n_c is the number of added control nodes and N is the total number of the state nodes.

when n_c reaches about 32% of N on OAN strategy, the damaged network gets repaired, whereas on RAN strategy, for achieving the goal, n_c needs to increase to more than 80% of N.

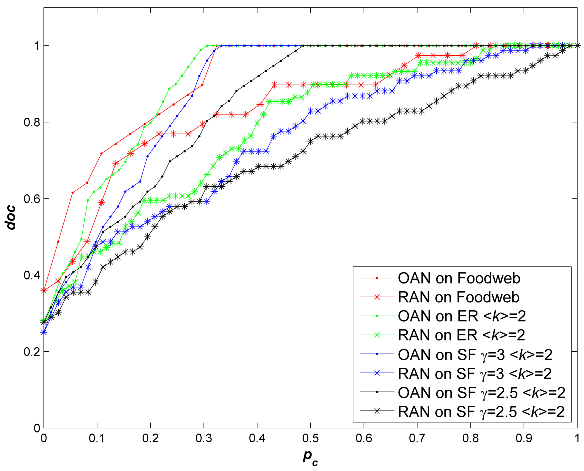

Fig. 7. **Recovering performance of OAN strategy and RAN strategy**.

Fig. 8 illustrates the recovering performance of both OAE strategy and RAE strategy on the four damaged networks by plotting the *doc* as the function of $p_e = \frac{n_e}{N*N}$, where n_e is the number of added edges and $N*N$ is the total number of the possible edges between the state nodes. In OAE strategy, the position of the added edge is calculated by Equation (15)-(18), whereas in RAE, the position of the added edge is randomized. Obviously, in terms of the recovering speed, OAE strategy is more efficient to get the controllability-damaged networks recovered than RAE strategy, e.g., in Foodweb, when n_e reaches less than 1% of the total edge number on OAE strategy, the damaged network gets repaired, whereas on RAE strategy, to achieve the goal, n_c needs to be around 3.7% of the total edge number.

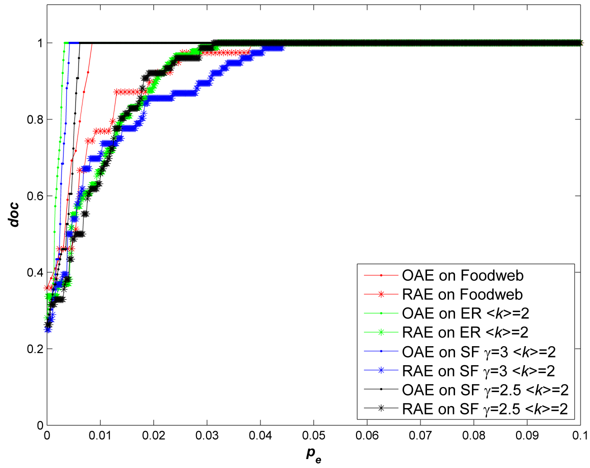

Fig. 8. **Recovering performance of OAE strategy and RAE strategy**.

6. CONCLUDING REMARKS

The complex networked systems are prone to lose the controllability due to the malicious attacks or random failures. In this paper, we first introduce the concept of degree of controllability for any arbitrary directed network, and if its degree of controllability is not full, then we propose two optimal recovering strategies, OAN and OAE, which add the minimum number of control nodes and edges, respectively, to help recover the damaged network's controllability. The simulation results on various real and model networks show the effectiveness of the two strategies and the superior performance compared to their randomized counterparts, RAN and RAE strategies. The research results presented here are of great significance from both theoretical and practical perspectives, e.g., it can be applied to repairing the power grid which is one of the most complex networked dynamical systems in the real world, and very vulnerable to the surrounding environment. In the future work, we will investigate the relationship among a network's underlying topology, its degree of controllability and the efforts needed to recover its damaged controllability. Also, we will endeavor to devise the heuristic approaches for OAN and OAE strategies to provide computationally efficient solutions, especially on the networks of hundreds of thousands of nodes and edges, where the traditional branch-and-bound technique is computationally prohibitive.

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