Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Multi-Model Adaptive Regulation for a Family of Systems Containing Different Zero Structures

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Abstract: An adaptive regulator is proposed for parameter dependent families of linear systems subject to changes in the zero structure. Adaptation is required for the parameter dependent family of plants but continuous adaptive regulation is limited by relative degree and right half plane zeros. A form of adaptive regulation is presented that accommodates parameter induced changes in the zero structure. The conditions for regulation divide the parameter space into disjoint sets thereby defining subfamilies of plants. These plant subfamilies guide controller design. Controller stability is guaranteed by Linear Matrix Inequalities (LMI) and a switch logic based on Lyapunov functions.

Keywords: Regulation; System Structure; Adaptive Control; LMI; Switched Linear system

1. INTRODUCTION

A single controller may be inadequate for systems that experience change in their zero structure. Such systems may be modeled by a structurally diverse family of plants. At any given time the appropriate plant model is uncertain. Multiple model adaptive techniques have been proposed to accommodate such systems Anderson (2000); Angeli and Mosca (2002); Boskovic (2008). Multiple model adaptation selects a controller from a predefined set. In general, the set of controllers is finite although the family of plants may be continuous.

The importance of the open loop zero structure for closed loop regulation has long been known, c.f. Kwakernaak and Sivan (1991); Francis (1977); Kwatny et al. (1991). For a parameter-dependent family of plants, points in the parameter space that do not satisfy the open loop existence conditions for regulator design are called singular points. Singular points form codimension-1 submanifolds that divide the parameter space into disjoint sets. These disjoint sets of the parameter space form subfamilies of plants that have the same zero-structure. A regulator designed for one subfamily will generically fail to regulate a plant in a different subfamily Berg and Kwatny (1994). This bound on *simultaneous regulation* of subfamilies is the basis for the novel multiple model adaptive control design technique presented here.

The design of a finite set of controllers to guarantee stability across the family of plants, called the *covering problem*, is fundamental. Several authors have considered covering from the perspective of controller robustness, Anderson (2000); Boskovic (2008). These designs start with a finite set of plant models and employ robustness metrics to cover the family of plants. We propose a covering method that starts with plant subfamilies and obtains controllers for convex regions of the subfamily's parameter space.

The design of a switch logic to select a stabilizing controller from the set of controllers is the second fundamental problem of multi-model adaptive control. The design method proposed here unifies switch logic and control covering into a single computation. Recall that a single algebraic Riccati equation (ARE) obtains a quadratic Lyapunov function matrix and linear quadratic regulator (LQR) gains. And the quadratic ARE can be written as a convex linear matrix inequality (LMI) to facilitate fast solution. A set of algebraic Riccati inequalities for a convex region of the subfamily's parameter space may be solved for a common LQR state feedback gain and a common quadratic Lyapunov function (CQLF). By choosing Lyapunov function based switch logic and LQR control gains, the multi-model covering and switch logic design computations are unified into a set of LMIs.

This paper is organized as follows. Section 2 defines the specific problem considered herein. Section 3 summarizes the regulation problem and details the relationship between zero dynamics and simultaneous regulation. Section 4 presents our conception of multiple model adaptive regulation (MMAR). Section 5, our main results, details MMAR. Section 6 gives simulation results and Section 7 summarizes the main conclusions.

2. PROBLEM DEFINITION

Define a parameter dependent family of linear plants

$$\begin{aligned} \dot{x} &= A_\theta x + B_\theta u \\ e &= C_\theta x \end{aligned} \tag{1}$$

as $P(\theta) \in \mathcal{P}$ where $x \in R^n$, $u \in R^m$, $e \in R^p$. The parameter dependent matrices are $A_\theta = A(\theta), B_\theta = B(\theta), C_\theta = C(\theta)$ where $\theta \in R^k$ is a vector of unknown but bounded constant parameters. The goal is to regulate the plant with respect to a set of exogenous signals generated by the model

$$\dot{\vartheta} = Z\vartheta \tag{2}$$

where $\vartheta \in R^r$. The set of exogenous signals considered in this paper are step commands and constant disturbances such that $Z = 0_r$. The exogenous signals are assumed to drive the plant (1) through matrices E and F; accordingly

$$\begin{aligned} \dot{x} &= A_\theta x + B_\theta u + E\vartheta \\ \dot{\vartheta} &= Z\vartheta \\ e &= C_\theta x + F\vartheta \end{aligned} \tag{3}$$

As is well known, such disturbance models can effectively characterize command signals and disturbances. The problem of designing robust regulators for systems described by (3) is well studied, e.g., Davison (1972); Francis (1977); Kwatny and Kalnitsky (1978). In this work an *adaptive* regulator is sought that associates an appropriate robust regulator with the actual occurring member of the plant family. The two central problems in doing this are:

- *Covering Problem*: Given a range of plant parameters θ, design a set of controllers $\mathcal{C}$ such that each $P(\theta) \in \mathcal{P}$ is stabilized by at least one $C_i \in \mathcal{C}$.
- *Switch Logic Design*: Given a plant family $\mathcal{P}$ and a finite control covering, design a switching logic that guarantees convergence to a stabilizing regulator for the actual occurring plant.

In subsequent sections, these issues will be addressed.

3. REGULATION

Before proceeding with adaptive regulation of the parameter dependent system defined in (3) it is necessary to summarize some general results for the regulation of an individual linear system.

3.1 The Linear Regulator Problem

Consider a parameter independent linear system with disturbance state vector ϑ

$$\begin{aligned} \dot{x} &= Ax + Bu + E\vartheta \\ \dot{\vartheta} &= Z\vartheta \\ e &= Cx + F\vartheta \end{aligned} \tag{4}$$

It will be assumed that B and C are of full rank.

Definition 1. Regulation requires both $\lim_{t\to\infty} e(t) = 0$ and internal stability. Regulation in the presence of variation in the plant matrices A, B, C is known as *robust regulation* or *structurally stable regulation.*

Structurally stable regulation uses error feedback and incorporates an internal model of the external signals to be tracked and disturbances to be rejected.

Theorem 2. Francis (1977) Necessary and sufficient conditions for structurally stable regulation are

(1) (A, B) stabilizable
(2) (C, A) detectable
(3) Rank $\begin{bmatrix} \lambda_i - A & B \\ C & 0 \end{bmatrix} = n + r$ for λ_i an eigenvalue of Z

The third condition requires the plant transmission zeros to be different than the spectrum of Z. Furthermore, there must be at least as many controls as there are outputs. Since it is always possible to reduce the number of controls, we will henceforth assume $r = m$, so the system is square.

3.2 Loss of Simultaneous Regulation

Theorem 2 specifies the open loop system $\{A, B, C\}$ for which robust regulation is possible. Now consider robust regulation failure. The system matrix for $\{A_\theta, B_\theta, C_\theta\}$ is

$$\Gamma_\theta(s) = \begin{bmatrix} sI - A_\theta & B_\theta \\ C_\theta & 0 \end{bmatrix}$$

Definition 3. The set of points in parameter space on which regulation fails is the *singular surface*,

$$\left\{\theta \in R^k : \det \Gamma_\theta(0) = 0\right\}$$

The system matrix $\Gamma_\theta(s)$ can lose rank due to a zero at the origin and also due to a defect in the input B_θ or output C_θ matrices. The singular surface is dimension $k-1$, or codimension one in the parameter space. Since Γ_θ is either a regular or singular pencil for fixed θ, the singular surface partitions the parameter space into disjoint sets. Theorem 4 parallels Berg and Kwatny (1994).

Theorem 4. Consider a region of the parameter space bisected by the singular surface. A robust regulator designed for one half of the space will be unstable in the adjacent half space for generic systems.

The singular surface divides the original family of plants into sub-families. A robust regulator designed for (4) and applied to (3) will fail to stabilize adjacent sub-families.

Proof: Loss of simultaneous regulation at a singular surface is introduced in Kwatny et al. (1991) and proved in Berg and Kwatny (1994). Loss of stability at a singular surface for a state feedback regulator design is detailed in Section 5.1.

Traversing a singular surface is a sufficient but not a necessary condition for loss of stability. Loss of stability is certain at the singular surface. Loss of stability is possible within an open region of the parameter space. In summary, the singular surface partitions the parameter space. The resulting disjoint regions are a starting point for multiple model controller selection.

4. MULTIPLE MODEL ADAPTIVE REGULATION

4.1 Covering

Due to Theorem 4, a multiple model approach is employed to regulate the family of plants $\mathcal{P}$. A generic multiple model control structure is illustrated in Figure 1. This generic structure can support numerous control design methods for C_i, even within the same set $\mathcal{C}$.

Here each C_i regulates some region of $\mathcal{P}$ and each $P(\theta) \in \mathcal{P}$ is regulated by at least one $C_i \in \mathcal{C}$. Previous authors, for example Anderson (2000); Boskovic (2008), design controllers for a finite set of plant models and then employ robustness metrics to ensure $\mathcal{P}$ is covered. In this paper, controllers are designed for a finite set of convex polytopes

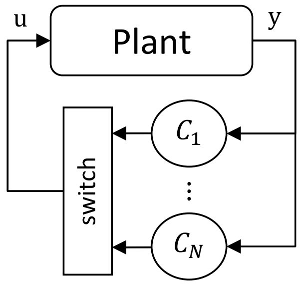

Fig. 1. Multiple Model Control Structure

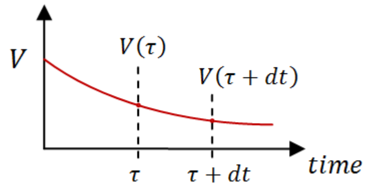

Fig. 2. Lyapunov Function V with $\frac{d}{dt}V < 0$

Ω_i in the parameter space as defined in Section 4.3.2 such that $\mathcal{P} \subseteq \{P(\theta)\,|\theta \in \bigcup \Omega_i\}$. This covering method accommodates singular surfaces in the parameter space.

Advantages of convex regions Ω_i include flexibility in shaping regions, less overlap of regions than operator norms (e.g. $\|\cdot\|_2$), and efficient computation due to underlying convexity.

A disadvantage of this covering method is that a Common Quadratic Lyapunov Function (CQLF) must exist for all $P(\theta) \in \Omega_i$. This CQLF requirement is stronger than exponential stability and may increase cardinality of $\mathcal{C}$. However, the CQLF undergirds the convex algorithm for covering.

4.2 Switching

A CQLF produces a scalar metric that can be tested to ensure stability. A Lyapunov function for the i'th convex polytope Ω_i with final state x_f,

$$V_i = (x - x_f)^T P_i (x - x_f) \tag{5}$$

is monitored for the "in the loop" controller. If the Lyapunov function is decrescent, ie

$$V_i(\tau + dt) < V_i(\tau) \tag{6}$$

as shown in Figure 2, then the correct controller has been identified. If the "on" Lyapunov function ceases to be decrescent, a different controller is switched on. At least one stabilizing controller exists by design. The controllers are tried "in the loop." For more detail see the prerouted switch logic in Angeli and Mosca (2002). When applied to linear systems, inequality (6) may be strengthened. A Lyapunov function with bounds

$$0 \le V(x) \le k_2 \|x\|_2\,,\ \tfrac{d}{dt}V(x) \le -k_3 \|x\|_2\,,$$

has a time rate of change Khalil (2002)

$$\dot{V} \le -\frac{k_3}{k_2} V \tag{7}$$

For the linear quadratic case $k_2 = \lambda_{max} P$ and $k_3 = \lambda_{min} C_z^T C_z$ where P and $C_z^T C_z$ are symmetric positive definite matrices defined in Sec. 4.3. Combine (5), (6), and (7) for the switch logic inequality

$$\tilde{x}(\tau + dt)^T P_i \tilde{x}(\tau + dt) \le \gamma\, \tilde{x}(\tau)^T P_i \tilde{x}(\tau) \tag{8}$$

where $\tilde{x} = x - x_f$ and

$$e^{-\frac{\lambda_{min} C_z^T C_z}{\lambda_{max} P} \cdot dt} < \gamma < 1 \tag{9}$$

If (8) is False a different controller is switched on. In general, the final state $x_f = \lim_{t\to\infty} x(t)$ of (8) is a function of unknown parameters, controller gains, and the exogenous input. Although the observer based synthesis method (Section 5.2) provides an estimate of the exogenous input ϑ, estimation of $x_f(\vartheta)$ is beyond the scope of this note. Since the regulator design of Section 5 is independent of ϑ and $x_f(\vartheta)$, this paper assumes that the unknown component of ϑ is arbitrarily small in relation to (7).

4.3 Common Quadratic Lyapunov Function

A Common Quadratic Lyapunov Function (CQLF) is sought for subfamilies of the parameter dependent family of plants (1). The next two sections adapt the LMI method in Boyd et al. (1994), pg. 115 to provide state feedback gains and CQLFs for systems described by (3).

LQR as LMI The quadratic LQR problem can be expressed as a Linear Matrix Inequality (LMI). Given a linear system

$$\dot{x} = Ax + Bu,\ z = C_z x + D_z u \tag{10}$$

with state feedback control $u = Kx$ the LQR problem of minimizing the energy

$$\int_0^\infty z^T z\, dt$$

can be solved by the inequality

$$\begin{bmatrix} AQ + QA^T + BY + Y^T B^T & (C_z Q + D_z Y)^T \\ C_z Q + D_z Y & -I \end{bmatrix} < 0 \tag{11}$$

which is a function of system parameters A & B, design weights C_z & D_z, and is convex in the symmetric matrix variable $Q > 0$. Here $Y = -\left(D_z^T D_z\right)^{-1} B^T$ with Lyapunov matrix $P = Q^{-1}$. The above LMI is equivalent to the quadratic Riccati matrix inequality via the Shur complement

$$A^T P + PA + C_z^T C_z - PB\left(D_z^T D_z\right)^{-1} B^T P \le 0$$

The controller is $K = YQ^{-1} = -\left(D_z^T D_z\right)^{-1} B^T P$.

Polytopic LMI Now apply (11) to obtain a common regulator for a subfamily of $\mathcal{P}$. Consider first the parameter space θ. Specify values in the parameter space as θ_{ij} where the index j signifies the vertex and the index i signifies the convex polytope Ω_i. This polytope is defined by the convex hull (Co) of its vertices, i.e.

$$\Omega_i \equiv Co\{\theta_{ij}\} \tag{12}$$

A family of plants with subfamilies on each side of the singular surface $|\Gamma_\theta(0)| = 0$ will need polytopes Ω_i, $i \in \{1, \ldots, N\}$ where $N \ge 2$. Figure 3 is an example covering

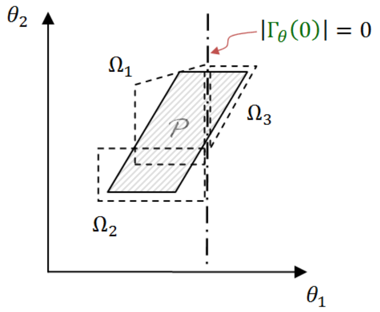

Fig. 3. Convex Regions Cover the Parameter Space

of a parameter space containing singular surface where $N = 3$. The index j for a vertex θ_{ij} in polytope Ω_i takes values in the set $\{1, \ldots, v_i\}$ where for example $v_i = 2^k$ for rectangular polytopes in $\theta \in R^k$.

The singular surface is a natural partition for plant subfamilies. Per Theorem 4, if for any $\theta \in \Omega_i$, $|\Gamma_\theta(0)| = 0$, then Ω_i contains singular surface and neither a common regulator nor a CQLF exist for all $\theta \in \Omega_i$.

The map $\theta \mapsto P(\theta)$ is assumed to be linear in θ; if not, a re-parameterization is required. Thus convexity in the parameter space is preserved in the family of plants of (1). Now extend the inequality (11) to a convex subfamily of plants.

Definition 5. The image of the set of vertices $\{\theta_{ij}\}$ under $P(\cdot)$ is the set of plants $\{P(\theta_{ij})\}$. A set of LMIs for $\{P(\theta_{ij})\}$ is a *polytopic LMI.*

In particular, consider the polytopic LMI where the inequality (11) is enforced at each vertex θ_{ij} of Ω_i,

$$\begin{bmatrix} LMI(\theta_{i1}) & & \\ & LMI(\theta_{i2}) & \\ & & \ddots \end{bmatrix} < 0 \tag{13}$$

This polytopic LMI is formed by substituting

$$A \to A(\theta_{ij}),\; B \to B(\theta_{ij})$$

into (11) at vertices $\theta_{ij} \in \{\theta_{i1}, \theta_{i2}, \ldots, \theta_{iv_i}\}$ of polytope Ω_i to obtain a set of inequalities for simultaneous solution. Details of $A(\theta_{ij})$, $B(\theta_{ij})$ are left to Section 5.

Each polytopic LMI (13) obtains a stabilizing controller $C_i \in \mathcal{C}$ with state feedback control K_i and Lyapunov function matrix P_i for the continuous set of plants $\{P(\theta) | \theta \in \Omega_i\}$. The state feedback control is

$$K_i = -\left(D_z^T D_z\right)^{-1} B_i^T P_i$$

with Lyapunov function matrix $P_i = Q_i^{-1}$. Choose B_i,

$$B_i \in Co\{B(\theta_{ij})\} \tag{14}$$

to obtain a constant K_i. It can be shown that if (13) holds for all $B(\theta_{ij})$ and B_i is chosen according to (14), then a solution P_i of the polytopic LMI

$$A_{CL}^T(\theta_{ij}) P_i + P_i A_{CL}(\theta_{ij}) + Q_i < 0 \tag{15}$$

with $A_{CL}(\theta_{ij}) = A(\theta_{ij}) + B(\theta_{ij}) K_i$ exists. The P_i of (15) is used in place of the P_i of (13) for the switch logic of section 4.2 if either $\{B(\theta_{ij})\}$ is nonsingleton or if the $LMI(\theta_{ij})$ in (13) are assigned different C_z, D_z, ie $D_z(\theta_{ij})$.

The distinction between LMI and polytopic LMI is convenient for assembling the inequalities. A numerical solver makes no distinction between an LMI and a set of LMIs.

5. MULTI-MODEL ADAPTIVE REGULATION

Now apply (13) to design the set of controllers $\{C_1, \ldots, C_N\}$ of the multi-model controller in Figure 1. Two types of regulator design are identified in Kwatny and Kalnitsky (1978):

- Error Augmentation, Davison (1972): The disturbance model (2) is used explicitly as the controller. This controller is driven by the error dynamics, has r states, and an observer is not required.
- Disturbance Estimation, Francis (1977): The disturbance model (2) is embedded in the observer. For a full order observer, the controller has $n + r$ states.

The design of Error Augmentation type controllers for MMAR is completed below. The design of Disturbance Estimation type controllers is shown to be a Bilinear Matrix Inequality.

5.1 *Error Augmentation & Full State Feedback*

Design A robust regulator design procedure for a parameter dependent plant of type (3) is given in Kwatny and Kalnitsky (1978), Davison (1972):

- Define an r dimensional, error driven dynamic system with state η that incorporates the disturbance model,

$$\dot{\eta} = Z\eta + Je$$

where J is chosen such that (J, Z) is controllable.

- Form the composite system from (4)

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & 0 \\ JC & Z \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

and solve for the stabilizing state feedback control

$$u = [\, K_x \;\; K_\eta \,] \cdot \begin{bmatrix} x \\ \eta \end{bmatrix} \tag{16}$$

Common Quadratic Solution The plant matrices for use in (13) are

$$A(\theta_{ij}) = \begin{bmatrix} A_{\theta_{ij}} & 0 \\ J_i C_{\theta_{ij}} & Z \end{bmatrix},\; B(\theta_{ij}) = \begin{bmatrix} B_{\theta_{ij}} \\ 0 \end{bmatrix} \tag{17}$$

Loss of Simultaneous Regulation According to Theorem 4, traversal of the singular surface is sufficient for loss of stability. The plant and compensator closed loop system $\dot{x}_{cl} = A_{cl} x_{cl}$ with state $x_{cl} = [\, x \;\; \eta \,]^T$ has dynamics matrix A_{cl}:

$$\begin{bmatrix} A_\theta + B_\theta K_x & B_\theta K_\eta \\ JC_\theta & Z \end{bmatrix} \tag{18}$$

Recall that $Z = 0_r$ for step reference signals and constant disturbances. Factor (18) as

$$\begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \cdot \begin{bmatrix} A_\theta & B_\theta \\ C_\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ K_x & K_\eta \end{bmatrix} \tag{19}$$

where J is dimension $r \times p$, K_η is dimension $p \times r$, and both J, K_η are full rank by design. Generically the singular surface is intersected transversely and one parameter at a time. Thus (19) losses rank when $\det\Gamma_\theta(0) = 0$ by application of rank inequalities.

5.2 Disturbance Estimation

Implement feedback control (4) using observed states $\hat{x}$, $\hat{\vartheta}$,

$$u = \bar{K}\begin{bmatrix}\hat{x}\\ \hat{\vartheta}\end{bmatrix}$$

The closed loop dynamics of the plant, exogenous dynamics, and observer is

$$\dot{x}_{cl} = A_{cl}x_{cl}, \qquad e = C_{cl}x_{cl}$$

with state vector

$$x_{cl} = [\, x \ \hat{\bar{x}} \ \vartheta \,] = [\, x \ \hat{x} \ \hat{\vartheta} \ \vartheta \,]$$

where

$$A_{cl} = \left[\begin{array}{cc|c} A & B\bar{K} & E \\ -LC & \bar{A}+L\bar{C}+B\bar{K} & -LF \\ \hline 0 & 0 & Z \end{array}\right],$$

$$C_{cl} = [\, C \ 0 \ F \,]$$

and

$$\bar{A} = \begin{bmatrix} A & E\\ 0 & Z\end{bmatrix}, \ \bar{B} = \begin{bmatrix} B\\ 0\end{bmatrix},$$

$$\bar{C} = [\, C \ F \,], \ \bar{K} = [\, K_x \ K_\vartheta \,].$$

Let A_{11} be the top left block of the partitioned A_{cl}. Account for plant (θ) and control(i)/observer(o) mismatch,

$$A_{11} = \begin{bmatrix} A_\theta & B_\theta\bar{K}_i \\ -L_iC_\theta & \bar{A}_o + L_i\bar{C}_o + \bar{B}_o\bar{K}_i\end{bmatrix}$$

where the composite matrices are

$$\bar{A}_o = \begin{bmatrix} A_o & E_o\\ 0 & Z\end{bmatrix}, \ \bar{B}_o = \begin{bmatrix} B_o\\ 0\end{bmatrix}, \ \bar{C}_o = [\, C_o \ F_o \,],$$

and the key requirement is that the spectrum of Z is contained in the spectrum of the observer dynamics,

$$\sigma(Z) \subset \sigma\left(\bar{A}_o + L_i\bar{C}_o + \bar{B}_o\bar{K}_i\right) \tag{20}$$

The design parameters are controller gains $\bar{K}_i$, L_i and observer matrices $\bar{A}_o$,$\bar{B}_o$,$\bar{C}_o$ subject to the internal model condition (20). The algebraic Riccati equation

$$A_{11}^T P + PA_{11} < 0 \tag{21}$$

is linear in both the independent variable P and design parameters $\bar{K}_i$, L_i, $\bar{A}_o$, $\bar{B}_o$, $\bar{C}_o$. Hence, (21) is a Bilinear Linear Matrix Inequality (BLMI).

Remark 6. Were the observer model an exact copy of the true plant dynamics, ie $A_o = A_\theta$ & etc., the separation principle would hold and A_{11} could be transformed to an upper triangular form. Then (11) could be solved for K_i in $A_o + B_oK_i$ and again for the dual problem of L_i in $\bar{A}_o^T + \bar{C}_o^T L_i^T$. In this idealized case a CQLF is certain for A_{11} in upper triangular form.

6. EXAMPLE

Consider a simplification of the aircraft longitudinal dynamics model in Kwatny et al. (1991). The states x_1, x_2, and x_3 approximate angle of attack, pitch, and pitch rate respectively. The angle of attack has relative degree one and the zero structure of pitch & pitch rate is preserved. The zero structure changes with operation at low and high angles of attack.

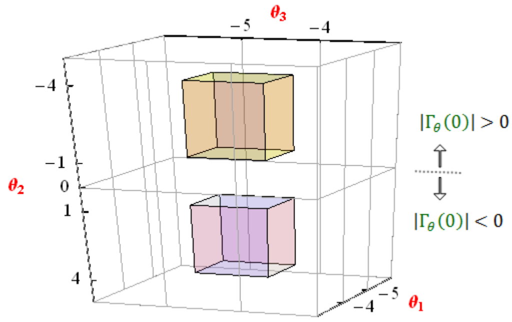

Fig. 4. Example Problem Parameter Space with Two Subfamilies

$$A = \begin{bmatrix} \theta_1 & 0 & 1\\ 0 & 0 & 1\\ \theta_2 & 0 & \theta_3\end{bmatrix}, \ B = \begin{bmatrix}1\\0\\0\end{bmatrix},$$

$$C = [\, 1 \ 1 \ 0 \,], \ D = 0, \ x^T = [\, x_1 \ x_2 \ x_3 \,],$$

$$Z = 0, \ F = -1, \ E^T = [\, 0 \ 0 \ 1 \,]$$

Here θ_1, θ_2 and θ_3 are unknown but constant parameters. When $\theta_2 = 0$ the system is structurally unstable with

- a transmission zero at $s = 0$,
- a pole at the origin,
- (A, B) is uncontrollable

Parameter variation for the damping terms are

$$-5 < \theta_1 < -4, \ -5 < \theta_3 < -4$$

The two subfamilies (top & bottom) are a function of θ_2 and may be defined by the sign of $\det\Gamma_\theta$. Figure 4 shows the two families in the parameter space.

$$\det\Gamma_\theta(0) = \begin{cases} >0 & \text{Top (T)} : \theta_2 \in [-4,-1] \\ <0 & \text{Bottom (B)} : \theta_2 \in [1,4]\end{cases}$$

Substitute (17) into (13) for A, B and solve for K_i, P_i, $i \in \{B, T\}$, since a single controller fails to regulate plants from both subfamilies.

Let the LQR design weights (10) be

$$[\, C_z \ D_z \,] = \begin{bmatrix} Q_z & 0\\ 0 & R_z\end{bmatrix}$$

where for both subfamilies

$$Q_z = \tfrac{1}{10}\,diag\left([1\,1\,25\,25]\right), \ R_z = 1$$

The state feedback (16) is

$$K_B = [\, -3.5 \ -8.2 \ -2.6 \ -2.7 \,]$$

$$K_T = [\, -17.6 \ 174 \ 38.8 \ 25.0 \,]$$

Let θ^*, σ^* be the correct values for the parameter and switch logic index respectively. The switch index specifies the "on" controller. Set

$$\theta^* = (\theta_1^*, \theta_2^*, \theta_3^*) = (-4, 1, -4)$$

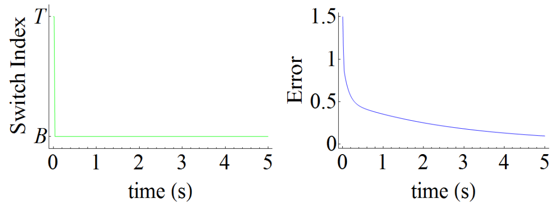

Fig. 5. (a) Switch index σ (left); (b) Regulation error e

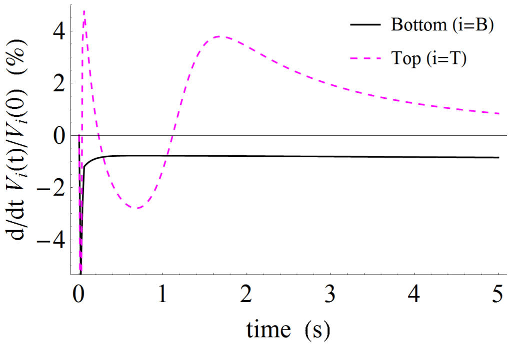

Fig. 6. Normalized Lyapunov derivative $\frac{d}{dt}V_i(t)/V_i(0)$ (%)

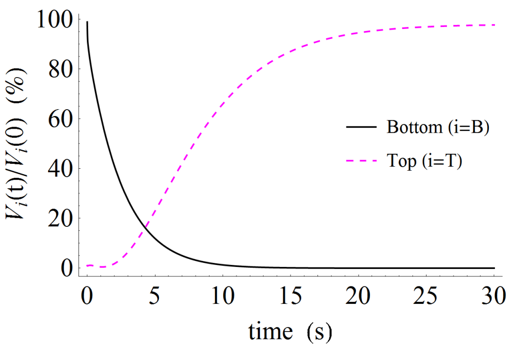

Fig. 7. Normalized Lyapunov function $V_i(t)/V_i(0)$ (%)

such that $P(\theta^*) \in \Omega_B$ and $\sigma^* = B$. Initialize the plant (x), controller (η), exogenous disturbance (ϑ), and switch logic index (σ) states

$$\begin{aligned}[x_1(0)\ x_2(0)\ x_3(0)\ \eta(0)] &= [\,1\ 0\ 0\ 0\,] \\ \vartheta(0) &= -0.5 \\ \sigma(0) &= T\end{aligned}$$

The multiple model adaptive regulation system was simulated in Mathematica. Observe in Figure 5(a) that switching is fast, $t \approx 0.03$, and accurate since $\dot{V}$ with the "wrong" controller in the loop (i.e. $\sigma = T$) quickly ceases to be negative definite as shown in Figure 6.

The threshold γ for the percent decrease of the "on" Lyapunov function can be scaled by changing the dt between sample periods as shown in (9). Evaluating the switch inequality less frequently, ie larger dt, is more robust to noise but may retard model identification. A prerouted switch logic is acceptable when the cardinality of $\mathcal{C}$ is small.

Output regulation $\lim_{t\to\infty} e(t) = 0$ succeeds as shown in Figure 5(b). The bounded Lyapunov functions V_i, $i \in \{B, T\}$, of Figure 7 show internal stability.

The final state, x_f is a function of the plant parameters, controller gains, and the disturbance state. For state vector $x = [x_1,\ x_2,\ x_3,\ \eta]^T$, feedback gain $K = [k_1,\ k_2,\ k_3,\ k_4]$, and set point ϑ,

$$x_f = \left[-\frac{\vartheta}{\theta_2}\ \ \vartheta\left(1 + \frac{1}{\theta_2}\right)\ \ 0\ \ \frac{\vartheta(\theta_1 + k_1 - k_2(1+\theta_2))}{k_4\theta_2} \right]^T$$

A priori knowledge of the final state x_f is an undesirable feature of this generalized energy (Lyapunov) switch logic.

7. CONCLUSIONS

Multiple model adaptive regulation of parameter dependent systems with zero structure change is implemented. A finite set of controllers is obtained from linear matrix inequalities for subfamilies of the parameter dependent plant with equivalent zero structure. Switch logic ensures that the appropriate stabilizing controller is chosen.

In summary, it is well known that regulation is not possible at the singular point of a zero structure change. Less well known is that regulation by a common controller is not possible across points of zero structure change. Thus, a common controller can only be used as parameters vary within a family having equivalent zero structure. The finite set of controllers with switch logic enables robust adaptive regulation.

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