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Adaptive Continuous Higher Order Sliding Mode Control

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Abstract: This paper proposes an adaptive continuous higher order sliding mode control scheme. The control law comprises a structure which provides smooth finite time stabilization of the origin for a nominal disturbance-free system formed from a chain of integrators, together with a super-twisting term to guarantee robustness to a class of twice differentiable uncertainty. The super-twisting scheme is adaptive in nature and seeks to minimize the size of the gains whilst still ensuring sliding. The adaptive scheme has a novel dual-layer structure based on equivalent injection concepts, which necessitates that the first and second derivatives of the uncertainty exist and are bounded, but knowledge of these bounds is not required. Simulation examples are provided to show the effectiveness of the proposed scheme.

1. INTRODUCTION

Sliding mode control systems have two significant features: the potential to create closed-loop systems with finite time convergence properties, and the ability to provide closed-loop insensitivity to a class of uncertainty (known as matched uncertainty). Both these properties are highly desirable but they come at a cost: invariably, the control laws include, possibly buried within the controller dynamics, discontinuous injection terms, which lead to the phenomenon of chattering [Shtessel et al., 2013]. The advent of so-called higher order sliding modes [Fridman and Levant, 2002] was heralded as ushering-in an era of chattering-free sliding mode controllers. Families of higher-order sliding schemes have been identified of arbitrary order, and these schemes have specific 'templates' and have a recursive or nested structure [Levant, 2003, 2006]. Many of the proofs of the results in this area rely on the concept of homogeneity [Kawski, 1995]. Whilst certainly such methods have lead to continuous sliding mode controllers – such as the very popular super-twisting controller – it has been subsequently shown that it is better to claim these controllers significantly mitigate chattering (if designed properly) rather than totally eliminating chattering. Extensive research has focussed on frequency based approaches to design such schemes to minimize chattering through the selection of the design freedom [Boiko, 2009]. Related to these advances in higher-order sliding mode schemes has been the discovery of classes of Lyapunov functions – particularly for 2-order sliding mode schemes (2SM) [Moreno and Osorio, 2006]. This in turn has revived interest in the development of adaptive schemes for the twisting and super-twisting control structures; see [Plestan et al., 2010].

Finite time controllers have also been extensively explored outside of the sliding mode literature. In optimal control theory for example there are several examples of closed-loop systems which converge to the origin in finite time [Ryan, 1983]. (For example a double integrator with time-optimal bang-bang feedback control.) Often these controllers have discontinuities – although they do not necessarily induce sliding modes. Finite-time closed-loop systems with *continuous dynamics* are considered in [Bhat and Bernstein, 1998] and the references therein. In [Bhat and Bernstein, 2005] finite-time stability of homogeneous systems is considered. This paper is the source of the celebrated result that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has a negative homogeneity degree. In [Bhat and Bernstein, 2005] this result is used to demonstrate the existence of a continuous finite-time stabilizing feedback controller for a chain of integrators, and provides an explicit representation for such a controller involving a small scalar parameter. The theory is elegant and the controller is very appealing, but it can be easily shown that in the presence of unknown and uncompensated disturbances, finite time convergence to the origin is lost: i.e. it lacks robustness.

This paper aims to create a continuous state feedback higher order sliding mode control law which robustly provides finite time convergence of the states to the origin despite the presence of unknown disturbances/uncertainties. It employs the continuous control law from [Bhat and Bernstein, 2005] in conjunction with an adaptive super-twisting structure to create a robust finite time scheme. In its most generic form the adaptive structure requires only the knowledge that the second derivative of the disturbance is bounded – although an explicit value for this bound is not required by the controller.

2. A CONTINUOUS HOSMC SCHEME

Consider the sliding variable dynamics $\sigma \in \mathbb{R}$ satisfying

$$\sigma^{(r)}(t) = a(t) + u(t) \tag{1}$$

where $u(t) \in \mathbb{R}$ is the manipulated variable and the disturbance $a(t) \in \mathbb{R}$ is unknown but satisfies $|\dot{a}(t)| \leq a_1$, $a_1 > 0$.

It is well known that a continuous control law that drives $\sigma, \dot{\sigma}, ..., \sigma^{(r)} \to 0$ in finite time for the unperturbed system in (1) can be designed using the results from the work of [Bhat and Bernstein, 2005]. The relevant result can be summarized in the following theorem:

Theorem 1 [Bhat and Bernstein, 2005] Let the scalar coefficients $\gamma_1, \gamma_2, ..., \gamma_r$ be such that the r-th order polynomial $p^r + \gamma_r p^{r-1} + ... + \gamma_2 p + \gamma_1$ is Hurwitz, and consider the system in (1) with $a(t) \equiv 0$. Then there exists an $\epsilon \in (0, 1)$ such that for every $\alpha \in (1-\epsilon, 1)$ the origin $\sigma, \dot{\sigma}, ..., \sigma^{(r)} = 0$ is a finite time stable equilibrium under the feedback

$$u(t) = -\gamma_1 |\sigma|^{\alpha_1} \operatorname{sgn}(\sigma) \ldots - \gamma_r |\sigma^{(r-1)}|^{\alpha_r} \operatorname{sgn}(\sigma^{(r-1)}) \quad (2)$$

where the scalars $\alpha_1, \alpha_2, ..., \alpha_r$ satisfy

$$\alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}, i = 2, ..., r \quad (3)$$

with $\alpha_{r+1} = 1$ and $\alpha_r = \alpha$.

Remark 1 It can be easily demonstrated that if $a(t) \neq 0$, the system in (1)-(3) may not even converge to the origin.

Assuming that the smooth disturbance $a(t)$ is not identically equal to zero and its derivative is bounded $|\dot{a}(t)| \leq a_1$, the objective is to design a sliding mode disturbance observer to exactly reconstruct $a(t)$ in finite time, and then compensate for it by means of control. The results are formulated in the following lemma and theorems.

Lemma 1: Define an auxiliary sliding variable s given by

$$s(t) = \sigma^{(r-1)}(t) - z(t) \quad (4)$$

$$\dot{z}(t) = u(t) + v(t) \quad (5)$$

then s and its derivative $\dot{s}$ are driven to zero in finite time and remain at zero for all subsequent time by the feedback (super-twisting) control law

$$v(t) = \lambda |s(t)|^{1/2} \operatorname{sgn}(s(t)) + v_1(t) \quad (6)$$

$$\dot{v}_1(t) = \beta \operatorname{sgn}(s(t)) \quad (7)$$

where the design scalars $\lambda = 1.5a_1^{1/2}$, $\beta = 1.1a_1$ and the $\sigma-$dynamics are defined by (1).

Proof: Taking into account equations (1) and (4)-(5), the s-dynamics are described by

$$\dot{s}(t) = a(t) - v(t) \quad (8)$$

It is well know that choosing $\lambda = 1.5a_1^{1/2}$ and $\beta = 1.1a_1$ induces a 2SM in finite time, (see for example [Shtessel et al., 2013]) and the Lemma is proven. ■

Remark 2 Another possible choice of gains for the super-twisting structure in (6)-(7) is

$$\beta > a_1 \quad (9)$$

$$\lambda > \sqrt{\frac{2}{\beta - a_1}} \frac{(\beta + a_1)(1+q)}{1-q} \quad (10)$$

where $|a(t)| < q\Omega_m$, $0 < q < 1$, $|v(t)| \leq \Omega_m$.

In order to enforce the condition $|v(t)| \leq \Omega_m$ the term $\dot{v}_1$ in (7) can be modified to be

$$\dot{v}_1(t) = \begin{cases} -v_1(t) & \text{if } |v(t)| > \Omega_m \\ \beta \operatorname{sgn}(s(t)) & \text{otherwise} \end{cases} \quad (11)$$

Lemma 2: The sliding mode disturbance observer in (4)-(7) exactly reconstructs $a(t)$ in finite time as

$$a(t) = v(t) \quad (12)$$

Proof: The control (injection) signal v in (6)-(7) induces and maintains a 2SM. Consequently $s = \dot{s} = 0$ in finite time. Therefore, during the 2-SM, the left hand side of (8) equals zero in finite time, and equation (12) holds for all subsequent time. ■

Theorem 2: Consider the system in (1) with $a(t)$ not identically equal to zero but with bounded derivative $|\dot{a}(t)| \leq a_1$. Let $\gamma_1, \gamma_2, ..., \gamma_r$ be such that the polynomial $p^r + \gamma_r p^{r-1} + ... + \gamma_2 p + \gamma_1$ is Hurwitz. Then there exists an $\epsilon \in (0, 1)$ such that for every $\alpha \in (1-\epsilon, 1)$ the origin $\sigma, \dot{\sigma}, ..., \sigma^{(r)} = 0$ is a finite time stable equilibrium under the feedback

$$u(t) = -\gamma_1 |\sigma|^{\alpha_1} \operatorname{sgn}(\sigma) \ldots - \gamma_r |\sigma^{(r-1)}|^{\alpha_r} \operatorname{sgn}(\sigma^{(r-1)}) - v \quad (13)$$

where v is defined by (6)-(7) and the scalars $\alpha_1, \alpha_2, ..., \alpha_r$ satisfy (3) with $\alpha_{r+1} = 1$ *and* $\alpha_r = \alpha$.

Proof: This follows from Lemmas 1 & 2 ■

Remark 3 The continuous controller (13) can be claimed to be a continuous HOSM controller for the system in (1), since it drives $\sigma, \dot{\sigma}, ..., \sigma^{(r)} \to 0$ in finite time in the presence of the smooth disturbance $a(t)$ with bounded derivative $|\dot{a}(t)| \leq a_1$, $a_1 > 0$.

3. A DUAL LAYER ADAPTIVE DISTURBANCE OBSERVER

As argued in the previous section, the continuous HOSM in (13) contains the term v that represents a reconstruction of the unknown disturbance term $a(t)$. The super-twisting injection term in (6) contains a component v_1, whose derivative is a discontinuous high frequency switching function with gain $\beta > a_1$. In order to reduce chattering, it is desirable to make β as close to a_1 as possible whilst ensuring $\beta > a_1$. Therefore, assuming that the gain λ can be selected large enough so that the second equation in (9) holds, the aim is to adapt $\beta(t)$ in equation (7) so that $\beta(t)$ is close to a_1 whilst satisfying condition in (9). This reduces the amplitude of the high frequency part of the super-twisting term in equation (7), which mitigates chattering. In this paper a scheme building on the 'equivalent control' approach of [Utkin and Poznyak, 2013] will be employed.

Remark 4 Since the gain λ is selected large enough and Ω_m is not a control parameter, the original notation of the super-twisting control in (6)-(7) has been retained.

3.1 Equivalent control in super-twisting control adaptation

Consider the super-twisting structure arising from equations (6), (7) and (8) written out in familiar form as

$$\dot{s}(t) = -\lambda |s(t)|^{1/2} \operatorname{sgn}(s(t)) + v_2(t) \tag{14}$$

$$\dot{v}_2(t) = \varphi(t) - w(t) \tag{15}$$

where $\varphi(t) := \dot{a}(t)$ and

$$w(t) = k(t)\operatorname{sgn}(s(t)) \tag{16}$$

The objective is to devise an adaptive scheme for the (now) time-varying gain $k(t)$ so that a 2SM is achieved and maintained, but which also attempts to ensure $k(t)$ is as small as possible. It will be further assumed that $|\dot{\varphi}(t)| \leq a_2$, $a_2 > 0$ (which implies that the original disturbance term $a(t)$ must be twice differentiable). During a 2-SM, $s = v_2 = 0$ (which is exactly equivalent to the condition $s = \dot{s} = 0$), and the 'equivalent control' $w_{eq}(t) = \varphi(t)$. Although the notion of 'equivalent control' was conceived as an abstraction to facilitate the understanding and analysis of the dynamics during the sliding mode, an arbitrary close approximation can be created by appropriate filtering of the switching signal. For the purposes of the developments which follow, it is assumed that an approximation $\bar{w}_{eq}(t)$ is available for use in the adaption schemes.

In the remainder of the paper, whilst the bound a_1 for the absolute values of the derivative of $a(t)$ is assumed to exist, its precise value is not known. Two situations will now be considered: firstly the case when a_2 is known; and secondly the case when a_2 is unknown.

3.2 An adaptive super-twisting observer with known a_2

Here it is assumed that $w_{eq}(t)$ in (16) is available (for instance, by filtering $w(t)$ in (16)). Define

$$\delta(t) = k(t) - \frac{1}{\epsilon_1} |w_{eq}(t)| - \epsilon_0 \tag{17}$$

where $0 < \epsilon_1 < 1$ and $\epsilon_0 > 0$ is a small real number. Define the *first layer* of the *dual-layer* adaptation algorithm, associated with the gain $k(t)$ in (16) according to

$$\dot{k}(t) = -\left(\rho_0 + \rho(t)\right) \operatorname{sgn}(\delta(t)) \tag{18}$$

where $\rho_0 > 0$ is a small scalar. The adaptive gain $\rho(t)$ associated with the *second layer* of the adaptation algorithm is chosen to satisfy

$$\dot{\rho}(t) = \gamma |\delta(t)| + \rho_0 \sqrt{\gamma} \operatorname{sgn}(e(t)) \tag{19}$$

where γ is a positive scalar (to be selected by the designer) and the term

$$e(t) = \frac{a_2}{\epsilon_1} - \rho(t) \tag{20}$$

where a_2 is the upper bound on the absolute value of the derivative of $\varphi(t)$ (which is assumed to be known).

Theorem 3: Consider the system in (1) with a twice differentiable disturbance $a(t)$ subject to $|\dot{a}(t)| \leq a_1$ and $|\ddot{a}(t)| \leq a_2$, where a_2 is known. Then the sliding mode observer in equations (4)-(5), (14) and (16), with the dual layer gain-adaptation in equations (18)-(20) reconstructs the disturbance $a(t)$ in finite time as

$$\hat{a}(t) = v(t) = -\lambda |s(t)|^{1/2} \operatorname{sgn}(s(t)) - \int k(t)\operatorname{sgn}(s(t))dt \tag{21}$$

Furthermore the variables $\delta(t)$ and $e(t)$ converge to zero in finite time and the gains $k(t)$ and $\rho(t)$ remain bounded.

Proof: It is easy to verify that the δ-dynamics from (17) satisfy

$$\dot{\delta}(t) = \dot{k}(t) - \frac{1}{\epsilon_1}\frac{d}{dt} |w_{eq}(t)| = \dot{k}(t) - \frac{1}{\epsilon_1}\frac{d}{dt} |\varphi(t)| \tag{22}$$

and therefore

$$\delta\dot{\delta} \leq -\delta(\rho_0 + \rho(t))\operatorname{sgn}(\delta(t)) + |\delta| \frac{a_2}{\epsilon_1} = (-\rho_0 + e) |\delta| \tag{23}$$

Since a_2 is a constant, from the definition of $e(t)$ in (20),

$$\dot{e}(t) = -\gamma |\delta| - \rho_0 \sqrt{\gamma} \operatorname{sgn}(e(t)) \tag{24}$$

and

$$e\dot{e} = -\gamma e |\delta| - \rho_0 \sqrt{\gamma} |e| \tag{25}$$

Consider a Lyapunov function candidate for the dynamics in (22) and (24) of the form

$$V = \frac{1}{2}\delta^2 + \frac{1}{2\gamma} e^2 \tag{26}$$

Taking the derivative of V along the trajectories of (22) and (24), and using the expressions in (23) and (25), it follows that

$$\begin{aligned}\dot{V} &\leq (-\rho_0 + e) |\delta| - e |\delta| - \rho_0 \frac{1}{\sqrt{\gamma}} |e| \\ &= -\rho_0 \sqrt{2} \left(\frac{1}{\sqrt{2}} |\delta| + \rho_0 \frac{1}{\sqrt{2\gamma}} |e| \right) \\ &\leq -\rho_0 \sqrt{2} V^{1/2}\end{aligned} \tag{27}$$

Inequality (27) guarantees *finite time* convergence $V \to 0$, which means the finite time convergence of $\delta, e \to 0$. As a consequence both δ and e remain bounded. Furthermore since $k(t) = \delta(t) + \frac{1}{\epsilon_1}|w_{eq}(t)| + \epsilon_0$ and $\rho(t) = \frac{a_2}{\epsilon_1} + e(t)$, the variables $k(t)$ and $r(t)$ also remain bounded as claimed. Since $\delta = e = 0$ in finite time, and from the definition of $\delta(t)$ in (17) the following equality holds (in finite time)

$$k(t) = |w_{eq}(t)| + \frac{1 - \epsilon_1}{\epsilon_1} |w_{eq}(t)| + \epsilon_0 \tag{28}$$

and therefore

$$k(t) = |\dot{a}(t)| + \frac{1 - \epsilon_1}{\epsilon_1} |\dot{a}(t)| + \epsilon_0 > |\dot{a}(t)| \tag{29}$$

This means that equation (9) holds, and therefore selecting λ sufficiently large, guarantees the finite time convergence to zero of the dynamics in (14), (16). During the 2-SM, equation (12) holds, and therefore $\hat{a}(t)$ perfectly reconstructs $a(t)$ in finite time. ■

3.3 An adaptive super-twisting observer with unknown a_2

The formulation in the previous section relied on the fact that a_2 is known because the adaptive scheme employed the variable $e(t) = \frac{a_2}{\epsilon_1} - \rho(t)$. If a_2 is not available a different scheme is required. Consider instead, as the second layer of the adaptive law,

$$\dot{\rho}(t) = \gamma |\delta(t)| \tag{30}$$

where γ is a positive scalar. Clearly using (30) in conjunction with the first adaptive layer (17) means the overall adaptive scheme is now independent of knowledge of both a_1 and a_2. From the definition of $e(t)$ in (20), and using the new adaptive term in (30), it follows

$$e\dot{e} = -\gamma e |\delta| \tag{31}$$

Theorem 4: Consider the system in (1) with a twice differentiable disturbance $a(t)$ subject to $|\dot{a}(t)| \leq a_1$,

$|\ddot{a}(t)| \leq a_2$. Suppose that both a_1 and a_2 are unknown. Then the sliding mode observer in equations (4)-(5), (14), (16) with the dual layer gain-adaptation in equations (18),(30) reconstructs the disturbance $a(t)$ in *finite time* in accordance with (21). Furthermore the error variable $\delta(t)$ converges to zero asymptotically and the gains $k(t)$ and $\rho(t)$ remain bounded.

Proof: As in the proof of Theorem 3 consider a Lyapunov function candidate for the dynamics in (17) and (20) of the form

$$V = \frac{1}{2}\delta^2 + \frac{1}{2\gamma}e^2 \tag{32}$$

Taking the derivative along the trajectories of $e(t)$ and $\delta(t)$, it follows from (23) and (31) that

$$\dot{V} \leq (-\rho_0 + e)\,|\delta| - e\,|\delta| \;\; = \;\; -\rho_0|\delta(t)| \tag{33}$$

Since $\dot{V} \leq 0$, it follows immediately that the variables $e(t)$ and $\delta(t)$ are bound for all time. Using arguments similar to those in the proof of Theorem 3, the variables $k(t)$ and $\rho(t)$ also remain bounded as claimed. Furthermore using LaSalle's invariance principle [Khalil, 1992], the error variable $\delta(t) \to 0$ (asymptotically) as claimed. Since $\delta \to 0$ asymptotically, consequently, in *finite time* t_0 (say), $|\delta(t)| < \epsilon_0/2$, and this inequality holds for all time $t > t_0$. Since by definition

$$|\delta(t)| = \left| k(t) - \frac{1}{\epsilon_1}|w_{eq}(t)| - \epsilon_0 \right|$$

it follows that

$$k(t) - \frac{1}{\epsilon_1}|w_{eq}(t)| - \epsilon_0| > -\epsilon_0/2$$

for all $t > t_0$. Therefore

$$\begin{aligned} k(t) &= |w_{eq}(t)| + \frac{1-\epsilon_1}{\epsilon_1}\,|w_{eq}(t)| + \epsilon_0/2 \\ &= |\dot{a}(t)| + \frac{1-\epsilon_1}{\epsilon_1}\,|\dot{a}(t)| + \epsilon_0/2 \\ &> |\dot{a}(t)| \end{aligned} \tag{34}$$

This means that equation (9) holds, and therefore selecting λ sufficiently large, guarantees the *finite time* convergence to zero of the dynamics in (14)-(16). During the 2-SM, equation (12) holds and reconstructs $a(t)$ in finite time. ■

4. ADAPTIVE CONTINUOUS HOSMC

Again, consider the sliding variable dynamics in (1). In Theorem 2 a continuous HOSM control driven by the super-twisting-based disturbance observer, which drives $s, \dot{s} \to 0$ in the presence of the smooth bounded disturbance $a(t)$, was formulated. Theorems 3 and 4 give formulations of adaptive super-twisting equivalent control-based disturbance observers that reconstruct the disturbance $a(t)$ in (1) depending on whether or not the bound a_2 on the second derivative of this disturbance is known.

The main results are formulated in the following theorems.

Theorem 5: Consider the system (1) with a twice differentiable disturbance $a(t)$ satisfying $|\dot{a}(t)| \leq a_1$, $|\ddot{a}(t)| \leq a_2$ and suppose whilst a_1 is unknown, a_2 is known. Let the coefficients $\gamma_1, \gamma_2, ..., \gamma_r$ be such that the polynomial $p^r + \gamma_r p^{r-1} + ... + \gamma_2 p + \gamma_1$ is Hurwitz. Then there exists an $\epsilon \in (0,1)$ such that for every $\alpha \in (1-\epsilon, 1)$ the origin $\sigma, \dot{\sigma}, ..., \sigma^{(r)} = 0$ is a finite time stable equilibrium under the feedback control in (13), where $\alpha_1, \alpha_2, ..., \alpha_r$ satisfy (3) with $\alpha_{r+1} = 1$ and $\alpha_r = \alpha$; v is defined in (21) and the adaptive scheme is given by equations (17)-(20).

Theorem 6: Consider the system in (1) and suppose the disturbance $a(t)$ has first and second derivatives satisfying $|\dot{a}(t)| \leq a_1$, $|\ddot{a}(t)| \leq a_2$ where both a_1 and a_2 are unknown. Let the coefficients $\gamma_1, \gamma_2, ..., \gamma_r$ be such that the polynomial $p^r + \gamma_r p^{r-1} + ... + \gamma_2 p + \gamma_1$ is Hurwitz. Then there exists an $\epsilon \in (0,1)$ such that for every $\alpha \in (1-\epsilon, 1)$ the control (13) drives $\sigma, \dot{\sigma}, ..., \sigma^{(r)} \to 0$ in finite time where $\alpha_1, \alpha_2, ..., \alpha_r$ satisfy (3) with $\alpha_{r+1} = 1$ and $\alpha_r = \alpha$; v is defined by equation (21) and the adaptive scheme is given by (17), (18), (20) and (30).

The proof of the Theorems 5 and 6 follows straightforwardly from Theorems 3 and 4.

Remark 5 The adaptive equivalent control-based control formulated in Theorems 5 and 6 can be interpreted as continuous HOSM control since after being applied to the perturbed system (1) of relative degree r it drives $\sigma, \dot{\sigma}, ..., \sigma^{(r)} \to 0$ in the presence of a non-vanishing disturbance $a(t)$ with bounded first and second derivatives.

5. SIMULATIONS

The system in (1) is simulated with relative degree ($r = 3$) plant. The coefficients of the underlying Hurwitz polynomial $p^3 + \gamma_3 p^2 + \gamma_2 p + \gamma_1$ associated with the controller are selected as $\gamma_1 = 8$, $\gamma_2 = 12$, $\gamma_3 = 6$. The exponents α_1, α_2, α_3 in equation (3) are calculated based on the seed $\alpha = 0.8$. Explicitly they are given by $\alpha_1 = 0.56$, $\alpha_2 = 0.66$ and $\alpha_3 = 0.8$. In the simulations which follow the initial conditions are selected as $\sigma(0) = 1$, $\dot{\sigma}(0) = 0.5$, $\ddot{\sigma}(0) = 0$. The disturbance in (1) is taken as $a(t) = 2\sin(t)$. The parameter λ in the injection term (7) is supposed to be sufficiently large and is taken as $\lambda = 4.75$. The other parameters of the observer (18), (19) that are used in simulations are $\rho_0 = 1$, $\gamma = 10$ and $\epsilon_1 = 0.99$.

First the system in (1) is simulated using the control law $u(t)$ in (2) with and without the disturbance $a(t)$.

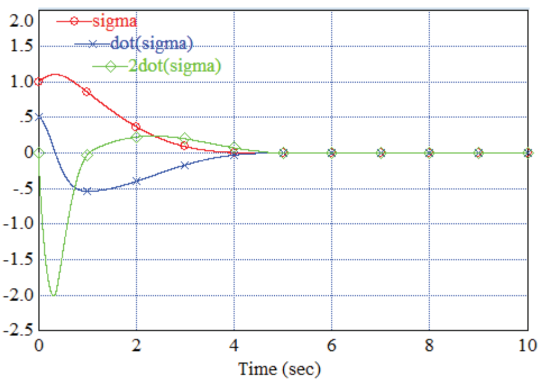

Fig. 1. Finite-time stabilization in unperturbed system

In Fig. 1, finite time convergence to the origin can be seen after 4.5 secs. Fig 2 shows that finite time stability of the origin is not achieved in the presence of the disturbance.

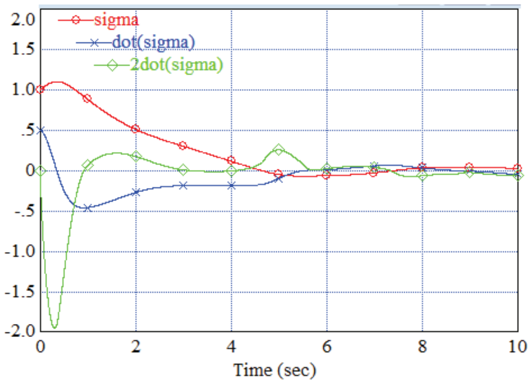

Fig. 2. Output stabilization in perturbed system

5.1 A case with known a_2

In this subsection the perturbed system in (1) is simulated using the control in equations (13)-(20). In this case the upper bound a_2 on the disturbance's second derivative is assumed to be known. During the simulation it is assumed that $|\ddot{a}| \leq a_2$ with $a_2 = 2$.

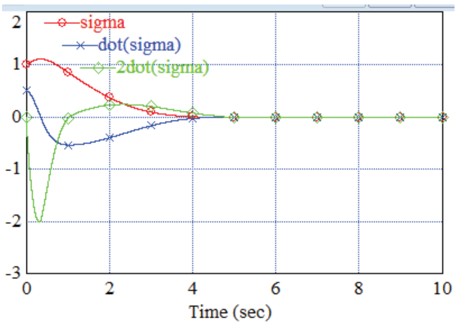

Fig. 3. Finite-time output stabilization of perturbed system when a_2 is known

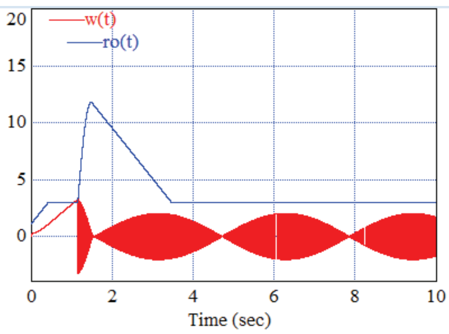

Fig. 4. The second layer of adaptation: $\rho(t)$ and $w(t)$ when a_2 is known

Fig. 3 shows that the nominal disturbance-free performance is recovered, and finite time convergence is achieved with the disturbance estimation scheme included. Fig. 4 shows the high frequency switching component $w(t)$ and also the adaptive term $\rho(t)$ which is shown to converge to the upper-bound on the 2nd derivative of the disturbance term (in this case $a_2 = 2$). Fig. 5 shows the modulation

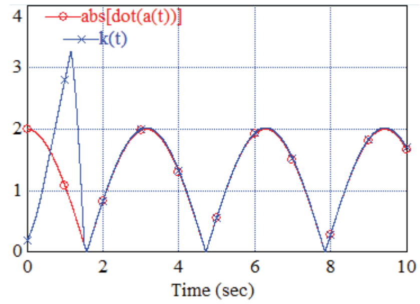

Fig. 5. The first layer of adaptation: $k(t)$ and $|a(t)|$ when a_2 is known

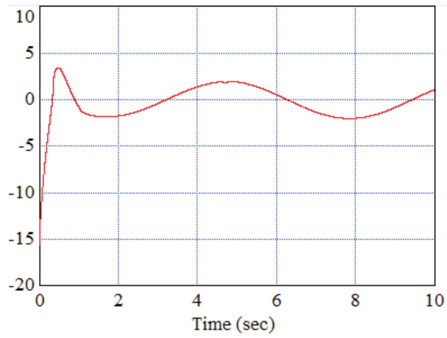

Fig. 6. The continuous HOSMC when a_2 is known

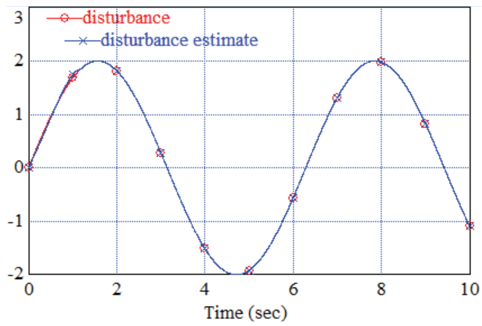

Fig. 7. The disturbance estimation when a_2 is known

term $k(t)$ tracking $|a(t)|$ very accurately (since in this case $\epsilon_1 = 0.99 \approx 1$). Fig. 6 shows the control signal $u(t)$, which is clearly smooth and chatter-free, and yet compensates for the disturbance $a(t)$. Fig. 7 shows that $\hat{a}(t)$ accurately tracks/re-creates the unknown disturbance $a(t)$.

5.2 A case with unknown a_2

In this subsection the perturbed system (1) is simulated while controlled by $u(t)$ in equations (17), (18), (20) and (30). In this case the bound a_2 on the second derivative of the disturbance is assumed to be unknown.

Fig. 8 shows that, as in the previous subsection, the nominal disturbance-free closed-loop performance is recovered, and finite time convergence is achieved. Fig. 9 shows the high frequency switching component $w(t)$ and also the adaptive term $\rho(t)$. It can be seen that $\rho(t)$ remains bounded and asymptotically converges to a value close

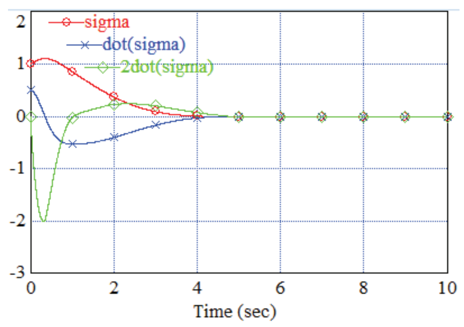

Fig. 8. Finite-time output stabilization of the perturbed system with gain adaptation (a_2 is unknown)

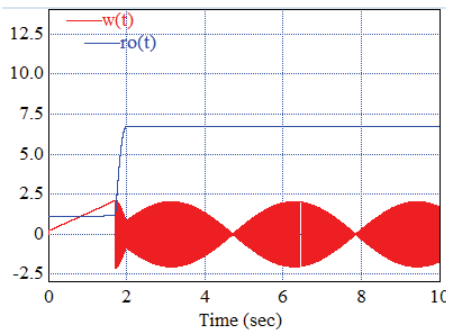

Fig. 9. The second layer of adaptation: $\rho(t)$ and $w(t)$ (a_2 is unknown)

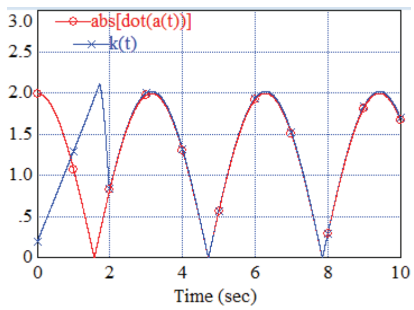

Fig. 10. The first layer of adaptation: $k(t)$ and $|a(t)|$ (a_2 is unknown)

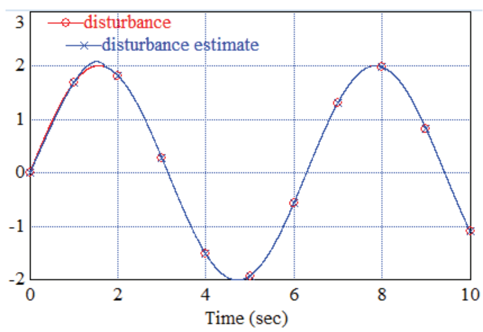

Fig. 11. The disturbance estimation (a_2 is unknown)

to 7.0 (which is greater than the true value of $a_2 = 2$). This is perfectly consistent with the theory, since the dual-layer adaptive scheme in (17), (18), (20) and (30) cannot guarantee that $\rho(t) \rightarrow a_2$. However Fig. 10 shows the modulation term $k(t)$ still tracks $|a(t)|$ very accurately (since in this case $\epsilon_1 = 0.99 \approx 1$). Finally Fig. 11 shows that $\hat{a}(t)$ accurately tracks the unknown disturbance $a(t)$.

6. CONCLUSIONS

This paper has proposed an adaptive continuous higher order sliding mode control scheme. The control law comprises a structure which provides smooth finite time stabilization of the origin for a nominal disturbance-free system formed from a chain of integrators, together with a super-twisting term to ensure robustness to a class of twice differentiable uncertainty. The super-twisting scheme is adaptive in nature and seeks to minimize the size of the gains whilst still ensuring sliding. The adaptive scheme has a novel dual-layer structure based on knowledge of the equivalent injection and necessitates that only the first and second derivatives exist and are bounded, but knowledge of these bounds is not required.

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