Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Large deviations in continuous-time linear single-input control systems ⋆

Boris T. Polyak * **Georgi V. Smirnov** **

* *Institute for Control Sciences of Russian Academy of Sciences, 65 Profsoyuznaya st., Moscow 117997, Russia, and ITMO University, St-Petersburg, Russia (e-mail: boris@ipu.ru)*
** *Centre of Physics, Department of Mathematics and Applications, University of Minho, Campus de Gualtar, 4710-057 Braga, Portugal (e-mail: smirnov@math.uminho.pt)*

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Abstract: By the pole assignment theorem for linear controllable systems, there exists a linear feedback yielding the closed-loop system with any given set of eigenvalues. Systems with non-zero initial conditions are studied, and we prove that large deviations of trajectories arise for large eigenvalues as well as for small ones. Invariant ellipsoids techniques to minimize the maximal deviation are provided and its properties are examined.

1. INTRODUCTION

Consider a single-input continuous time linear control system

$$\dot{x} = Ax + bu, \ u \in \mathbf{R}, \ b \in \mathbf{R}^n. \qquad (1)$$

If the vectors $b, Ab, \ldots, A^{n-1}b$ are linearly independent, then the system is controllable and, by the pole assignment theorem, there exists a linear feedback defined by a vector K such that the equilibrium position $x = 0$ of the linear differential equation

$$\dot{x} = Ax + b\langle K, x\rangle \qquad (2)$$

is asymptotically stable. Moreover, one can generate a linear system with any given set $\Lambda \in \mathbf{C}$ of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Therefore, by choosing an appropriate linear feedback it is possible to obtain a closed-loop system with an arbitrary given damping speed. However, the trajectories of the closed-loop system with fast damping significantly deviate from the equilibrium position during the initial phase of the stabilization for some non-zero initial conditions. This phenomenon is called the "peak" effect and the large deviation is referred to as an overshoot. The existence of the peak effect was discovered in Izmailov [1987]. More precisely, Izmailov proved that there exists a constant $\gamma = \gamma(A, b) > 0$, such that, if $\{\lambda_1, \ldots, \lambda_n\}$, are the eigenvalues of system (2), then the condition $\mathrm{Re}\lambda_j \leq -\sigma < 0$, $j = \overline{1, n}$, implies

$$\sup_{0 \leq t \leq \frac{1}{\sigma}} \sup_{|x_0|=1} |x(t, x_0)| \geq \gamma\sigma^{n-1}. \qquad (3)$$

⋆ The research was supported by the Portuguese Foundation for Science and Technologies (FCT), the Portuguese Operational Programme for Competitiveness Factors (COMPETE), the Portuguese National Strategic Reference Framework (QREN), and the European Regional Development Fund (FEDER) through Project VARIANT (PTDC/MAT/111809/2009). The work of the first author was also supported by grants RFFI 14-07-00067, 14-08-01230 and Government of Russian Federation, Grant 074-U01, Ministry of Education and Science of Russian Federation, Project 14.Z50.31.0031.

The proof given by Izmailov was significantly simplified in Bushenkov and Smirnov [1997]. In Sussmann and Kokotovic [1991] Izmailov's result was generalized in order to obtain estimates for so-called peaking exponent of an output.

Surprisingly this effect is not specific for large eigenvalues only. In Smirnov et al. [2009] it was shown that for a set of small eigenvalues we meet large values of $|x(t)|$ for large time instances t. This is why we prefer term "large deviations" instead of traditional "peak". Moreover, the situation with "large deviations" is well known in numerical analysis. In the famous paper Moler and Van Loan [2003] there are many examples of the norm of matrix exponent $\varphi(t) = ||e^{At}||_2 = \max_{|x(0)|_2=1} |e^{At}x(0)|_2$ having huge "hump" (or several "humps"), see e.g. Figures 3 and 5 there.

In this work we analyze the situation with transition process in linear systems for nonzero initial conditions more deeply. First we treat the systems in companion form and provide examples with all eigenvalues equal real $-\sigma < 0$ and estimate deviations of the trajectory for specific initial conditions. We show that the large deviation effect is present both for σ large or small. Then we calculate constant γ in (3). Next, the cases of other locations of eigenvalues are examined, for instance if some (at least two) of the eigenvalues have very big or very small moduli. We conclude analysis of systems in companion form with the important open problem. It reads: *Independently on closed-loop eigenvalues, if there exist* $\alpha_n > 1, |x_0| = 1$ *such that trajectory deviation exceeds* α_n. Then we extend the results for systems in general (not canonical) form. In Section 3 we describe and justify a numerical method to design a feedback guaranteeing small overshooting. The method consists of construction of invariant ellipsoids for the closed-loop system using semidefinite programming (SDP) approach (Boyd et al. [1994]).

2. ESTIMATES FOR LARGE DEVIATIONS

The Euclidean norm is denoted by $|x|_2$. We fix the symbol $|x|$ to denote the $|x|_\infty = \max\left\{|x_m| : m = \overline{1,n}\right\}$ norm of $x \in R^n$. The set $\Lambda = \{\lambda_1, \dots, \lambda_n\}$, with $\mathrm{Re}\lambda_j < 0$, $j = \overline{1,n}$ is the set of the eigenvalues of system (2). The k-th column of the identity matrix I is denoted by e_k.

2.1 Motivating examples

The nature of large deviations during the transition process can be easily seen from the following simple examples with all equal eigenvalues. Consider the differential equation $(d/dt + \sigma)^n y(t) = 0$, where $\sigma > 0$ and $n \geq 2$. Its solution satisfying the initial conditions $y(0) = 1$, $y^{(k)}(0) = 0$, $k = \overline{1, n-1}$ is given by the formula

$$y(t) = \left(\sum_{j=0}^{n-1} \frac{(\sigma t)^j}{j!}\right) e^{-\sigma t}.$$

Since $y'(t) = -\frac{-\sigma^n t^{n-1}}{(n-1)!} e^{-\sigma t}$, we get $y^{(n-1)}(t) = -\frac{\sigma^n t e^{-\sigma t}}{n-1} L^{(1)}_{n-2}(\sigma t)$, where $L^{(\alpha)}_k(\theta) = \frac{\theta^{-\alpha} e^\theta}{k!} \frac{d^k}{d\theta^k}\left(\theta^{n+\alpha} e^{-\theta}\right)$ is the generalized Laguerre polynomial. Thus we have

$$\left| y^{(n-1)}\left(\frac{\theta_n}{\sigma}\right)\right| = \beta_n \sigma^{n-1}, \beta_n = \frac{\left|\theta_n e^{-\theta_n} L^{(1)}_{n-2}(\theta_n)\right|}{n-1},$$

where $\theta_n = \mathrm{argmax}_{\theta \geq 0}\left|\theta e^{-\theta} L^{(1)}_{n-2}(\theta)\right|$. Notice that β_n does not depend on σ.

Now consider another solution

$$y(t) = \left(\sum_{j=0}^{n-1} \frac{((1+\sigma)t)^j}{j!}\right) e^{-\sigma t}. \quad (4)$$

It is easy to check that $y(\cdot)$ satisfies the initial conditions $y^{(k)}(0) = 1$, $k = \overline{0, n-1}$. Obviously we have

$$y\left(\frac{n}{\sigma}\right) \geq \left(1 + \frac{1}{\sigma}\right)^n \left(\frac{n}{e}\right)^n \frac{1}{n!} \geq \frac{1}{e\sqrt{n}\sigma^n}.$$

Thus for small values of σ we have $|y(n/\sigma)| = O\left(\sigma^{-n}\right)$.

Going back to original variables $x = (y, y', \dots, y^(n-1))^T \in \mathbf{R}^n$ we conclude that

$$\sup_{t>0} \sup_{|x_0|=1} |x(t, x_0)| \geq \gamma_n \max\{\sigma^{n-1}, \sigma^{-n+1}\}, \quad (5)$$

thus for equal eigenvalues large deviations are unavoidable both for σ large and σ small.

2.2 Large deviations: the case of systems in canonical form

The above examples can be treated as estimates for deviations when the system is given in canonical form while closed-loop system has identical eigenvalues. Now we proceed with more general eigenvalue location Λ.

The first two results obtained in this paper improve the estimates from Smirnov et al. [2009]. Consider system (1) with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (6)$$

and $b = e_n$. Then there exists K yielding linear system (2), (6) with any given set of eigenvalues Λ. In the sequel $K \in \mathbf{R}^n$ is fixed and we use the notation $x(t, x_0) = x(t, K, x_0)$ to denote the solution to system (2), (6) starting at x_0.

Spectrum with large absolute values

Theorem 1. If $\sigma = \min\{|\lambda_j| : j = \overline{1,n}\}$, and $\omega = \max\{|\lambda_j| : j = \overline{1,n}\}$, then the inequality

$$\left| x\left(\frac{\log 2}{n\omega}, e_1\right)\right| \geq \frac{2\log 2 - 1}{n} \sigma^{n-1}$$

holds.

This is a specification of Izmailov's theorem.

Spectrum with small absolute values

Theorem 2. If $\omega = \max\{|\lambda_j| : j = \overline{1,n}\}$, and γ is the smallest positive number satisfying $2(n-1) = (n-1+\gamma)e^\gamma$, then the inequality

$$|x(\gamma/\omega, e_n)| \geq c_n \omega^{-(n-1)}, \; c_n = \gamma^{n-1}(2 - e^\gamma)$$

holds.

Spectrum with large and small absolute values Assume that

$$\omega > |\lambda_1| \geq \ldots \geq |\lambda_\nu| \geq \eta > \xi > |\lambda_{\nu+1}| \geq \ldots \geq |\lambda_n|,$$

where $2 \leq \nu \leq n$ and $\xi < 1$. Let γ be the smallest positive number satisfying $1 = \gamma(1 + q\gamma\omega n/2)e^{q\gamma\omega n}$, $q = (\omega n + \xi 2^n)^{-1}$.

Theorem 3. Under above assumptions the inequality

$$|x(q\gamma, e_{n-\nu+1})| \geq q\gamma|\lambda_1 \ldots \lambda_\nu|(1 - (\gamma/2)e^{q\gamma\omega n})$$

holds.

Consider the closed-loop system with the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & -20000.02 & -1000040.0001 & -2000.02 \end{pmatrix}. \quad (7)$$

The eigenvalues of the matrix are $\lambda_1 = \lambda_2 = -1000$ and $\lambda_3 = \lambda_4 = -0.01$. The Euclidean norm of the solution starting at $(0, 0, 1, 0)$ is shown in Fig. 1, while for initial condition $(0, 1, 0, 0)$ it is shown in Fig. 2. Notice that that large deviations occur at different moments t.

Another similar result under above assumptions is:

Theorem 4. If $\xi < \frac{\omega}{2}\left(1 - \frac{1}{(1+\omega)^n}\right) \frac{t^{n-\nu-1}}{(n-\nu-1)!} e^{-(1+\omega)^n t}$ then the following inequality holds:

$$|x(t, e_{n-\nu})| \geq \frac{t^{n-\nu-1}}{2n(n-\nu-1)!}.$$

From this theorem we see that if $\nu \leq n-2$, then the effect of large deviation occurs, provided ξ is sufficiently small.

The behaviour of the solution to 10-dimensional closed-loop system with

$$\Lambda = \{-2, -2, -2, -2, -2, -0.2, -0.2, -0.2, -0.2, -0.2\} \quad (8)$$

and $x(0) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ is presented in Fig. 3. One can see, that for $n = 10$ large deviations are met even for the case of all eigenvalues relatively close to 1.

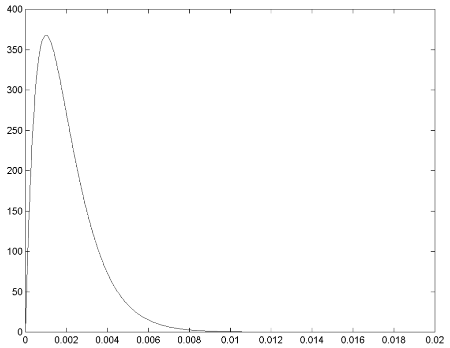

Fig. 1. System (7), $x(0) = (0, 0, 1, 0)$, function $|x(t)|_2$

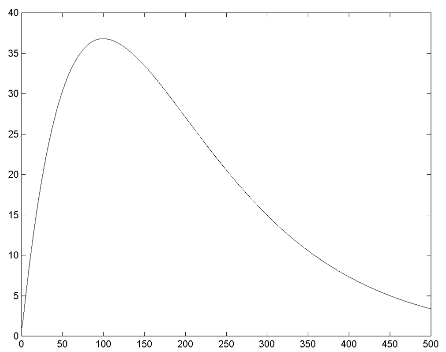

Fig. 2. System (7), $x(0) = (0, 1, 0, 0)$, function $|x(t)|_2$

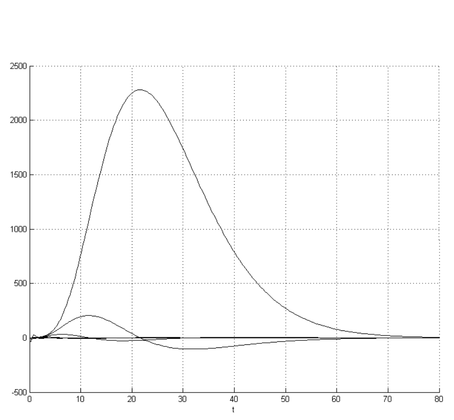

Fig. 3. System (8), coordinates $x_1(t)$ - $x_{10}(t)$

2.3 Unavoidable deviations: the open problem

Now the main question, relating to systems in canonical form, is: *independently on closed-loop eigenvalues, if there exist* $\alpha_n > 1, |x_0| = 1$ *such that trajectory deviation exceeds* α_n. More rigorously we are interested in calculation of

$$\alpha_n = \inf_{\Lambda} \sup_{t \geq 0} \sup_{|x_0|=1} |x(t, x_0)|$$

where $x(t, x_0)$ is the solution of (2) in canonical form with initial condition $x(0) = x_0$ and the feedback yielding the set of the eigenvalues equal Λ. Then we can guarantee that for fixed order n and any feedback K there exist $x_0^*, |x_0^*| = 1, t^* \geq 0$ (depending on K) such that $|x(t^*, x_0^*)| \geq \alpha_n$. It is natural to expect that $\alpha_n > 1$ grows with n. The first conjecture for the above problem was:

(1) The best choice for eigenvalue location is $\Lambda^* = \{-\sigma, \ldots, -\sigma\}$.
(2) The worst initial condition is $x_0^* = (1, \ldots, 1)^*$.

Thus we are in the framework of example in Section 2.1. Now for fixed n and $\sigma > 0$ we calculate $v_n(\sigma) = \min_t \max_k \{|y^{(k)}(t)|\}$, where $y(t)$ is given by (4). Minimizing over σ we get $\alpha_n = v_n(\sigma^*) = \min_\sigma v_n(\sigma)$. Values of α_n, σ^* as functions of n are given in Table 1.

n	3	4	5	6	7	8
α_n	1.58	2.77	4.93	8.82	15.88	28.81
σ^*	1.35	1.16	1.11	1.09	1.08	1.08

n	9	10	11	12	13	14
α_n	52.68	97.06	180.06	341.37	653.19	1251.8
σ^*	1.07	1.07	1.07	1.07	1.06	1.06

For such systems the smallest deviation grows with n:

$$\alpha_n \approx 1.9^n$$

while the best σ is close to 1, being a bit larger than 1.

Unfortunately, the above conjecture is wrong. There are numerical counterexamples, where the leftmost eigenvalue becomes very large, and the largest deviation (over all $|x(0)| = 1$) is smaller α_n in above tables. It can be explained qualitatively: arbitrary change of coefficients of a polynomial can reduce its order. Probably, correct estimation of unavoidable deviation is possible under additional assumption on boundedness of eigenvalues (or coefficients of the characteristic polynomial).

2.4 Large deviations: the case of systems in general form

Let $l \in R^n$ be such that $\langle l, A^{n-1}b \rangle = 1$ and $\langle l, A^{j-1}b \rangle = 0$, $j = \overline{1, n-1}$. In the coordinates $z_j(x) = \langle l, A^{j-1}x \rangle$ the closed-loop system $\dot{x} = Fx$, $F = A + bK^T$, has the matrix

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{pmatrix}. \tag{9}$$

Here a_j are the coefficients of the matrix F characteristic polynomial $\lambda^n + a_1\lambda^{n-1} + \ldots + a_n$. The matrix A in these coordinates has the form (6) and $b = e_n$. The coordinate system depends only on the pair (A, b) and does not depend on the choice of the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, see Wonham [1979]. Consider the matrix $\mathcal{M}$ with the rows $l^T A^k$, $k = \overline{0, n-1}$. Then we have $z = \mathcal{M}x$. Since $\mu(\mathcal{M})|x|_2 \geq |y|_2 \geq |y|$, where $\mu(\mathcal{M})$ is the maximal singular value of the matrix $\mathcal{M}$. Combining Theorem 2 with this result, we get

Theorem 5. For any solution of (2) the constraint $\mathrm{Re}\lambda_j \leq -\sigma < 0$, $j = \overline{1, n}$ for eigenvalues of the closed-loop system implies estimate (3) with

$$\gamma = \frac{2\log 2 - 1}{n\mu(\mathcal{M})}$$

This is Izmailov's theorem with specified bound of large deviations.

3. LMI TECHNIQUE TO REDUCE DEVIATIONS

3.1 LMI approach

Having in mind above results on bounds of deviations in closed-loop systems, the problem arises: how to design a linear feedback in order to guarantee the minimal possible deviations? The approximate solution can be found via LMI approach (Boyd et al. [1994]). Consider SDP problem

$$\min \alpha, \tag{10}$$

$$P(A + bK^T)^T + (A + bK^T)P \preceq -2\sigma P, \tag{11}$$

$$I \preceq P \preceq \alpha I. \tag{12}$$

Here $\alpha \geq 1$, $K \in \mathbf{R}^n$, and $P = P^T \in \mathbf{R}^{n\times n}$ are the variables. Lyapunov inequality (11) guarantees that the decay rate of the closed-loop system exceeds μ and that the ellipsoid $E = \{x \in \mathbf{R}^n : \langle x, P^{-1}x \leq 1\rangle\}$ is its invariant set, while conditions (10) and (12) imply that E has minimal eccentricities. Inequality (11) is nonlinear with respect to the unknown variables P and K. This difficulty can be easily overcome introducing new variable $y = PK$. In terms of variables P and y (11) reads

$$AP + PA^T + by^T + yb^T \preceq -2\sigma P. \tag{13}$$

This is a standard SDP problem (Boyd et al. [1994]) and its solution can be a good candidate for a feedback with small deviation for all nonzero initial conditions. The example below confirms this for $n = 2$, while Theorem 6 exhibits that asymptotically (for large σ) the deviations are of the same order as provided by Izmailov's lower bound (3).

3.2 Example

Solving problem (10)-(12) it is possible to give a trivial solution to the minimal overshooting problem for the oscillator (Seabra [2010]). Set

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Show that there exist vectors K such that I solves the Lyapunov inequality with $\sigma = 0$. This implies that the optimal values are $\hat{\alpha} = 1$ and $\hat{P} = I$. Indeed, with $P = I$ inequality (11) takes the form

$$\begin{pmatrix} 0 & 1+\xi \\ 1+\xi & 2\eta \end{pmatrix} \preceq 0.$$

This condition is equivalent to $\xi = -1$ and $\eta \leq 0$. The corresponding eigenvalues of the closed-loop system are $\lambda = (\eta \pm \sqrt{\eta^2 - 4})/2$, $\eta < 0$. From the engineering point of view, obviously, the best choice is $\eta = -2$. In this case the system is asymptotically stable, has maximum degree of stability, does not oscillate, and has minimal overshooting equal to 1.

3.3 Towards a rigorous justification of SDP approach

The following theorem together with Theorem 1 shows that for large absolute values of eigenvalues of the closed-loop system, the SDP approach gives a feedback guaranteeing, at least asymptotically, the minimal possible overshooting.

Theorem 6. For the feedback $K = P^{-1}y$, P and y being the solutions of SDP problem we have

$$\sup_{0\leq t}\sup_{|x_0|=1} |x(t, x_0)| \leq O(\mu^{n-1}), \quad \mu \to \infty. \tag{14}$$

4. OUTLINE OF THE PROOFS

4.1 Auxiliary results

To make the presentation self-contained recall the following result from Smirnov et al. [2009].

Lemma 7. Let $X(t)$ be the fundamental matrix of system (2). Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the system are different. Then the following representations hold:

$$x_{n,l}(t) = \sum_{\beta=0}^{l-1} (-1)^{n-\beta+1} \sum_{1\leq i_1<\ldots<i_{n-\beta}\leq n} \lambda_{i_1}\ldots\lambda_{i_{n-\beta}} \times \sum_{\alpha\geq l-\beta} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha+\beta-l} \lambda_1^{s_1}\ldots\lambda_n^{s_n}.$$

and

$$x_{1,n}(t) = \sum_{\alpha=n-1}^{\infty} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha-n+1} \lambda_1^{s_1}\ldots\lambda_n^{s_n}.$$

4.2 Proof of Theorem 1

Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of system (2) are different. Consider the solution $x(\cdot)$ to (2) with $x(0) = e_1$. From Lemma 7, we have

$$x_n(t) = (-1)^{n+1}\lambda_1\ldots\lambda_n \sum_{\alpha=1}^{\infty} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha-1} \lambda_1^{s_1}\ldots\lambda_n^{s_n}.$$

Let $\omega = \max\{|\lambda_j| : j = \overline{1, n}\}$ and $t_* = \gamma/(\omega n)$. Then we have $|x_n(t_*)| = |\lambda_1\ldots\lambda_n|\Delta(\gamma)/(\omega n)$, where

$$\Delta(\gamma) = \gamma\left|1 + \sum_{\alpha=2}^{\infty} \frac{t_*^{\alpha-1}}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha-1} \lambda_1^{s_1}\ldots\lambda_n^{s_n}\right|.$$

Observe that

$$\begin{pmatrix} \alpha+n-2 \\ n-1 \end{pmatrix} \leq n^{\alpha-1}.$$

Therefore we get

$$\Delta(\gamma) \geq \gamma\left(1 - \sum_{\alpha=2}^{\infty} \frac{\gamma^{\alpha-1}}{\alpha!(\omega n)^{\alpha-1}} \sum_{s_1+\ldots+s_n=\alpha-1} \omega^{\alpha-1}\right)$$

$$= \gamma\left(1 - \sum_{\alpha=2}^{\infty} \frac{\gamma^{\alpha-1}}{\alpha!(\omega n)^{\alpha-1}} \omega^{\alpha-1} \begin{pmatrix} \alpha+n-2 \\ n-1 \end{pmatrix}\right)$$

$$\geq \gamma \left(1 - \sum_{\alpha=2}^{\infty} \frac{\gamma^{\alpha-1}}{\alpha!}\right) = 2\gamma + 1 - e^{\gamma}.$$

Since the function $\phi(\gamma) = 2\gamma + 1 - e^{\gamma}$, $\gamma \geq 0$, attains its maximum at $\gamma = \log 2$, we obtain the result for different $\lambda_1, \ldots, \lambda_n$. Since the matrix of system (2) continuously depends on $\lambda_1, \ldots, \lambda_n$ and $x_n(t_*)$ continuously depends on the matrix, the result is true for arbitrary $\lambda_1, \ldots, \lambda_n$. □

4.3 Proof of Theorem 2

Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of system (2) are different. Consider the solution $x(\cdot)$ to (2) with $x(0) = e_n$. From Lemma 7 we obtain

$$x_1(t) = \sum_{\alpha=n-1}^{\infty} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha-n+1} \lambda_1^{s_1} \ldots \lambda_n^{s_n}$$

$$= \frac{t^{n-1}}{(n-1)!} + \sum_{\alpha=n}^{\infty} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha-n+1} \lambda_1^{s_1} \ldots \lambda_n^{s_n}.$$

Let $\omega = \max\{|\lambda_j| : j = \overline{1,n}\}$, $t_* = \gamma/\omega$. Then we have

$$|x_1(t_*)| \geq \frac{\gamma^{n-1}}{\omega^{n-1}(n-1)!} - \sum_{\alpha=n}^{\infty} \frac{\gamma^\alpha \omega^{\alpha-n+1}}{\alpha! \omega^\alpha} \begin{pmatrix} \alpha \\ n-1 \end{pmatrix}$$

$$= \frac{\gamma^{n-1}}{\omega^{n-1}(n-1)!} \left(2 - e^{\gamma}\right).$$

Let γ be the smallest positive number satisfying $2(n-1) = (n-1+\gamma)e^{\gamma}$. The function $\phi(\gamma) = \gamma^{n-1}\left(2 - e^{\gamma}\right)$ attains its maximum at this point. Arguing as at the end of the proof of Theorem 1 we obtain the result. □

4.4 Proof of Theorem 3

The following auxiliary estimate is almost obvious.

Lemma 8. The inequality holds

$$\left| \sum_{1 \leq i_1 < \ldots < i_\nu \leq n} \lambda_{i_1} \ldots \lambda_{i_\nu} \right| \geq |\lambda_1 \ldots \lambda_\nu| \geq \eta^\nu.$$

Proof of Theorem 3. Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ are different. Set $l = n - \nu + 1$. Then we have

$$x_{n,n-\nu+1}(t) = (-1)^{\nu+1} \sum_{1 \leq i_1 < \ldots < i_\nu \leq n} \lambda_{i_1} \ldots \lambda_{i_\nu} t$$

$$+(-1)^{\nu+1} \sum_{1 \leq i_1 < \ldots < i_\nu \leq n} \lambda_{i_1} \ldots \lambda_{i_\nu}$$

$$\times \sum_{\alpha \geq 2} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha-1} \lambda_1^{s_1} \ldots \lambda_n^{s_n}$$

$$+ \sum_{\beta=0}^{n-\nu-1} (-1)^{n-\beta+1} \sum_{1 \leq i_1 < \ldots < i_{n-\beta} \leq n} \lambda_{i_1} \ldots \lambda_{i_{n-\beta}}$$

$$\times \sum_{\alpha \geq n-\nu-\beta+1} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha+\beta-n+\nu-1} \lambda_1^{s_1} \ldots \lambda_n^{s_n}.$$

Since $\exp(a) - (1+a) \leq \exp(a)a^2/2$, $a \geq 0$, we have

$$\left| \sum_{\alpha \geq 2} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha-1} \lambda_1^{s_1} \ldots \lambda_n^{s_n} \right|$$

$$\leq \sum_{\alpha \geq 2} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha-1} \omega^{\alpha-1}$$

$$= \sum_{\alpha \geq 2} \frac{t^\alpha}{\alpha!} \omega^{\alpha-1} \begin{pmatrix} \alpha+n-2 \\ n-1 \end{pmatrix}$$

$$\leq \sum_{\alpha \geq 2} \frac{1}{\omega n} \frac{(\omega n t)^\alpha}{\alpha!} \leq \frac{\omega n t^2}{2} \exp(\omega n t)$$

(see the proof of Theorem 1). Similarly, we have

$$\left| \sum_{\alpha \geq n-\nu-\beta+1} \frac{t^\alpha}{\alpha!} \sum_{s_1+\ldots+s_n=\alpha+\beta-n+\nu-1} \lambda_1^{s_1} \ldots \lambda_n^{s_n} \right|$$

$$\leq \frac{t^{n-\nu-\beta+1}}{(n-\nu-\beta+1)!} \exp(\omega n t) \leq \frac{t^2}{2} \exp(\omega n t),$$

whenever $\beta \leq n - \nu - 1$ and $t < 1$. By Lemma 8 we get

$$\frac{\sum_{\beta=0}^{n-\nu-1} \left| \sum_{1 \leq i_1 < \ldots < i_{n-\beta} \leq n} \lambda_{i_1} \ldots \lambda_{i_{n-\beta}} \right|}{\left| \sum_{1 \leq i_1 < \ldots < i_\nu \leq n} \lambda_{i_1} \ldots \lambda_{i_\nu} \right|}$$

$$= \frac{\sum_{m=\nu+1}^{n} \left| \sum_{1 \leq i_1 < \ldots < i_m \leq n} \lambda_{i_1} \ldots \lambda_{i_m} \right|}{\left| \sum_{1 \leq i_1 < \ldots < i_\nu \leq n} \lambda_{i_1} \ldots \lambda_{i_\nu} \right|}$$

$$\leq \sum_{m=\nu+1}^{n} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \xi^{m-\nu} = \sum_{m=\nu+1}^{n} \begin{pmatrix} n \\ m \end{pmatrix} \xi^{m-\nu}$$

$$\leq \xi \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} = \xi 2^n.$$

Set $t_* = \gamma/(\omega n + \xi 2^n)$. Combining the above estimates with Lemma 8 we obtain

$$|x_{n,n-\nu+1}(t_*)| \geq \left| \sum_{1 \leq i_1 < \ldots < i_\nu \leq n} \lambda_{i_1} \ldots \lambda_{i_\nu} \right|$$

$$\times t_* \left(1 - \frac{t_* \exp(\omega n t_*)}{2} \left(\omega n + \xi 2^n\right)\right)$$

Arguing as at the end of the proof of Theorem 1 we obtain the result. □

4.5 Proof of Theorem 4

Along with system (2) consider an auxiliary system

$$\frac{d}{dt} \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -\alpha_\nu & \ldots & -\alpha_1 \end{pmatrix} \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n)} \end{pmatrix}. \quad (15)$$

We denote its matrix by F_0. The characteristic polynomial of F_0 is $\lambda^{n-\nu} \prod_{j=1}^{\nu} (\lambda - \lambda_j)$. The vectors $y^* = (y^{(1)}, \ldots, y^{(n-\nu)}, 0, \ldots, 0)$ form an invariant subspace of the matrix F_0. Obviously the solution $y(t, e_{n-\nu})$ to (15) with the initial condition $y(0, e_{n-\nu}) = e_{n-\nu}$ has the form

$$y^*(t, e_{n-\nu}) = \left(\frac{t^{n-\nu-1}}{(n-\nu-1)!}, \ldots, t, 1, 0, \ldots, 0 \right).$$

In the proof of this theorem we use the norm $|x|_1 = \sum_{k=1}^{n} |x^{(k)}|$. Recall that the corresponding norm of a matrix $L = (L_{ij})$ is given by $|L|_1 = \max_j \sum_i |L_{ij}|$.

Proof of Theorem 4. It is easy to see that the following inequalities hold: $|F|_1 \le (1+\omega)^n$ and $|F_0 - F|_1 \le (\xi/\omega)(1+\omega)^n$. From the Gronwall inequality we get

$$|x(t,e_{n-\nu}) - y(t,e_{n-\nu})|_1$$

$$\le \int_0^t e^{|F|(t-s)} |F_0 y(s,e_{n-\nu}) - F y(s,e_{n-\nu})|_1 ds$$

$$\le \frac{\xi}{\omega} \frac{(1+\omega)^n}{(1+\omega)^n - 1} e^{(1+\omega)^n t}.$$

Observe that $|y(t,e_{n-\nu})|_1 > t^{n-\nu-1}/(n-\nu-1)!$. Therefore we have

$$|x(t,e_{n-\nu})|_1 \ge |y(t,e_{n-\nu})|_1 - |x(t,e_{n-\nu}) - y(t,e_{n-\nu})|_1$$

$$\ge \frac{t^{n-\nu-1}}{(n-\nu-1)!} - \frac{\xi}{\omega} \frac{(1+\omega)^n}{(1+\omega)^n - 1} e^{(1+\omega)^n t} \ge \frac{t^{n-\nu-1}}{2(n-\nu-1)!}.$$

This ends the proof. □

4.6 Proof of Theorem 6

Assume that $|\lambda|$ is greater than the modulus of any matrix A eigenvalue. The proof is based on the following auxiliary lemmas.

Lemma 9. The vectors $b_k = (A - \lambda I)^{-k} b$, $k = \overline{0, n-1}$, form a basis in $\mathbf{R}^n$.

Any vector $x \in \mathbf{R}^n$ can be represented as $x = \sum_{k=0}^{n-1} \beta_k b_k$ and as $x = \sum_{m=0}^{n-1} \gamma_m A^m b$. The vectors $\beta = (\beta_0, \dots, \beta_{n-1})$ and $\gamma = (\gamma_0, \dots, \gamma_{n-1})$ satisfy the equality $\beta(\lambda,\gamma) = M(\lambda)\gamma$, where $M(\lambda)$ is an $(n \times n)$-matrix.

Lemma 10. The following representation takes place: $\beta_k(\lambda,\gamma) = \sum_{r=r_-}^{k} \beta_k^r(\gamma)(-\lambda)^r$, $k = \overline{0, n-1}$, whenever $|\lambda|$ is large enough.

As a special case of this lemma we get

Lemma 11. The coordinates α_k, $k = \overline{0, n-1}$, of the vector Ab with respect to the basis $b_k = (A - \lambda I)^{-k} b$, $k = \overline{0, n-1}$, are $O(|\lambda|^{n-1})$ as $|\lambda| \to \infty$.

Lemma 12. $||M^{-1}(\lambda)|| = O(1)$ as $|\lambda| \to \infty$.

Proof of Theorem 6. By Lemma 9 the vectors $b_k = (A - \lambda I)^{-k} b$, $k = \overline{0, n-1}$, form a basis in R^n. Let $x = \sum_{k=0}^{n-1} \beta_k b_k$ and $Ab = \sum_{k=0}^{n-1} \alpha_k b_k$. Put $u(x) = u(\beta) = \lambda\beta_0 - (\beta_1 + \beta_0\alpha_0)$. The system $\dot{x} = Ax + u(x)b$ in the coordinates β takes the form

$$\begin{aligned}
\dot{\beta}_0 &= \lambda\beta_0, \\
\dot{\beta}_1 &= \lambda\beta_1 + \beta_2 + \beta_0\alpha_1, \\
&\dots \\
\dot{\beta}_{n-2} &= \lambda\beta_{n-2} + \beta_{n-1} + \beta_0\alpha_{n-2}, \\
\dot{\beta}_{n-1} &= \lambda\beta_{n-1} + \beta_0\alpha_{n-1}.
\end{aligned}$$

Put $\delta^2 = (n-1)\max_{k=\overline{1,n-1}} \alpha_k^2$ and

$$W(\beta) = \sum_{k=1}^{n-1} \beta_k^2 + \delta^2\beta_0^2. \tag{16}$$

Then we have

$$\frac{dW(\beta)}{dt} = 2\sum_{k=1}^{n-2} \beta_k(\lambda\beta_k + \beta_{k+1} + \beta_0\alpha_k)$$

$$+2\left(\beta_{n-1}(\lambda\beta_{n-1} + \beta_0\alpha_{n-1}) + \lambda\delta^2\beta_0^2\right)$$

$$= 2\lambda W(\beta) + 2\left(\sum_{k=1}^{n-2}(\beta_k\beta_{k+1} + \beta_k\beta_0\alpha_k) + \beta_{n-1}\beta_0\alpha_{n-1}\right)$$

$$\le 2\lambda W(\beta) + \sum_{k=1}^{n-2}(\beta_k^2 + \beta_{k+1}^2) + \sum_{k=1}^{n-1}\left(\frac{\alpha_k^2\beta_k^2(n-1)}{\delta^2} + \frac{\delta^2\beta_0^2}{n-1}\right)$$

$$\le (2\lambda + 3)W(\beta). \tag{17}$$

Let $\{e_k\}_{k=0}^{n-1}$ be an orthonormal basis in $\mathbf{R}^n$. A vector $x \in \mathbf{R}^n$ can be represented as $x = \sum_{k=0}^{n-1} x_k e_k$. Assume that the vectors $x = (x_0, \dots, x_{n-1})$ and $\gamma = (\gamma_0, \dots, \gamma_{n-1})$ satisfy the equality $\gamma = Nx$, where N is an $(n \times n)$-matrix. Put $V(x) = W(M(\lambda)Nx)$. From (16), (17), and Lemmas 10 - 12 we see that the feedback $u(x)$ and the quadratic form $V(x)$ satisfy conditions the conditions $dV(x(t))/dt \le (2\lambda + 3)V(x(t))$ along the trajectories of system (2), $\kappa_{\min}|x|_2^2 \le V(x) \le \kappa_{\max}|x|_2^2$ and $\kappa_{\max}/\kappa_{\min} = O(|\lambda|^{2(n-1)})$, $|\lambda| \to \infty$. Now, we define P as $\langle x, P^{-1}x\rangle = V(x)$ and set $\sigma = -\lambda - 3/2$, where $\lambda < 0$. Obviously the eccentricities of the ellipsoid $\{x \in R^n \mid \langle x, P^{-1}x\rangle\}$ do not exceed $\sqrt{\kappa_{\max}/\kappa_{\min}} = O(\sigma^{n-1})$, $\sigma \to \infty$. □

ACKNOWLEDGEMENTS

The authors are grateful to A. Tremba, M.Khlebnikov, P.Scherbakov and Isabel Caiado for their help in preparation of this paper.

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