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Nonasymptotic Mean-Field Games

Hamidou Tembine *

* *KAUST Strategic Research Initiative Center for Uncertainty Quantification in Computational Science and Engineering, CEMSE, KAUST*
(e-mail: tembine@ieee.org).

Abstract: Mean-field games have been studied under the assumption of very large number of players. For such large systems, the basic idea consists to approximate large games by a stylized game model with a continuum of players. The approach has been shown to be useful in some applications. However, the stylized game model with continuum of decision-makers is rarely observed in practice and the approximation proposed in the asymptotic regime is meaningless for networked systems with few entities. In this paper we propose a mean-field framework that is suitable not only for large systems but also for a small world with few number of entities. The applicability of the proposed framework is illustrated through a dynamic auction with asymmetric valuation distributions.

Keywords: non-asymptotic, mean-field, game theory

1. INTRODUCTION

Recently there have been renewed interests in large-scale interactions in several research disciplines, with its uses in wireless networks, financial markets, biology, power grid, social and cloud networks. In mathematical physics, most of models are analyzed in the asymptotic regime when the size of the system grows without bounds. As an example, the McKean-Vlasov model [9, 10, 12] for interacting particles is analyzed when the number of particles tends to infinity. Such an approach is referred to as mean field approach. The seminal works of Sznitman [15] in the 1980s and the recent work of Kotolenez & Kurtz [14] show that the asymptotic system provides a good approximation of the finite system in the following sense: For any $\epsilon > 0$ there exists a population size n_ϵ such that for any $n \geq n_\epsilon$, the error gap between the solution of the infinite system and the system with size n is at most ϵ. Moreover, the work in [14] shows that the number n_ϵ is in order of $O\left(\log(\frac{1}{\epsilon})\frac{1}{\epsilon^{d+2}}\right)$ for a class of smooth functions, where d denotes the dimension of the space. Thus, for $n < n_\epsilon$ this current theory does not give an approximation that is meaningful.

In queueing theory, the number of customers is usually assumed to be large or follows a certain distribution with unbounded support (e.g., exponential, Poisson etc) and the buffer size (queue) can be infinite. However, many applications of interests such as airport boarding queues, supermarket queues, restaurant queue, iphone/ipad waiting queue involve a finite number of customers/travelers. Approximation by a continuum of decision-makers may not reflect the reality. For example the number of clients in the supermarket queue cannot exceed the size of available capacity of markets and there is a certain distance between the clients to be respected. In other words, human behaviors are not necessarily like standard fluid dynamics. In game theory, the rapidly emerging field of mean-field games [4] is addressing behavioral and algorithmic issues [1] for mathematical models with continuum of players. We refer the reader to [5] for a survey on (asymptotic) mean field games.

The classical works mentioned above provide rich mathematical foundations and equilibrium concepts in the asymptotic regime, but relatively little in the way of computational and representational insights that would allow for few number of players.

Our primarily goal in this article is to provide a simple and easy to check condition such that mean-field theory can be used for finite-scale which we call *non-asymptotic mean-field* approach. We investigate two basic conditions in the present paper. The first condition is indistinguishability (or interchangeability) of the payoff functions. The indistinguishability property is easy to verify. The indistinguishability assumption is implicitly used in the classical (static) mean-field analysis including the seminal works of Aumann 1964 [11], Selten 1970 [13], Schmeidler 1973. This assumption is also implicitly used in the dynamic version of mean-field games by Jovanovic & Rosenthal 1988[2], Benamou & Brenier 2000 [3] and Lasry & Lions 2007 [4]. The second condition is the (regularity) smoothness of the payoff functions. The regularity property is relatively easy to check.

Based on these two conditions, we present a simple approximation framework for finite horizon mean-field systems. The framework can be easily extended to infinite horizon case. The non-asymptotic mean field approach is based on a simple observation that the many effects of different actions cancel out when the payoff is indistinguishable. Nevertheless, it can lead to a significant simplification of mathematical mean-field models in finite regime. The approach presented here is non-asymptotic and is unrelated to the mean-field convergence that originates from law of large numbers (and its generalization to de Finetti-Hewitt-

Savage functional mean-field convergence) in large populations. The non-asymptotic mean field approach holds even when there are only few players in a game, or few nodes in a network.

The idea presented here is inspired from the works in [17, 18, 16] on the so-called averaging principle. These previous works are limited to static and one-shot games. Here we use that idea not only for static games but also for dynamic mean-field games. One of the motivations of the asymptotic mean field game approach is that it may reduce the complexity analysis of large systems. The present work here goes beyond that. We believe that if the complexity of the infinite system can be reduced easily then, the finite system can also be studied using a non-asymptotic mean-field approach.

In order to apply the mean-field approach to a system with arbitrary number of players, we shall exploit more the structure of objective function and the main assumption of the model which is the indistinguishability property i.e., the performance index is unchanged if one permutes the label of the players. This is what we will do in this work. The aggregative structure of the problem and the indistinguishability property of the players are used to derive an error bound for any number of players. Interestingly, our result holds not only for large number of players but also for few number of players. For example, for $n = 5$ players, there is no systematic way to apply the theory developed in the previous works [4, 11] but the non-asymptotic mean-field result presented here could be applied. The non-asymptotic mean-field result does not impose additional assumptions on the payoff function. We show that the indistinguishability property provides an accurate error bound for any system size. We show that the total equilibrium payoff with heterogeneous parameters can be approximated by the symmetric payoff where the symmetric is the respect to the mean of those parameters. These parameters can be a real number, vector, matrix or a infinite functional. The proof of the approximation error is essentially based on a Taylor expansion which cancels out the first order terms due to indistinguishability property.

We provide a basic auction example where the non-asymptotic mean-field interaction is required and the indistinguishability property could be exploited more efficiently. In models of first-price auctions, when bidders are ex ante heterogeneous, deriving explicit equilibrium bid functions is an open issue. Due to the boundary-value problem nature of the equilibrium, numerical methods remain challenging issue. Recent theoretical research concerning asymmetric auctions have determined some qualitative properties these bid functions must satisfy when certain conditions are met. Here we propose an accurate approximation based on non-asymptotic mean field game approach and examine the relative expected payoffs of bidders and the seller revenue (which is indistinguishable) to decide whether the approximate solutions are consistent with theory.

The remainder of the paper is structured as follows. In Section 2 we present a mean field system with arbitrary number of interacting entities and propose a nonasymptotic static mean field framework. In Section 3 we extend our basic results in a dynamic setup. In Section 4 we present computation of error bound of equilibrium bids in dynamic auction with asymmetric bidders.

We summarize some of the notations in Table 1.

Table 1. Summary of Notations

Symbol	Meaning
$\mathcal{N}$	set of potential minor players
n	cardinality of $\mathcal{N}$
$\mathcal{A}$	action space
a_j	action of player j
$r_g(a_1, \ldots, a_n)$	global payoff function of the major player
$\mathbb{1}_{\{.\}}$	indicator function.
$\bar{m}\bigotimes^n$	$= (\bar{m}, \ldots, \bar{m})$
τ_j	strategy of player j
$R_{j,T}$	long-term payoff of player j

2. MEAN-FIELD SYSTEM FOR ARBITRARY NUMBER OF ENTITIES

Consider an interactive system with $n + 1 \geq 2$ entities (players) consisting of n generic minor players and one major player (called designer). Each minor player has a payoff function. The major player has its own payoff function that could be the global performance of the minor players or another generic payoff. The payoff function of the major player is captured by a certain function $r_g(a_1, \ldots, a_n) \in \mathbb{R}$ which we call global payoff function. Each decision variable a_j belongs to a Polish [1] space $\mathcal{A}$.

2.1 Main Assumptions on the structure of payoff function

Assumption A0: Indistinguishability. We assume that the global payoff function is invariant by permuting the index of the players, i.e.,

$$r_g(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = r_g(a_{\pi(1)}, \ldots, a_{\pi(i)}, \ldots, a_{\pi(j)}, \ldots, a_{\pi(n)}),$$

for every permutation $\pi : \mathcal{N} \longrightarrow \mathcal{N}$, where $\mathcal{N} := \{1, 2, \ldots, n\}$.

To verify $A0$, it suffices to check for pairwise interchangeability, i.e., permutation of any two of the coordinates. In mathematics, the indistinguishability property is sometimes referred as symmetric function, i.e., one whose value at any n-tuple of arguments is the same as its value at any permutation of that $n-$tuple.

Assumption A1: Smoothness. We assume that the objective function r_g is (locally) twice differentiable with the respect to the variables.

It is important to notice that the assumption A0 can be easily checked by designers, engineers and non-specialists. In practice, A0 will result in functions that can be expressed in terms of the mean $\bar{m} = \frac{1}{n}\sum_{j=1}^n a_j,\ \frac{1}{n}\sum_{j=1}^n a_j^2,$ $\frac{1}{n}\sum_{j=1}^n \phi(a_j, \bar{m}) \ldots,\ \left(\prod_{j=1}^n a_j\right)$, etc. Assumption A0 is implicitly used in [11, 13, 2, 4].

[1] A Polish space E is a Separable topological space E for which there exists a compatible metric d such that (E, d) is a complete metric space. Here, "'separable"' means has a countable dense subset.

Our goal is to provide a useful approximation and error bound for the global payoff r_g in an equilibrium or in function of the parameters of the game.

2.2 Approximation for static games

Next we provide the basic results that hold for both non-asymptotic and asymptotic static systems.

Theorem 1. Assume that $A0$ and $A1$ hold. Then the following results hold:

- $\partial_{a_j} r_g(\bar{m}^{\bigotimes n}) = \partial_{a_1} r_g(\bar{m}^{\bigotimes n})$ where $\bar{m}^{\bigotimes n} := (\bar{m}, \ldots, \bar{m})$ and
$$\bar{m} = \frac{1}{n}\sum_{j=1}^{n} a_j = \int_{b\in\mathcal{A}} b \left[\frac{1}{n}\sum_{j=1}^{n}\delta_{a_j}\right](db), \quad (1)$$
δ_{a_j} is the Dirac measure concentrated at the point a_j,
$$m = \frac{1}{n}\sum_{j=1}^{n}\delta_{a_j} \quad (2)$$
- The structure of the payoff function implies that the first order term in the Taylor expansion is cancelled out.
- The cross-derivatives are independent of the labels: $\partial^2_{a_i a_j} r_g(\bar{m}^{\bigotimes n}) = \partial^2_{a_1 a_2} r_g(\bar{m}^{\bigotimes n})$

Note that this theorem can be used for games with continuous action space as well as for games with discrete action space via mixed extensions. Examples of games that satisfy A0-A1 includes Prisoner Dilemma, Battle of Sex, Hawk-Dove, coordination games, anti-coordination games, minority games, matching pennies, etc.

Theorem 2. Suppose that the payoff function r_g satisfies the assumptions $A0$ and $A1$. Assume that a is in a small neighborhood of the mean vector $\bar{m}^{\bigotimes n} := (\bar{m}, \ldots, \bar{m})$ i.e., there is a small positive number $c_{\bar{m}, r_g}$ which may depend on $\bar{m}$ and the function r_g such that $\| (a_1, \ldots, a_n) - \bar{m}^{\bigotimes n} \| \leq c_{\bar{m}, r_g}$ then $r_g(a) - \bar{r}(\bar{m}) = O\left(\| a - \bar{m}^{\bigotimes n} \|_2^2\right)$, and
$$\| r_g(a) - \bar{r}(\bar{m}) \| \leq \delta c^2_{\bar{m}, r_g}$$
where $\bar{r}(\bar{m}) := r_g(\bar{m}, \ldots, \bar{m}) = r_g(\bar{m}^{\bigotimes n}), \delta > 0$.

The proof of result 2 follows from the following result 3 which gives the explicit error bound:

Theorem 3. Assume that $A0 - A1$ hold. Then, the explicit error bound for arbitrary number of players is $r_g(a) - \bar{r}(\bar{m})$ is in order of
$$\delta_{\bar{m}, \bar{r}} \sum_{j=1}^{n} (a_j - \bar{m})^2,$$
where
$$\delta_{\bar{m}, \bar{r}} = | \frac{n}{2(n-1)} \left(-\frac{1}{n^2}\bar{r}''(\bar{m}) + \partial^2_{a_1 a_1} r_g(\bar{m}^{\bigotimes n}) \right) |$$

Remark 4. In order to compute the error bound, one needs only $\bar{r}$, and $\partial^2_{a_1 a_1} r_g(\bar{m}^{\bigotimes n})$. The expression of the function $r_g(a)$ is not required for vector with non-symmetric components. This allows us to provide an approximation result for unknown payoff function.

Remark 5. If $\bar{r}''$ is bounded by β and $\frac{\sum_{j=1}^{n}(a_j - \bar{m})^2}{n} \leq \sigma$ then
$$r_g(a) - \bar{r}(\bar{m}) \leq \frac{\beta}{2(n-1)} \frac{\sum_{j=1}^{n}(a_j - \bar{m})^2}{n} \quad (3)$$
$$\leq \frac{\beta\sigma}{2(n-1)} \leq \frac{\beta\epsilon^2}{2(n-1)}. \quad (4)$$

In particular, if the finite regime has a solution in a certain sense, that is $\epsilon-$close to a vector with symmetric component then, non-asymptotic mean field approach provides automatically an $O(\frac{\epsilon^2}{2(n-1)})-$solution for any number of players $n \geq 2$. This is a non-asymptotic result in the sense that it holds for all range of system size $n \geq 2$. Also, by choosing $\epsilon = \frac{1}{n^\alpha}, \ \alpha \geq 1$ one gets an error bound in order of $\frac{1}{2n^{2\alpha+1}}$. Note that ϵ can be very small even if n is not large. For example, with $n = 2$ players and $\alpha = 10$, one gets an error bound in order of $\frac{1}{2^{22}}$ which is satisfactory in terms of computational accuracy.

3. DYNAMIC SETUP

In this section we provide very useful approximation results for dynamic interactive systems. We consider a finite horizon with length $T \geq 1$.

3.1 Non-asymptotic mean-field optimization

Consider a designer who controls the action to be dictated to n minor entities. Assume that the designer aims to achieve a certain goal with objective function given by the choice variables that the designer dictates to the minor entities. The objective function is $R_{g,T}(\tau) = \sum_{t=0}^{T-1} r_g(a_t)$ where $r_g : \mathcal{A}^n \to \mathcal{A}^n$ satisfies assumptions $A0 - 1$ and $a_t = (a_{1,t}, \ldots, a_{n,t}) \in \mathcal{A}^n$ is the choice variable at time t. Let $\bar{m}_t = \frac{1}{n}\sum_{j=1}^{n} a_{j,t}$ be the sequence of mean actions and set $\bar{r}_T(\bar{m}) = \sum_{t=0}^{T-1} \bar{r}(\bar{m}_t)$.

Theorem 6. An explicit error bound for $R_{g,T}(\tau) - \bar{r}_T(\bar{m})$ with arbitrary number of minor entities is given by
$$\delta_{T, \bar{m}, \bar{r}} \| a - \bar{m} \|^2_{l^2_T},$$
where
$$\delta_{T, \bar{m}, \bar{r}} = \sup_{t\in\mathcal{T}} | \frac{n}{2(n-1)} \left(-\frac{1}{n^2}\bar{r}''(\bar{m}_t) + \partial^2_{a_1 a_1} r_g(\bar{m}_t^{\bigotimes n}) \right) |$$
and $l^2_T = \{(x_t)_{t \leq T-1} \mid \sum_{t=0}^{T-1} |x_t|^2 < +\infty\}$, and $\|a - \bar{m}\|_{l^2_T} = \sum_{t=0}^{T-1} |a_t - \bar{m}_t|^2$

See Result 7 below for a proof.

3.2 Non-asymptotic mean-field stochastic games

Consider a stochastic game [22] with n minor players and one major player (designer). Time is discrete. Time space is $\mathcal{T} = \{0, 1, \ldots, T-1\}$ where $T \geq 1$. Each player j has its individual state $s_{j,t}, \ t \in \mathcal{T}$ which evolves according to a Markovian process. The action space a player depends on its current space, $\mathcal{A}(s_{j,t}) \subset \mathbb{R}$. A pure strategy of player j at time t is a mapping from history $\mathcal{H}_{j,t}$ up to t to the current action space $\mathcal{A}(s_{j,t})$. Denote by $\tau_j = (\tau_{j,t})_t$ the strategy of player j. For $h_{j,t} \in \mathcal{H}_{j,t}, \ \tau_{j,t}(h_{j,t}) = a_{j,t} \in \mathcal{A}(s_{j,t})$. The instantaneous payoff function of player j is

$r(s_{j,t}, a_{j,t}, m_t)$ where m_t is the state-action distribution, which satisfies the indistinguishability property with the respect to the other players. We assume that the payoff function r is smooth.

The long-term payoff of player j is $R_{j,T}(s_0, \tau)$

$$= \mathbb{E}\left[\bar{g}(s_T) + \sum_{t\in\mathcal{T}} r(s_{j,t}, a_{j,t}, m_t) \mid (s_{j,0},\ \tau_j)_{j\in\mathcal{N}}\right]$$

where $\bar{g}$ is the terminal payoff. A strategy profile τ^* is a (Nash) equilibrium if no player can improve her payoff by unilateral deviation, i.e., for every player j,

$$R_{j,T}(s_0, \tau^*) = \max_{\tau_j} R_{j,T}(s_0, \tau_j, \tau^*_{-j}).$$

Let $V_{j,T}(s_0, \tau_{-j})$ be the value function of the bidder j , i.e., it is the supremum, over all possible bidding strategies, of the expectation of the payoff $R_{j,T}$ starting from an initial state s_0 when the other players strategy profile is τ_{-j}.

$$V_{j,T}(s_0, \tau_{-j}) = \max_{\tau_j}[R_{j,T}(s_0, \tau) \mid \tau_{-j}, s_0].$$

Taking the expectation over the other players state, the recursive Bellman-Kolmogorov equation is given by

$$V_{j,t}(s_j) = \sup_{a_j}\left[r(s_j, a_j, m_t) + \mathbb{E}_{s'_j} V_{j,t+1}(s'_j \mid s_j, a_j, \tau)\right],$$
$$m_{t+1} \sim \mathbb{P}(.\mid m_t, \tau_t)$$
$$s_{j,t+1} \sim q(.|s_{j,t}, a_{j,t}, m_t, \tau_t) = \mathbb{P}(.\mid s_{j,t}, a_{j,t}, m_t, \tau_t),$$

where $q, \mathbb{P}$ define the transition probabilities between states.

Theorem 7. Let $a_{j,t}(h_{j,t}) = \bar{m}_t(h_{j,t}) + \epsilon\gamma_{j,t}(h_{j,t})$, where

$$\gamma_{j,t}(h_{j,t}) = \frac{a_{j,t}(h_{j,t}) - \bar{m}_t(h_{j,t})}{\epsilon},$$
$$\bar{m}_t(h_{j,t}) = \frac{1}{n}\sum_{j'=1}^{n} a_{j',t}(h_{j',t}),$$

and

$$\epsilon = \max_j \sup_{t\in\mathcal{T}} \sup_{h_{j,t}\in\mathcal{H}_{j,t}} |a_{j,t}(h_{j,t}) - \bar{m}_t(h_{j,t})|,$$

Assume the state transition q is continuous. Then the total term payoff $R_g(s_0; \tau_1, \ldots, \tau_n)$ is in order of $R_g(s_0; \bar{m}, \ldots, \bar{m}) + O(\epsilon^2)$ for any $n \geq 2$.

Using similar lines as in result 3, one gets that any time $t \in \mathcal{T}$, $|r_j(s_t, a_t) - r_j(s_t, \bar{m}_t)|$ is bounded by $\delta_{t,\bar{m}_t,\bar{r}}(s_t)\|a_t - \bar{m}_t\|^2$. Now, a small changes in the action may change the state, and hence the term $\delta_{t,\bar{m}_t,\bar{r}}$ is changed. Using the continuity of the state transition q, we take a uniform bound by considering the supremum: $\bar{\delta}_{t,\bar{m}_t,\bar{r}} = \sup_s \delta_{t,\bar{m}_t,\bar{r}}(s)$. Then, the global error is bounded by $\epsilon^2 \left(\sup_{t\in\mathcal{T}} \bar{\delta}_{t,\bar{m}_t,\bar{r}}\right)$.

4. AUCTION WITH ASYMMETRIC BIDDERS

Static setup The theory of auctions as games of incomplete information originated in 1961 in the work of Vickrey. A seller has an object to sell. She adopted a first-price auction rule. Consider a first-price auction with asymmetric bidders. There are $n \geq 2$ bidders for the object. Each bidder independently submit a single bid without seeing the others' bids. If there is only one bidder with the highest bid, the object is sold to the bidder with biggest bid. The winner pays her bid, that is, the price is the highest (or first price bid). If there is more than one bidder, the object goes to each of these bidders with equal probability. The bidder v_j has a valuation of the object. The random variable $\tilde{v}_j$ has a C^1-cumulative distribution function with support $[\underline{v}, \bar{v}]$ where $\underline{v} < \bar{v}$. A strategy of bidder j is a mapping from valuation to a bid space: $v_j \longmapsto b_j(v_j)$. The risk-neutral payoff of bidder j is $(v_j - b)\mathbb{P}(\max_{j'\neq j} b_{j'}(\tilde{v}_{j'}) < b)$. Using the independence of the valuation $\tilde{v}_{j'}$, the risk-neutral payoff can written as $(v_j - b)\prod_{j'\neq j} F_{j'}(b^{-1}_{j'}(b))$. The information structure of auction game is as follows. Each bidder knows its value, bid but not the valuation of the other bidders. Each bidder knows the valuation cumulative distribution of the others. The structure of the game is common knowledge. We are interested in the equilibria, equilibrium payoffs and revenue of the seller. Existence of equilibrium of auction games have been widely studied ([6, 7, 8]).

Clearly, no bidder would bid an amount that is greater than her value because of negative payoff. By fixing the bidding strategy of the others one has attempted to compute the best response correspondence. Any increase in the bid will decrease the gain but increase the probability of winning. This is a sort of tradeoff between the profit and the probability of winning.

We differentiate the function

$$b \longrightarrow (v_j - b)\prod_{j'\neq j} F_{j'}(b^{-1}_{j'}(b)).$$

In order to find an equilibrium one needs to solve n Ordinary Differential Equations (ODEs) with $2n$ boundary conditions.

$$v'_j(b) = \frac{F_j(v_j(b))}{F'_j(v_j(b))}\left[\frac{1}{n-1}\sum_{j'=1}^{n}\frac{1}{v_{j'}(b) - b} - \frac{1}{v_j(b) - b}\right] \quad (5)$$

$$v_j(\underline{b}) = \underline{v},\ v_j(\bar{b}) = \bar{v} \quad (6)$$

The inverse of the function v is the optimal strategy b. There is no need to mention that this is intractable even with small number of bidders. Even for three bidders we do not understand clearly how the solutions behave in function of F_j.

Why this is not a simple ODE problem?

Non-standard existence theorem is needed: We cannot apply the standard local existence and uniqueness theorem to the ODE with initial value (lowest bid) $v_j(\underline{b}) = \underline{v}$ because by the right-hand-side terms $\frac{1}{v_j(b)-b}$ in the ODEs are unbounded at $\underline{v}$. In addition, the equilibrium satisfies $v_j(\bar{b}) = \bar{v}$ but the term $\bar{b}$ is unknown. Due to these difficulties, explicit solutions of (5) and (6) are not available.

A non-standard numerical method is needed:

Since explicit solutions are open issues, one may ask if it is possible to solve the problem numerically. According to the recent work in [17], the numerical implementation of the system (5) , (6) remains a challenging task. One of the well-known numerical methods consists to solve to find among the solutions of ODEs together with the initial conditions, that satisfy the highest equilibrium bid constraint. Such an approach is known as *forward-shooting method*. However, the forward-shooting method of Marshall et al. [19] do not converge to the solution due to approximation near $\underline{b}$, with the derivative $v_j'(\underline{b})$ It has been shown in [17] that for the special case of power law (i.e. $F(v) = v^{\alpha}$), a dynamical system approach can be used with the change of variable

$$v_j(b) = bV_j(b),\ b = e^w.$$

In the backward approach, one searches for the value of $\bar{b}$ by solving Equation (5) backward in b subject to the end condition $v_j(\bar{b}) = \bar{v}$ and looking for the value of $\bar{b}$ for which the initial value coincides with $\underline{v}$. However, the standard backward-shooting method is inherently unstable, specially when the bids are near $\underline{v}$. The authors in [20] showed that the backward-shooting method is unstable even in the symmetric case.

If all the functions $F_j(v)$ are the same, and hence equal to $\bar{m}(v)$ then, we know from Vickrey 1961 that the symmetric equilibrium is

$$b(s) = s - \frac{\int_{\underline{v}}^{s} F^{n-1}(x)dx}{F^{n-1}(s)},$$

which is obtained as follows:

Instead of n ODEs we have one ODE to solve. The ODE is

$$v'(b)(v(b) - b) = \frac{G(v(b))}{G'(v(b))}, \qquad (7)$$

where G the value distribution of the $n-1$ bidders. Using the bijection function and the fact that $(h^{-1})'(x) = \frac{1}{h'(h^{-1}(x))}$ Hence, $\frac{1}{h'(x)}(x - h(x))G'(x) = G(x)$ where $h(x)$ is the strategy. This means that $xG' = h'G + hG' = (hG)'$. By simple integration between the minimum value and v, one gets $h(v) = v - \frac{1}{G(v)}\int_{\underline{v}}^{v} G(x)\ dx$.

Nonasymptotic mean-field approach provides a useful error bound in this open problem

For asymmetric distribution we are able to get a precise error bound when the distribution F_j are close to their arithmetic mean, the equilibrium strategies and payoffs can be approximated in a perturbed range. To do so, we use a non-asymptotic mean field approach over function space. First remark that the revenue of the seller, satisfies the indistinguishability property, since it is, up to a constant, the integral of the product $\prod_{j\in\mathcal{N}} F_j$. We rewrite the function F_j as $F_j(v) = \bar{m}(v) + \epsilon\gamma_j(v)$, where $\gamma_j(v) = \frac{F_j(v) - \bar{m}(v)}{\epsilon}$, $\bar{m}(v) = \frac{1}{n}\sum_{j'=1}^{n} F_{j'}(v)$ and $\epsilon = \max_j \max_{[\underline{v},\bar{v}]}\ |F_j(v) - \bar{m}(v)|$.

Using result 3, one gets a good approximate of the asymmetric equilibrium strategies and the equilibrium payoff with deviation order of $O(\epsilon^2)$. The revenue of the auctioneer is

$$R(F_1, F_2, \ldots, F_n) = R(\bar{m}, \bar{m}, \ldots, \bar{m}) + O(\epsilon^2).$$

4.1 Fast algorithm for computing approximate equilibrium

We construct a fast algorithm for computing approximate equilibrium. Recall that the first optimality equation can be written as

$$1 + (b - v_j(b)) \sum_{j'\neq j} \frac{F'_{j'}(v_{j'}(b))}{F_{j'}(v_{j'}(b))} v'_{j'}(b) = 0.$$

Define the functional

$$H_j(\bar{b}, v) := 1 + (b - v_j(b)) \sum_{j'\neq j} \frac{F'_{j'}(v_{j'}(b))}{F_{j'}(v_{j'}(b))} v'_{j'}(b).$$

We consider polynomial expansion of inverse-bid functions. The function v_j is written in a flexible functional form $v_j(b) = \bar{b} - \sum_{k=0}^{+\infty} \mu_{j,k}(\bar{b} - b)^k$.

We truncate this polynomial to order $K \geq 2$ and replace it in the first order optimality equation. Denote $\hat{v}_{j,K}(b) = \bar{b} - \sum_{k=0}^{K} \mu_{j,k}(\bar{b} - b)^k$. Taking into account $2n$ boundary conditions, one gets that

$$L(\bar{b}, v) = \sum_{j=1}^{n} H_j(\bar{b}, v)^2 + \sum_{j=1}^{n}(v_j(\bar{b}) - \bar{v})^2 + \sum_{j=1}^{n}(v_j(\underline{b}) - \underline{v})^2 \geq 0$$

has a minimum 0 and the minimizer is the equilibrium inverse bid strategy v. Hence, it is reasonable to consider the functional H when each of function $\hat{v}_{j,K}$ belongs the subspace $\mathcal{D}_K$ the set of polynomial with degree at most K. This is space with dimension $K+1$. The problem becomes

$$\inf_{(\hat{v}_{j,K})_j \in \mathcal{D}_K} L(\bar{b}, \hat{v}) = \inf_{(\mu_{j,k})_{j,k}} L(\bar{b}, \hat{v})$$

Remember that $\bar{b}$ is the highest bid that is submitted in equilibrium. It is therefore an unknown. Thus, we add this into the optimization problem. Hence one has $1 + n(K+1)$ unknown variables to find. Using a grid decomposition of the domain $[\underline{v}, \bar{v}]$ with T points inside, we arrive at a nonlinear least-squares algorithm for selecting $\bar{b}$ and $(\mu_{j,k})_{j,k}$ by solving $\inf_{(\hat{\bar{b}}_t)_t, (\mu_{j,k})_{j,k}} \sum_{t=1}^{T} L(\hat{\bar{b}}_t, \hat{v}_K)$, which yields

$$\inf_{(\hat{\bar{b}}_t)_t, (\mu_{j,k})_{j,k}} \sum_{t=1}^{T}\sum_{j=1}^{n} H_j(\hat{\bar{b}}_t, \hat{v}_K)^2 +$$

$$T\sum_{j=1}^{n}(\hat{v}_{j,K}(\bar{b}_t) - \bar{v})^2 + T\sum_{j=1}^{n}(\hat{v}_{j,K}(\underline{b}) - \underline{v})^2$$

The points $\hat{\bar{b}}_t$ on a grid will be chosen uniformly spaced, i.e. $\hat{\bar{b}}_t = \bar{b} + \frac{t}{T}(\underline{v} - \bar{b})$

Standard Newton-Gauss-Seidel methods provide a very fast convergence rate to a solution if the initial guess if appropriately chosen. However the choice of initial data and guess need to be conducted.

For $n \geq 2$ we do not understand yet the behavior of $b^*_{j,\alpha}(v)$. However, we are able to provide a useful approximation for asymmetric distribution in function of their deviation to the mean. Moreover the approximation holds for the revenue of the seller (auctioneer).

5. CONCLUDING REMARKS

We have presented a mean field framework where the indistinguishability property can be exploited to cover not only the asymptotic regime but also the non-asymptotic regime. In other words, our approximation is suitable not only for large systems but also for a small system with few players. The framework can be used to approximate unknown functions in heterogeneous systems, in optimization theory as well as in game theory.

This work suggests several paths for future research. First, the approach introduced here can be used in a large number of applications, starting from other auctions formats in particular to private information models in general where strategies are functions of types. Second, more progress needs to be done by considering a less restrictive action and belief spaces that are far from the mean of the mean field. The smoothness condition on the objective function may not be satisfied in practice. Finally, we would like to understand how large the deviation of the non-asymptotic result is, compared to a symmetric vector.

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