Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Quadratic MPC with ℓ_0-input constraint ⋆

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Abstract: In this paper we propose a novel quadratic model predictive control technique that constrains the number of active inputs at each control horizon instant. This problem is known as sparse control. We use an iterative convex optimization procedure to solve the corresponding optimization problem subject to sparsity constraints defined by means of the ℓ_0-norm. We also derive a sufficient condition on the minimum number of active of inputs that guarantees the exponential stability of the closed-loop system. A simulation example illustrates the benefits of the control design method proposed in the paper.

1. INTRODUCTION

Controlling a process using a reduced number of inputs has been a topic of increasing interest in the control literature in the recent years (see e.g. Gallieri and Maciejowski [2013]). This problem is known as sparse control (Schuler et al. [2011], Gallieri and Maciejowski [2012]). A sparse discrete-time signal, or a sparse vector is characterized for having zeros on most of its elements. The use of sparse vectors has become important in many applications such as compressive sampling (Candès and Wakin [2008], Delgado et al. [2012]), system identification (Godoy et al. [2014]), channel estimation in wireless communications (Carvajal et al. [2012]) and over-actuated control systems (Gallieri and Maciejowski [2012]) among others.

Minimizing the number of active actuators (control inputs) has several advantages. For instance, in Hartley et al. [2013] sparse control is deployed to minimize propellant consumption in the spacecraft rendezvous and to accommodate the minimum impulse constraint. Moreover, when the transmission of the information (measurement and/or control inputs) is done using scarce communication resources (Chen et al. [2011]), sparse control have been used to minimize the network bandwidth usage (Nagahara et al. [2012]) and to minimize the energy consumption of the self-powered devices due to transmission (Haupt et al. [2008]).

The ℓ_0-norm (number of non-zero elements of a vector) is the natural measure of sparsity. However, to avoid computational complexity when dealing directly with ℓ_0-norm constraints in optimal control problems, two approaches have been typically used: i) the use of ℓ_1-norm (Schuler et al. [2011], Gallieri and Maciejowski [2013]) and ii) the use of a greedy algorithm called Orthogonal Matching Pursuit algorithm (Nagahara et al. [2012]).

Orthogonal Matching Pursuit (OMP) (Tropp and Gilbert [2007]) is an algorithm that computes a suboptimal solution that satisfies an ℓ_0 constraint. OMP is computationally inexpensive, however it is not straightforward to modify the algorithm in order to incorporate extra constraints properly. On the other hand, several methods developed for ℓ_1-norm optimization provide enough flexibility to handle several kind of constraints. Moreover, ℓ_1-norm regularization provides a convenient approach to promote sparsity, however in most applications ℓ_1-norm has no clear meaning, and the selection of the regularization parameter could be difficult, specially in problems where corrupted measurements are considered.

The control design techniques in Gallieri and Maciejowski [2013] and Nagahara et al. [2012] promote sparsity on the control actions. However, these techniques do not incorporate a hard constraint on the number of active control inputs at each control horizon instant. Moreover, in Nagahara et al. [2012] the single-input single-output case is analyzed and it is not clear how to include extra convex constraints in the corresponding optimization procedure.

In this paper, we propose the design of quadratic MPC controllers subject to ℓ_0-constraints on each control horizon instant. This technique allows us to reduce the number of actuators being used and the transmission bandwidth usage (when dealing with networked control systems). We formulate the problem with an ℓ_0-constraint optimization problem that allows us to include any convex constraint and also to consider multivariable systems. We also derive a sufficient condition that allows to find the minimum number of active inputs for guaranteeing global exponential stability. The layout of remainder of the paper is as follows. In Section 2, we discuss the problem of interest. In Section 3, we present an iterative convex optimization procedure to solve the optimization problem in a quadratic model predictive strategy subject to constraints in the number of active inputs at each control horizon instant. In Section 4, we establish a sufficient condition on the number of active inputs that guarantees stability. A simulation

⋆ This work was partially supported by MICINN DPI2011-27845-C02-02 and grants PREDOC/2011/37 and E-2013-02 from *Universitat Jaume I.*

study is given in Section 5, and finally we draw conclusions in Section 6.

Notation and Basic Definitions Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the real and non-negative real number sets. We represent the transpose of a given matrix A and a vector x via $(Ax)' = x'A'$. The Euclidean norm is denoted via $|\cdot|$ while the weighted Euclidean norm (squared) is denoted by $|x|_P^2 = x'Px$. Additionally, the induced norm of a given matrix A is its largest singular value. The maximum and minimum eigenvalues of a given matrix A are represented via $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ respectively. $\mathcal{I}$ denotes an identity matrix of appropriate dimension. $\mathbf{0}_m$, and $\mathbf{1}_m$ denote vectors with only zero or one entries respectively. The operator $\text{diag}_n(x)$ transforms $x \in \mathbb{R}^n$ into a diagonal matrix $A \in \mathbb{R}^{n \times n}$.

2. PROBLEM DESCRIPTION

Consider the following discrete-time linear time-invariant system:

$$x_{k+1} = Ax_k + Bu_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state , $u_k \in \mathbb{R}^m$ is the control input vector. The pair (A, B) is assumed to be stabilizable where the matrix A is not necessarily Schur stable. We are seeking to control system (1), if it is possible, with a reduced number of active inputs, $\gamma \leq m$. To this end, one needs to design a controller which can provide the best possible actuation considering only γ active inputs while the remaining $\gamma - m$ inactive inputs will take a null value.

$\sigma \in \mathbb{R}^m$ denotes a binary vector which indicates the active and inactive inputs, i.e., the $i-th$ component of σ is given by:

$$\sigma_i = \begin{cases} 1 & \text{if } u_i \text{ is active,} \\ 0 & \text{otherwise } (u_i = 0), \end{cases} \tag{2}$$

for all $i \in \{1, \ldots, m\}$. Thus, the number of non-zero elements of vector σ (ℓ_0-norm) is $|\sigma|_0 = \gamma$.

To formulate the MPC optimal problem, we first consider the following quadratic cost function

$$V_N(x, \boldsymbol{u}) = |\hat{x}_N|_P^2 + \sum_{j=0}^{N-1} |\hat{x}_j|_Q^2 + |\hat{u}_j|_R^2, \tag{3}$$

where $\hat{x}$ and $\hat{u}$ stand for the predicted values of the system state and input respectively, and N is the prediction horizon. The matrices Q, R, and P are assumed to be positive definite. The vector $\boldsymbol{u}$ contains the tentative control actions over the prediction horizon, i.e.,

$$\boldsymbol{u} = \left[\hat{u}_0', \ldots, \hat{u}_{N-1}'\right]' \in \mathbb{R}^{Nm},$$

The optimization of interest for the current state, $x_k = x$, is given as

$$\mathbb{P}_N(x): \quad V_N^{op}(x) = \min_{\boldsymbol{u} \in \mathbb{R}^{Nm}} \{V_N(x, \boldsymbol{u})\}, \tag{4}$$

subject to:

$$\hat{x}_{j+1} = A\hat{x}_j + B\hat{u}_j, \tag{5}$$

$$|\hat{u}_j|_0 \leq \gamma, \tag{6}$$

$$|\hat{x}_1|_G^2 \leq J(x), \tag{7}$$

for all $j \in \{0, \ldots, N-1\}$, where $\hat{x}_0 = x_k$, $\gamma \leq m$, G is a positive definite matrix and $J(x)$ is a positive function decreasing in x, i.e., $J(x) > 0$ for all $x \neq 0$.

Here constraint (6) encompasses the ℓ_0-norm for the control input along the prediction horizon, while (7) is a constraint introduced to guarantee stability. Therefore, the design of G and $J(x)$ will be studied in Section 4.

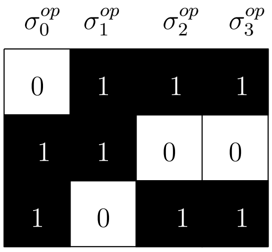

Fig. 1. Vector of active inputs in the horizon.

Consequently, the optimal input sequence, $\boldsymbol{u}^{op}(x)$, is the one which minimizes the cost function,

$$\boldsymbol{u}^{op}(x) \triangleq \arg\left\{\min_{\boldsymbol{u} \in \mathbb{R}^{Nm}} V_N(x, \boldsymbol{u})\right\}. \tag{8}$$

Thus, the resulting optimal solution is the, so-called, input control sequence

$$\boldsymbol{u}^{op}(x) = \left[(\hat{u}_0^{op})', \ldots, (\hat{u}_{N-1}^{op})'\right]', \tag{9}$$

while the resulting optimal state sequence is

$$\boldsymbol{x}^{op}(x) = [x', (\hat{x}_1^{op})', \ldots, (\hat{x}_N^{op})']'.$$

Additionally, for this particular problem, we also obtain the resulting optimal active input sequence, given by

$$\boldsymbol{\sigma}^{op}(x) = \left[(\sigma_0^{op})', \ldots, (\sigma_{N-1}^{op})'\right]'. \tag{10}$$

Notice that the elements of $\sigma^{op}(x)$ may differ from each other. However, $|\sigma_j^{op}|_0 \leq \gamma$ for all $j \in \{0, \ldots, N-1\}$. For example, if $N = 4$, $m = 3$, and $\gamma = 2$ a possible $\boldsymbol{\sigma}^{op}(x)$ is shown in Figure 1.

Finally, we use a *receding horizon* technique, i.e., only the first element of $\boldsymbol{u}^{op}(x)$ is applied to the system at each sampling instant (see e.g. Rawlings and Mayne [2009]). The solution of the optimal problem, $\mathbb{P}_N(x)$ in (4), yields the MPC control law, $\kappa_N(\cdot): \mathbb{R}^n \to \mathbb{R}^m$,

$$\kappa_N(x) \triangleq \hat{u}_0^{op}. \tag{11}$$

Thus, the resulting MPC loop can be represented via

$$x_{k+1} = Ax_k + B\kappa_N(x_k). \tag{12}$$

In the following section, we will present a general method to solve an optimization problem subject to ℓ_0-norm constraints. This solution is then used to solve the quadratic MPC problem in (4)-(6).

3. ℓ_0-CONSTRAINED BASED SOLUTION

The optimization problem that includes ℓ_0-norm in the cost function or as a constraint is known to be computationally demanding since a combinatorial optimization is typically considered. Many approaches have been developed that approximate or relax the original problem. One of such approaches is ℓ_1-norm relaxation, which provides a convenient way to promote sparsity. Several methods developed for ℓ_1-norm minimization are flexible enough to handle convex constraints. However, in ℓ_1-norm minimization problems there is no way to know in advance the number of non-zero elements of the solution.

Another approach to relax the original ℓ_0-norm problem, is by computing a suboptimal solution. In this framework, greedy algorithms such as OMP provide a computationally inexpensive solution.

Recently, an interesting approach to deal with ℓ_0-norm constraints has been proposed in Dattorro [2005]. This approach is based on the idea that imposing a ℓ_0-norm constraint $|u|_0 < \gamma$ in a vector $u \in \mathbb{R}^m_{\geq 0}$, is equivalent to impose the constraint that the sum of the smallest $m - \gamma$ components of the vector u is equal to zero.

An important ingredient of the method to impose an ℓ_0 constraint is that the computation of the sum of the $m - \gamma$ smallest elements of u can be re-formulated as a minimization problem. Consider that $\pi(u)$ is the descend sorting function, and $\pi_i(u)$ denotes the $i - th$ largest element of u. The sum of the $(m-\gamma)$ smallest components of a vector $u \in \mathbb{R}^m_{\geq 0}$, i.e $\sum_{i=\gamma+1}^{m} \pi_i(u)$, can be written as the following minimization problem

$$\begin{aligned} \sum_{i=\gamma+1}^{m} \pi_i(u) = \min_{w \in \mathbb{R}^m_{\geq 0}} \quad & w'u \\ \text{subject to} \quad & \mathbf{0}_m \leq w \leq \mathbf{1}_m \\ & w'\mathbf{1}_m = m - \gamma \end{aligned} \tag{13}$$

Problem (13) has a closed form solution that correspond to the vector w that has ones in the elements corresponding to the $m - \gamma$ smallest values of u, and zeros elsewhere.

In Dattorro [2005] the above idea has been used to impose ℓ_0-norm constraints, i.e. to solve the following feasibility problem

$$\begin{aligned} \underset{u \in \mathbb{R}_{\geq 0}}{\text{find}} \quad & u \\ \text{subject to} \quad & u \in \mathcal{C} \\ & |u|_0 \leq \gamma \end{aligned} \tag{14}$$

where $\mathcal{C}$ is a convex set. This feasibility problem has the same set of solutions that the following optimization problem,

$$\min_{u \in \mathbb{R}^m_{\geq 0}} \min_{w \in \mathbb{R}^m_{\geq 0}} w'u \tag{15}$$

$$\begin{aligned} \text{subject to} \quad & \mathbf{0}_m \leq w \leq \mathbf{1}_m \\ & w'\mathbf{1}_m = m - \gamma \\ & u \in \mathcal{C} \end{aligned} \tag{16}$$

where $\mathcal{C}$ is a convex set. Problem (15)-(16) can be solved by alternating minimization between w and u. In more detail, given a current value for u, i.e. $\hat{u}^k$ at the iteration k, the optimization update is given by

$$\hat{w}^{k+1} = \arg \left\{ \min_{w \in \mathbb{R}^m_{\geq 0}} w'\hat{u}^k \qquad s.t\ (16) \right\} \tag{17}$$

$$\hat{u}^{k+1} = \arg \left\{ \min_{u \in \mathbb{R}^m_{\geq 0}} (\hat{w}^{k+1})'u \qquad s.t\ (16) \right\} \tag{18}$$

In problem (15), the constraint $|u|_0 \leq \gamma$ is satisfied if and only if

$$w'u = 0 \tag{19}$$

Thus, this condition can be used to test if the ℓ_0-norm constraint has been satisfied.

Notice that the minimization steps in (17)-(18) can be easily implemented using standard solvers such as CVX (Grant et al. [2011]).

3.1 Method 1

Problem (15) provides an effective method to impose ℓ_0-norm constraints. However, in several problems such as MPC, it is desirable to also minimize a cost function $f(u)$. Thus, the following optimization problem is proposed

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & f(u) \\ \text{s. t.} \quad & u \in \mathcal{C} \\ & |u|_0 \leq \gamma \end{aligned} \tag{20}$$

where $\mathcal{C}$ is a convex set. In order to achieve both: minimize $f(u)$ and satisfy the constraint $|u|_0 \leq \gamma$, in this paper, problem (20) is solved by using the iterative solution of ℓ_0 constrained feasibility problem. In more detail, given a current control input $\hat{u}^k$, a new control input $\hat{u}^{k+1}$ is obtained by solving the following problem

$$\begin{aligned} \min_{z \in \mathbb{R}^m_{\geq 0}, u \in \mathbb{R}^m} \min_{w \in \mathbb{R}^m_{\geq 0}} \quad & w'z \\ \text{subject to} \quad & f(u) \leq f(\hat{u}^k)\ (1 - \varepsilon) \\ & u \in \mathcal{C} \\ & \mathbf{0}_m \leq w \leq \mathbf{1}_m \\ & w'\mathbf{1}_m = m - \gamma \\ & -z \leq u \leq z \end{aligned} \tag{21}$$

where $0 < \varepsilon < 1$ is an user supplied parameter that manages the reduction required in the cost function and z is a dummy variable to handle negative values for u. When the solution of problem (21) does not satisfy condition (19), it means that at the current iteration, the method couldn't find an $u \in \mathcal{C}$ that reduce the cost by as required by ε. Thus, ε is reduced to $\varepsilon = 0.5\varepsilon$. The algorithm continues until condition (19) couldn't be satisfied for an small enough ε.

Method 1 could be computationally expensive, in the sense, that each iteration requires to solve problem (21), which is also solved by an iterative procedure. In the following section, a relaxed method that is computationally less expensive is considered.

3.2 Method 2

In the method proposed in this section, problem (20) is relaxed by allowing that $\sum_{i=\gamma+1}^{m} \pi_i(u)$ to be non-zero, but forced to be small. Thus, the relaxed problem is as follows

$$\begin{aligned} \min_{z \in \mathbb{R}^m_{\geq 0}, u \in \mathbb{R}^m} \min_{w \in \mathbb{R}^m_{\geq 0}} \quad & f(u) + \alpha w'z \\ \text{s. t.} \quad & u \in \mathcal{C} \\ & \mathbf{0}_m \leq w \leq \mathbf{1}_m \\ & w'\mathbf{1}_m = m - \gamma \\ & -z \leq u \leq z \end{aligned} \tag{22}$$

where $\alpha > 0$ is a regularization parameter that manages the tradeoff between minimizing $f(u)$ and minimizing the sum of the $(m-\gamma)$ smallest entries of u. The optimization problem (22) is solved by using an iterative procedure like the one in (17)-(18) to solve (15)-(16).

Remark 1. Note that the proposed algorithm can easily handle ℓ_0-norm constraints over a selection in the vector, i.e. $|S_i u|_0 \leq \gamma$, where S_i is a given diagonal matrix with entries $\{0, 1\}$. We use this approach to solve problem (4)-(6), where ℓ_0-norm constraints are imposed on several selections of vector $\boldsymbol{u}$ i.e. to promote sparseness of control signals in space (most actuators remain still) and/or time (actuators stay still most of the time). In addition, we can also minimize the ℓ_0-norm of the whole optimal input vector, i.e., $|\boldsymbol{u}| \leq \gamma$. Thus, this optimization strategy can be applied to the data-rate limited network control problem, see e.g., Nagahara et al. [2012].

4. STABILITY ANALYSIS

In this section, sufficient conditions to guarantee stability of the MPC loop in (12) are established. These results provide a guide to choose the value of γ in the optimal control problem, $\mathbb{P}_N(x)$, in order to ensure stability of the MPC loop.

Firstly, we define the predicted state sequence as

$$\boldsymbol{x}_{[1:N]} = [\hat{x}_1', \ldots, \hat{x}_N']'.$$

Considering an initial system state $\hat{x}_0 = x$, from (5), we obtain

$$\boldsymbol{x}_{[1:N]} = \Lambda x + \Phi \boldsymbol{u},$$

where

$$\Phi \triangleq \begin{bmatrix} B & 0 & \cdots & 0 & 0 \\ AB & B & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & AB & B \end{bmatrix}, \quad \Lambda \triangleq \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}.$$

Thus, the cost function (3) can be re-written as

$$V_N(x, \boldsymbol{u}) = \nu(x) + \boldsymbol{u}' H \boldsymbol{u} + 2\boldsymbol{u}' F x,$$

where the term $\nu(x)$ is independent of $\boldsymbol{u}$ and

$$H \triangleq \Phi' \mathcal{Q} \Phi + \mathcal{R} \in \mathbb{R}^{Nm \times Nm},$$
$$F \triangleq \Phi' \mathcal{Q} \Lambda \in \mathbb{R}^{Nm \times n},$$

with

$$\mathcal{Q} \triangleq \operatorname{diag}\{Q, \ldots, Q, P\} \in \mathbb{R}^{Nn \times Nn}$$
$$\mathcal{R} \triangleq \operatorname{diag}\{R, \ldots, R\} \in \mathbb{R}^{Nm \times Nm}.$$

Notice that, since Q, R, and P are positive definite, so is H. Based on this representation, the following unconstrained optimal input, $\boldsymbol{u}_{uc}^{op}(x)$, can be defined, see Rawlings and Mayne [2009].

Lemma 2. (Unconstrained Solution). If constraints (6) and (7) are not taken into account in $\mathbb{P}_N(x)$ in (4), i.e., $\gamma = m$, then $V_N(x, \boldsymbol{u})$ is minimized when

$$\boldsymbol{u}_{uc}^{op}(x) \triangleq \arg\left\{ \min_{\boldsymbol{u} \in \mathbb{R}^{Nm}} V_N(x, \boldsymbol{u}) \right\} \triangleq -H^{-1} F x. \quad (23)$$

■

The next theorem establishes a sufficient condition for exponential stability of the MPC loop.

Theorem 3. Suppose that the matrix P in the cost function, $V_N(x, \boldsymbol{u})$, is chosen to be the solution to the algebraic Riccati equation

$$A_K' P A_K - P + Q + K' R K = 0, \quad (24)$$

where

$$K = -(B'PB + R)^{-1} B'PA, \quad A_K = A + BK. \quad (25)$$

If in (6) a fixed γ is chosen such that

$$Q + K'RK - \Psi_\sigma \succ 0 \quad (26)$$

where

$$\Psi_\sigma = A_K' W_\sigma A_K + (2A_K + \Delta_\sigma)'(P + W_\sigma)\Delta_\sigma, \quad (27)$$
$$\Delta_\sigma = B(L_\sigma - \mathcal{I})K, \quad (28)$$
$$W_\sigma = F' H^{-1} (\mathcal{L}_\sigma - \mathcal{I}) H (\mathcal{L}_\sigma - \mathcal{I}) H^{-1} F, \quad (29)$$

with

$$L_\sigma = \operatorname{diag}_m\{\sigma\} \in \mathbb{R}^{m \times m},$$
$$\mathcal{L}_\sigma = \operatorname{diag}\{L_\sigma, \ldots, L_\sigma\} \in \mathbb{R}^{Nm \times Nm}, \quad (30)$$

and matrix G and function $J(x)$ are chosen as

$$G = P + W_\sigma,$$
$$J(x) = x'(A_K + \Delta_\sigma) G (A_K + \Delta_\sigma) x \quad (31)$$

Then, the MPC closed-loop system (12) is globally exponentially stable.

Proof. Considering that matrix P in (3) satisfies (24), the unconstrained solution, $\boldsymbol{u}_{uc}^{op}(x)$ in (23) can be expressed via:

$$\boldsymbol{u}_{uc}^{op}(x) = [(K\hat{x})' \ (K\hat{x}_1)' \ \ldots \ (K\hat{x}_{N-1})']' \quad (32)$$

Now, the optimal cost function, $V_N^{op}(x) = V_N^{op}(x, \boldsymbol{u}^{op})$, with $x_k = x$, can be rewritten as:

$$V_N^{op}(x) = x'Px + (\boldsymbol{u}^{op}(x) - \boldsymbol{u}_{uc}^{op}(x))' H (\boldsymbol{u}^{op}(x) - \boldsymbol{u}_{uc}^{op}(x))$$

Notice that when the constraint (6) is saturated, i.e., $\gamma = m$, we have that $\boldsymbol{u}^{op}(x) = \boldsymbol{u}_{uc}^{op}(x)$. Thus, $V_{uc}^{op}(x) = x'Px$. Therefore, it follows that

$$V_N^{op}(x) \geq x'Px \geq a_1 |x|^2, \quad (33)$$

where $a_1 = \lambda_{\min}(P)$.

Then, we obtain an upper bound for the cost function for the case when $\gamma \leq m$. To do this, we use the following suboptimal solution:

$$\tilde{\boldsymbol{u}} = \mathcal{L}_\sigma \boldsymbol{u}_{uc}^{op}(x) = [(L_\sigma K\hat{x})' \ (L_\sigma K\hat{x}_1)' \ \ldots \ (L_\sigma K\hat{x}_{N-1})']' \quad (34)$$

for a given feasible σ which satisfies that $|\sigma|_0 \leq \gamma$.

Thus, the optimal cost function satisfies that:

$$\begin{aligned} V_N^{op}(x) &\leq V_N(x, \tilde{\boldsymbol{u}}) \\ &= x'Px + \boldsymbol{u}_{uc}^{op}(x)'(\mathcal{L}_\sigma - I)' H (\mathcal{L}_\sigma - I) \boldsymbol{u}_{uc}^{op}(x) \end{aligned}$$

Then, considering (23), we obtain that

$$V_N(x, \tilde{\boldsymbol{u}}) \leq x'(P + W_\sigma)x \leq a_2 |x|^2, \quad (35)$$

where $a_2 = \lambda_{\max}(G)$. From (33) with (35), we have that

$$\begin{aligned} \Delta V_N^{op}(x) &= V_N^{op}(x_{k+1}) - V_N^{op}(x_k) \\ &\leq V_N(x, \tilde{\boldsymbol{u}}) - V_N^{op}(x_k) \\ &\leq x_{k+1}' G x_{k+1} - x_k' P x_k. \end{aligned} \quad (36)$$

Taking into account (7), it follows that

$$\Delta V_N^{op}(x) \leq J(x_k) - x_k' P x_k. \quad (37)$$

Then, (37) can be expanded as:

$$\begin{aligned} \Delta V_N^{op}(x) &\leq x'((A_K + \Delta_\sigma)' G (A_K + \Delta_\sigma) - P)x \\ &= x'(A_K' G A_K - P + 2A_K' G \Delta_\sigma + \Delta_\sigma' G \Delta_\sigma)x \end{aligned}$$

Since P satisfies (24), it follows that

$$\Delta V_N^{op}(x) \leq -x'(Q + K'RK - \Psi_\sigma)x \quad (38)$$

Considering that condition (26) holds, we have that the cost function is monotonically decreasing in k, i.e.,

$$\Delta V_N^{op}(x_k) < -a_3 |x_k|^2 < 0, \quad (39)$$

where $a_3 = \lambda_{\min}(Q + K'RK - \Psi_\sigma)$. Now, taking into account (35) and (39), it is posible to establish that

$$\begin{aligned} V_N^{op}(x_{k+1}) &\leq V_N^{op}(x_k) - a_3 |x_k|^2 \\ &\leq V_N^{op}(x_k) - \frac{a_3}{a_2} V_N^{op}(x_k) \leq \rho V_N^{op}(x_k) \end{aligned}$$

where $\rho = 1 - \frac{a_3}{a_2} \in [0, 1)$ and $a_3 > 0$ by (26). Hence, the optimal cost function will be exponentially bounded by

$$V_N^{op}(x_k) \leq \rho^k V_N^{op}(x_0), \quad \forall k \geq 0, x_0 \in \mathbb{R}^n. \quad (40)$$

Finally, considering (33) and (35), we obtain that

$$|x_k| \leq \frac{a_2}{a_1} \rho^k |x_0|, \quad (41)$$

for all $k \geq 0, x_0 \in \mathbb{R}^n$.

Consequently, the MPC closed-loop system (12) is globally exponentially stable. ■

Remark 4. Notice that decay rate ρ in Theorem 3 depends on the binary variable σ. One can use the results of this theorem to reduce the number of input to guarantee stability of the closed-loop while obtaining a desired performance in terms of the decay rate ρ.

Remark 5. The proposed algorithm can be used to minimize the ℓ_0-norm of the input by minimizing $\gamma_k \leq m$. To do this, at each sampling instant, k, one can initiate the algorithm with $\gamma = 0$ and then increases it until obtain the minimum value of γ which guarantees that $\Delta V_N(x) \leq 0$. Notice that, from (38), for the nominal case when $\gamma = m$, we have that $\Delta V_N^{op}(x) \leq -x'Qx$. Therefore, there is always possible to find a positive constant $\gamma \leq m$ which allows one to obtain a stabilizing predictive controller $\kappa_N(x)$.

Remark 6. It is important to highlight that the suboptimal input sequence, $\tilde{\boldsymbol{u}}(x)$, we have that $x'_{k+1}Gx_{k+1} = J(x_k)$. This implies that $\tilde{\boldsymbol{u}}(x)$ is a feasible stabilizing input sequence. Hence, there is always possible to find an optimal input sequence,$\boldsymbol{u}^{op}(x)$, that satisfies condition (7).

Remark 7. It is important to clarify that the suboptimal input sequence, $\tilde{\boldsymbol{u}}$, used in this proof, is generated using the same active inputs over the prediction horizon, see (30). Nevertheless, the active inputs in the resulting optimal input sequence, $\tilde{\boldsymbol{u}}^{op}(x)$, in general, may vary over the prediction horizon, as depicted in Fig. 1. This generates discontinuities in the system dynamics and, therefore, in the cost function. To establish stability of MPC with discontinuous cost functions, one must guarantee that the cost function is a uniformly strict Lyapunov function, i.e., $\Delta V^{op}(x) \leq -x'\Gamma x$, with Γ a positive definite matrix, which is the case of our problem, see Lazar et al. [2009].

Remark 8. The optimization algorithms used in this work may provide, in some cases, only a local optimum, $\boldsymbol{u}^{lop}(x)$, since they are solved via a heuristic approach. However, we initialize the optimization algorithm using the proposed suboptimal input sequence, $\tilde{\boldsymbol{u}}(x)$ in (34). Thus, it follows that $V_N(x, \boldsymbol{u}^{lop}(x)) \leq V_N(x, \tilde{\boldsymbol{u}}(x))$. Consequently, by applying optimality in (37), resulting local optima also guarantee stability.

5. SIMULATION STUDY

To verify the performance of the proposed control strategy (using method 2 in Section 3), we apply our results to the linear system (1) considering that $x \in \mathbb{R}^4$, $u \in \mathbb{R}^3$, and

$$A = \begin{bmatrix} 0.6122 & 0.2349 & -0.0021 & 0.1362 \\ -0.0366 & 0.7871 & 0.2047 & -0.1814 \\ -0.1941 & -0.1420 & 1.1499 & -0.2657 \\ -0.1864 & 0.0280 & 0.2942 & 1.2742 \end{bmatrix}, \quad (42)$$

$$B = \begin{bmatrix} -0.0151 & 0.2338 & 0.2710 \\ -0.3032 & -0.1504 & 0.0087 \\ 0.8390 & -0.0009 & -0.3200 \\ -0.0878 & -0.4431 & 0.0016 \end{bmatrix}. \quad (43)$$

Notice that matrix A has 2 unstable eigenvalues.

To design the quadratic cost function, $V_N(x, \boldsymbol{u})$, we choose $Q = \mathcal{I}_{4\times 4}$, $R = \mathcal{I}_{3\times 3}$, and a prediction horizon of $N = 4$. Then, the matrix P is obtained by solving the Riccati equation in (24), yielding to

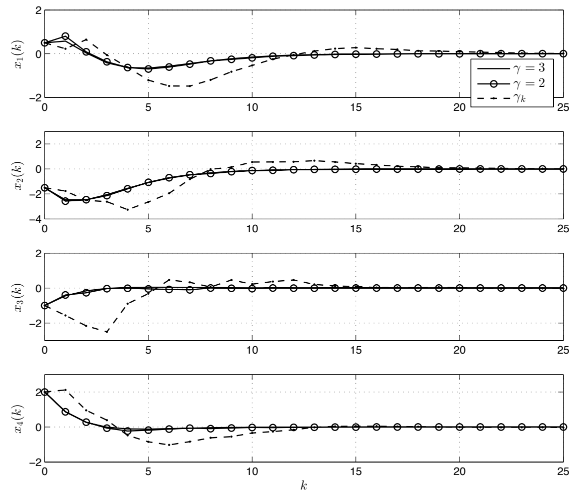

Fig. 2. State stabilization $x(k)$ for $\gamma = 2$ and variable $\gamma(k)$.

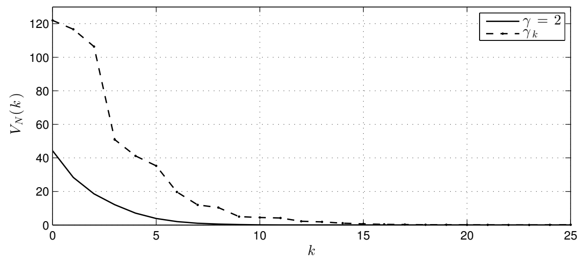

Fig. 3. Cost function $V_N(k)$ for $\gamma = 2$ and variable $\gamma(k)$.

$$P = \begin{bmatrix} 1.5578 & 0.2584 & -0.3077 & -0.4575 \\ 0.2584 & 2.5204 & 0.7421 & -1.1341 \\ -0.3077 & 0.7421 & 3.0840 & 0.4355 \\ -0.4575 & -1.1341 & 0.4355 & 6.8907 \end{bmatrix}, \quad (44)$$

$$K = \begin{bmatrix} 0.1405 & 0.1430 & -0.7880 & 0.1778 \\ -0.3576 & -0.0909 & 0.4489 & 1.5468 \\ -0.2499 & -0.0051 & 0.3768 & -0.2046 \end{bmatrix}. \quad (45)$$

In this case, based on Theorem 3, we found that $\sigma = [1\ 1\ 0]'$ satisfies the stabilizing condition (26) in Theorem 3. Thus, we know a priori that, it is possible to stabilize the system using only $\gamma = 2$ inputs. With this value of σ, we obtain that

$$\Psi = \begin{bmatrix} 0.3139 & 0.1144 & -0.3233 & 0.4152 \\ 0.1960 & 0.1787 & -0.0572 & 0.3868 \\ -0.2092 & 0.0682 & 0.4107 & -0.1013 \\ 0.5400 & 0.3226 & -0.3828 & 0.8892 \end{bmatrix}. \quad (46)$$

Therefore, it is possible to check that $Q + K'RK - \Psi \succ 0$. Consequently, the cost function and the system state can be bounded by (40) and (41) respectively, where in this case $a_1 = 1.3228$, $a_2 = 8.0175$, $a_3 = 0.3043$, and $\rho = 0.962$.

We are going to analyze the implementation of two MPC controllers. The first one, compute the optimal control actions provided a time-invariant ℓ_0 constraint, $\gamma = 2$ (given by the sufficient stability condition of Theorem 3). For the second controller, we let γ to be time variant, i.e., $\gamma(k)$ on compute the optimal controller that guarantee stability with the minimum number of active inputs.

Fig. 2 and Fig. 3 show that the controller using $\gamma = 2$ stabilizes the system quicker than the one with $\gamma(k)$ at the

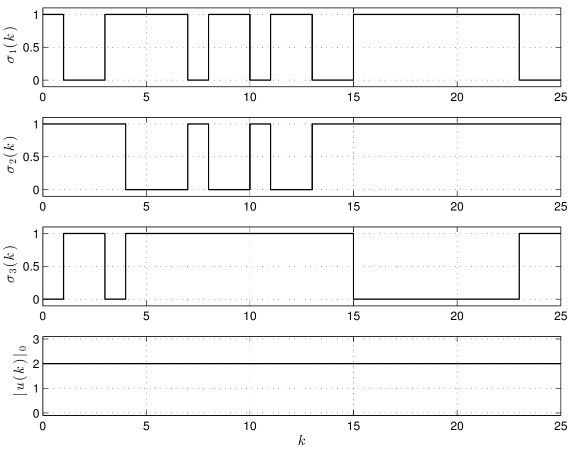

Fig. 4. $\sigma(k)$ and $|u(k)|_0$ for $\gamma = 2$.

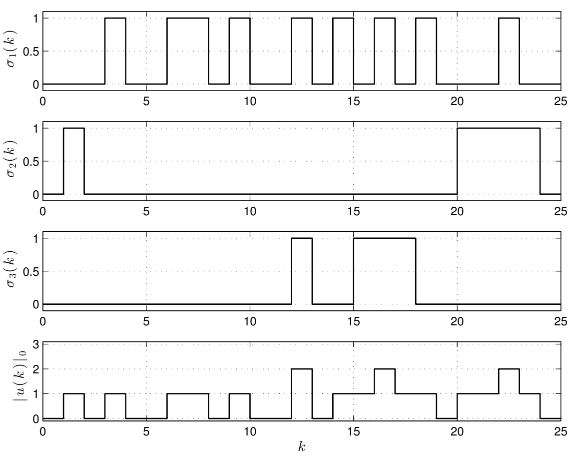

Fig. 5. $\sigma(k)$ and $|u(k)|_0$ for minimum $\gamma(k)$.

expense of using more active inputs at each instant (see Fig. 4 and Fig. 5). This fact reveals the trade-off between control performances ($V_N(k)$) and number of active control inputs ($|u(k)|_0$): the control performances get worse when the number of active control inputs is lower and vice versa. Note that the in some instants the system can be stabilized even with $|u(k)|_0 = 0$.

Although the controller with $\gamma(k)$ uses less active control inputs, in general, the computational effort (and so time calculation) is higher than for the case with a fixed γ because more combinations have to be explored. This makes the choice of the time-invariant γ resulting from Theorem 3 a good trade-off between performance, usage of non-zero control inputs and on-line computational effort.

When the system state is close to the origin, in some cases, a chattering effect may be produced. To avoid this, one can apply directly the suboptimal solution, i.e., $\kappa_N(x) = L_\sigma K x$, which provides that $|u(k)|_0 = \gamma$ while guaranteeing stability.

6. CONCLUSIONS

In this paper, we have considered the problem of designing a quadratic MPC technique that constrains the number of active inputs at each control horizon instant. We rewrite the MPC optimization problem with ℓ_0 input constraints as an iterative convex optimization procedure. We have also established a sufficient condition on the number of active inputs in order to guarantee exponential stability.

Further research may include studying robustness against disturbances and model uncertainties. The proposed technique provides the flexibility of incorporating any convex constraint such as bounded signals, $x \in \mathbb{X}$ and $u \in \mathbb{U}$ that can represent the control input saturation and/or model uncertainties. However, the stability analysis has to be extended to deal with this more complex problem.

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