Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Edge-Event Based Consensus in Networks with Common Time-Varying Delays ⋆

Feng Xiao * **Tongwen Chen** ** **Huijun Gao** *

* *Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150080, China (Emails: fengxiao@hit.edu.cn; hjgao@hit.edu.cn)*
** *Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Alberta T6G 2V4, Canada (Email: tchen@ualberta.ca)*

Abstract: This paper studies the delay robustness of a class of periodically edge-event driven synchronous consensus protocols in time-invariant networks. These protocols have the benefits of improved performance at reduced communication and computation costs. Under the assumption that all information links share a common time-varying transmission delay, we give non-conservative estimates of the maximum allowable time-delay and event-detecting period for solving the average consensus problem in terms of the algebraic structure of interaction topologies. Furthermore, rigorous stability analysis shows that the proposed technique is also applicable to the asynchronous consensus with multiple time-delays.

Keywords: Multi-agent systems; event-based consensus; sampled-data consensus; time-delays.

1. INTRODUCTION

As an important research topic in multi-agent coordination, sampled-data consensus has been intensively studied by researchers in the past several years. It involves the design and stability analysis of various kinds of distributed algorithms/protocols, which coordinate the behaviour of each agent through local interactive data-samplings to get the information shared by all agents (Ren and Beard, 2005).

Typically, inter-agent data-samplings are scheduled in a synchronous periodic manner (Xie et al., 2009a,b; Cao and Ren, 2010; Gao and Wang, 2010; Qin and Gao, 2012). This scheme makes it easier for the protocol design and convergence analysis and it also serves as the basis for further development, such as synchronous aperiodic sampled-data consensus (Liu et al., 2012), asynchronous periodic sampled-data consensus (Gao and Wang, 2011) and aperiodic sampled-data consensus (Lin et al., 2004; Cao et al., 2008; Xiao and Wang, 2008). Preliminary results show that the algebraic property of underlying interaction graphs plays a key role in choosing the maximal data-sampling periods and it also determines the system robustness against communication delays (Xie et al., 2009a). The employed analysis tools include the spectrum analysis of graph Laplacian, nonnegative matrix theory (Cao and Ren, 2010) and Linear Matrix Inequalities (LMIs) (Gao and Wang, 2010). Nevertheless, these time driven systems usually have constant data-sampling rates no matter whether necessary or not.

To reduce the number of unnecessary state-samplings and actuator updates, event based approach is an alternative option of scheduling data-sampling actions. It has many favorable advantages over the pure time driven control including lower communication and controller-updating costs (Åström and Bernhardsson, 2002; Åström, 2008; Lemmon, 2010). Event based consensus protocols specify that each agent activates the actions of data-samplings and controller update only when its observable measurement errors exceed certain thresholds. For each agent, if each of its involved events triggers the inter-agent data-samplings of itself with all its neighbors, we call this kind of events "agent-events", which were indeed widely adopted in the literature (Dimarogonas et al., 2012; Seyboth et al., 2013; Fan et al., 2013). Clearly, the agent-event based approach aims reducing controller-updating costs; and communication costs are likely to be further reduced if triggering conditions of data-samplings on different information links are checked independently. This observation motivates the study of so-called "edge-event based" protocols, which were originally proposed in Xiao et al. (2012). In this setup, the communication link between any pair of adjacent agents is modeled by an edge of the interaction graph; edge-events are introduced independently to information links; their triggering conditions are checked collectively by the corresponding two linked agents; and their occurrences activate the mutual state-samplings and controller updates. These protocols could be easily applied in a distributed asynchronous environment and guarantee an improved performance at reduced communication and computation costs (Xiao et al., 2012).

This paper performs the delay robustness analysis of a class of periodically edge-event driven consensus protocols in time-invariant networks, which combine edge-event

⋆ Corresponding Author: Feng Xiao. This work was supported by the Natural Sciences and Engineering Research Council of Canada, NSFC (61273030), and NCET (NCET-13-0178).

based samplings with local time driven event-detections. Note that the time driven feature of event-detections can easily ensure a lower bound of inter-event times over each information link, which is hard to be ensured in the traditional event based framework (Dimarogonas et al., 2012; Seyboth et al., 2013; Fan et al., 2013). We assume that all information links share a common time-varying transmission delay with an upper bound. By Lyapunov methods, the trade-off between maximum allowable time-delay and event-detecting period is characterized in terms of the algebraic structure of interaction topologies and non-conservative estimate of maximum allowable time-delay is given for solving the average consensus problem. Furthermore, the analysis technique is still valid in the more general setting with multiple time-varying delays.

This paper is organized as follows: the problem is formulated in Section 2; the main result is presented in Section 3; a simulation example is given in Section 4; finally, the paper is concluded in Section 5; some necessary lemmas for proving the main result are attached in the Appendix.

2. PROBLEM FORMULATION

In this paper, we study a multi-agent system with n single-integrators. Label these agents with 1 through n. The information links between agents are assumed to be bidirectional and the *interaction topology* is modeled by an undirected simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ without self-loops. $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the vertex set, modeling the n integrators respectively. Any edge (v_i, v_j) in the edge set $\mathcal{E}$ is an unordered pair of vertices, which implies the existence of an information link connecting agents i and j. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called *connected* if for any two different vertices i and j, there exists a sequence of vertices such that the sequence begins at v_i and ends at v_j and any two consecutive vertices are adjacent (making up an edge) in $\mathcal{G}$. All the agents j, satisfying $(v_i, v_j) \in \mathcal{E}$, are usually defined as the *neighbors* of agent i and indexed by $\mathcal{N}_i$ (Olfati-Saber et al., 2007; Ji et al., 2012).

Let $x_i(t) \in \mathbb{R}$ denote the state of agent i, $i = 1, 2, \ldots, n$. The dynamics of each agent is described by the following equation:

$$x_i(t) = u_i(t), \quad i = 1, 2, \ldots, n,$$

where $u_i(t)$ is a local state feedback, called *protocol*, to be designed based on the information received by agent i from its neighbors.

In Olfati-Saber et al. (2007), the following protocol with a fixed time-delay was studied:

$$u_i(t) = \sum_{j \in \mathcal{N}_i} \Big(x_j(t-\tau) - x_i(t-\tau) \Big).$$

It was shown that $\tau < \frac{\pi}{2\lambda_n}$ is a necessary and sufficient condition for the solvability of the average consensus problem, where λ_n is the largest eigenvalue of the Laplacian of graph $\mathcal{G}$. If the above protocol is considered in the traditional framework of sampled-data consensus, we have the following revised form:

$$u_i(t) = \sum_{j \in \mathcal{N}_i} \Big(x_j(t_k) - x_i(t_k) \Big), \quad t \in [t_k + \tau, t_{k+1} + \tau), \tag{1}$$

where $t_0, t_1, t_2, \ldots,$ is a sequence of time with $t_{k+1} = t_k + h$, $k = 0, 1, 2, \ldots$. Here h is the data-sampling period. Xie, Liu, and Jia (2009a) showed that $h < \frac{2}{\lambda_n}$ is necessary and sufficient for solving the consensus problem in the absence of time-delay; and allowable delay τ is also determined by the eigenvalues of the underlying graph Laplacian.

To reduce the unnecessary data-samplings at t_k, $k = 0, 1, \ldots,$ we invoke the *edge-event* based technique:

(1) Assume that there exists a common transmission time-delay $\tau(t_k)$, less than h, on all information links at time t_k, $k = 0, 1, 2, \ldots$.
(2) For any pair of adjacent agents i, j, they initialize their mutual data-sampling at time t_0. Then $x_j(t_0) - x_i(t_0)$ is available to agents i and j at time $t_0 + \tau(t_0)$. Denote
$$\hat{x}_{ij}(t) = x_j(t_0) - x_i(t_0), t \in [t_0, t_1)$$
and index the *most recent data-sampling time* by $\kappa_{ij}(t) = \kappa_{ji}(t) = 0$, $t \in [t_0, t_1)$.
(3) At time t_k, $k = 1, 2, \ldots,$ agents i and j check the following inequalities, respectively:
$$\begin{cases} |x_i(t_k) - x_i(t_{\kappa_{ij}(t_{k-1})})| < \dfrac{\alpha}{2} |\hat{x}_{ij}(t_{k-1})| \\ |x_j(t_k) - x_j(t_{\kappa_{ji}(t_{k-1})})| < \dfrac{\alpha}{2} |\hat{x}_{ji}(t_{k-1})| \end{cases} \tag{2}$$
where parameter α with $0 < \alpha < 1$ is a threshold, shared by all agents. If either of the above two inequalities does not hold, the mutual data-sampling between agents i and j is triggered and set
$$\begin{cases} \hat{x}_{ij}(t) = -\hat{x}_{ji}(t) = x_j(t_k) - x_i(t_k) \\ \kappa_{ij}(t) = \kappa_{ji}(t) = k \end{cases}, t \in [t_k, t_{k+1});$$
otherwise, set
$$\begin{cases} \hat{x}_{ij}(t) = -\hat{x}_{ji}(t) = \hat{x}_{ij}(t_{k-1}) \\ \kappa_{ij}(t) = \kappa_{ji}(t) = \kappa_{ij}(t_{k-1}) \end{cases}, t \in [t_k, t_{k+1}).$$

Within the above data-sampling scheme, h is called the *event-detecting period* and the protocol is given as follows:

$$u_i(t) = \sum_{j \in \mathcal{N}_i} \hat{x}_{ij}(t - \tau(t_{\kappa_{ij}(t)})). \tag{3}$$

Remark. We should note that $\hat{x}_{ij}(t - \tau(t_{\kappa_{ij}(t)}))$ is available to both agents i and j and the event-detecting condition (2) only depends on local information of agents i and j. So no communication between agents i and j is needed in checking inequality (2) at t_k. Moreover, it is easy to check that if $\alpha = 0$ and the time-delay is fixed, then the above protocol becomes protocol (1). In Xiao, Meng, and Chen (2012), it was shown that if we choose event-detecting period h with $0 < h < \frac{1-\alpha}{\lambda_n}$ and time-delays are all 0, then the average consensus problem will be solvable. In fact, the upper bound $\frac{1-\alpha}{\lambda_n}$ can be further relaxed by $\frac{2(1-\alpha)}{\lambda_n}$.

3. MAIN RESULT

Let M_I denote an incidence matrix of graph $\mathcal{G}$ with some given edge orientations and let λ_n be the largest eigenvalue of *graph Laplacian* $M_I {M_I}^T$. Matrix ${M_I}^T M_I$ is called *Edge Laplacian* and denoted by $L^E = [l_{ij}^E]$ in this paper. Note that the definition of $M_I {M_I}^T$ is independent of the choice of M_I, and matrices $M_I {M_I}^T$ and L^E share the same non-zero eigenvalues (Mesbahi and Egerstedt, 2010). Denote

$$\begin{cases} \tau_{\max} = \max_k \tau(t_k) \\ \tau_{asyn} = \max\{\tau(t_{\kappa_{ij_1}(t)}) - \tau(t_{\kappa_{ij_2}(t)}) : \\ \qquad (v_i, v_{j_1}), (v_i, v_{j_2}) \in \mathcal{E}, t \geq 0\} \\ \tau_{var} = \max\{\tau(t_{\kappa_{ij}(t)}) - \tau(t_{\kappa_{ij}(t)-1}) : \\ \qquad (v_i, v_j) \in \mathcal{E}(\mathcal{G}), t \geq 0\} \end{cases}$$

and let σ_Ξ denote the least upper bound of singular values of matrices in set

$$\Xi = \Big\{L^E \circ \Theta : \text{each entry of } \Theta \text{ is nonnegative and not larger than } 1\Big\},$$

where $\circ$ denotes the entrywise product (Hadamard product) of matrices. Clearly, $\sigma_\Xi \geq \lambda_n$ and it also depends on the algebraic structure of graph $\mathcal{G}$.

Theorem 1. Suppose that the interaction graph $\mathcal{G}$ is connected and time-delay $\tau(t_k)$, $k = 0, 1, \ldots$, is smaller than event-detecting period h. If

$$1-\alpha-(h+\tau_{var})\lambda_n-(\sqrt{2h\tau_{\max}}+\sqrt{2}\tau_{asyn})\sigma_\Xi > 0, \quad (4)$$

then protocol (3) solves the average consensus problem; that is, the states of agents all converge to their average value as time goes on; particularly, if

$$h < \frac{1-\alpha}{2\lambda_n + 2\sqrt{2}\sigma_\Xi}, \quad (5)$$

then for any time-varying delay smaller than h, protocol (3) solves the average consensus problem.

Remark. By the assumption that $\tau(t_k)$, $k = 0, 1, \ldots$, is smaller than h, we have that $\tau_{\max}$, τ_{asyn}, and τ_{var} are all smaller than h and thus inequality (5) implies inequality (4), which means that inequality (4) is always solvable.

3.1 Technical Proof

Let m be the total number of edges in $\mathcal{G}$ and denote them by $e_1, e_2, \ldots, e_m$. For each edge e_p with $e_p = (v_i, v_j)$, if it is orientated from v_j to v_i in the definition of M_I, denote $y_p(t) = x_i(t) - x_j(t)$ and $\hat{y}_p(t) = \hat{x}_{ji}(t)$. With abuse of notations, we use notation $\kappa_p(t)$ instead of $\kappa_{ij}(t)$ and use notation τ_k^p instead of $\tau(t_{\kappa_{ij}(t_k)})$ in what follows. Denote vectors $y(t) = [y_1(t), y_2(t), \ldots, y_m(t)]^T$ and $\hat{y}(t) = [\hat{y}_1(t), \hat{y}_2(t), \ldots, \hat{y}_m(t)]^T$. Then we have

$$\begin{cases} y(t) = {M_I}^T x(t) \\ \dot{x}(t) = -M_I[\hat{y}_1(t-\tau(t_{\kappa_1(t)})), \hat{y}_2(t-\tau(t_{\kappa_2(t)})), \\ \qquad \ldots, \hat{y}_m(t-\tau(t_{\kappa_m(t)}))]^T \\ \dot{y}(t) = -{M_I}^T M_I[\hat{y}_1(t-\tau(t_{\kappa_1(t)})), \hat{y}_2(t-\tau(t_{\kappa_2(t)})), \\ \qquad \ldots, \hat{y}_m(t-\tau(t_{\kappa_m(t)}))]^T \end{cases} \quad (6)$$

Denote the state average $\varrho = \frac{1}{n}\sum_{i=1}^n x_i(t)$. Under protocol (3) and the assumption of undirected interaction graph, it can be shown that ϱ is a constant. Denote $\delta_i(t) = x_i(t) - \varrho$, $i = 1, 2, \ldots, n$, and consider the following Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^n \delta_i(t)^2.$$

By equation (6),

$$\frac{dV(t)}{dt} = -\sum_{i=1}^m \hat{y}_i(t-\tau(t_{\kappa_i(t)}))y_i(t), \quad (7)$$

and the third equation in (6) is equivalent to

$$\dot{y}_i(t) = -\sum_{j=1}^m l_{ij}^E \hat{y}_j(t-\tau(t_{\kappa_j(t)})), i = 1, 2, \ldots, m. \quad (8)$$

We collect all possible times $t_k, t_k + \tau_k^i$, $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots$, in the increasing order and label them by $s_0, s_1, t_2, \ldots$. For any edge e_i and time s_k, let ϱ_k^i denote the index number such that $t_{\varrho_k^i} + \tau_{\varrho_k^i}^i \leq s_k < t_{\varrho_k^i+1} + \tau_{\varrho_k^i+1}^i$.

Now, suppose that $t_k = s_{k_0}$ and $t_k + \tau_k^i = s_{k_1} < s_{k_2} \leq t_{k+1} + \tau_{k+1}^i$. By the properties that $\hat{y}_i(t-\tau(t_{\kappa_i(t)})) = \hat{y}_i(t_k)$ for $t \in [s_{k_1}, s_{k_2})$, $\varrho_p^i = k$ for $p = k_1, k_1+1, \ldots, k_2-1$, and $\hat{y}_j(t-\tau(t_{\kappa_j(t)})) = \hat{y}_j(t_{\varrho_p^j})$ for $t \in [s_p, s_{p+1})$ and all j and p, and by equation (8), we have that

$$\begin{aligned}
&\int_{s_{k_1}}^{s_{k_2}} \hat{y}_i(t-\tau(t_{\kappa_i(t)}))y_i(t)dt \\
&= \int_{s_{k_1}}^{s_{k_2}} \hat{y}_i(t_k)y_i(s_{k_1})dt \\
&\quad - \sum_{j=1}^m \sum_{p=k_1}^{k_2-1} l_{ij}^E \int_{s_p}^{s_{p+1}} \hat{y}_i(t_{\varrho_p^i}) \int_{s_{k_1}}^{s} \hat{y}_j(t-\tau(t_{\kappa_j(t)}))dtds \\
&=(s_{k_2}-s_{k_1})\hat{y}_i(t_k)\Big(y_i(t_k) - \sum_{p=k_0}^{k_1-1}(s_{p+1}-s_p)\sum_{j=1}^m l_{ij}^E \hat{y}_j(t_{\varrho_p^j})\Big) \\
&\quad - (h+\tau_{var})\sum_{j=1}^m \sum_{p=k_1}^{k_2-1} \frac{s_{p+1}-s_p}{h+\tau_{var}} \\
&\quad \times \Big(\frac{s_{p+1}-s_p}{2} + s_{k_2} - s_{p+1}\Big) l_{ij}^E \hat{y}_i(t_{\varrho_p^i})\hat{y}_j(t_{\varrho_p^j}) \\
&\geq (1-\alpha)\sum_{p=k_1}^{k_2-1}(s_{p+1}-s_p)\hat{y}_i(t_{\varrho_p^i})^2 - (s_{k_2}-s_{k_1})\hat{y}_i(t_k) \\
&\quad \times \sum_{j=1}^m l_{ij}^E \Big(d_{ij}^-(t_k)\hat{y}_j(t_{k-1}) + d_{ij}(t_k)\hat{y}_j(t_k)\Big) \\
&\quad - (h+\tau_{var})\sum_{p=k_1}^{k_2-1}(s_{p+1}-s_p) \\
&\quad \times \sum_{j=1}^m \frac{1}{h+\tau_{var}}\Big(\frac{s_{p+1}-s_p}{2} + s_{k_2} - s_{p+1}\Big) l_{ij}^E \hat{y}_i(t_{\varrho_p^i})\hat{y}_j(t_{\varrho_p^j}),
\end{aligned}$$

where the last inequality follows from Lemma 3, $d_{ij}^-(t_k) = \min\{\tau_k^i, \tau_k^j\}$, $d_{ij}(t_k) = \max\{\tau_k^i - \tau_k^j, 0\}$, and $\frac{1}{h+\tau_{var}}\Big(\frac{s_{p+1}-s_p}{2} + s_{k_2} - s_{p+1}\Big) < 1$ by that $s_{k_2} - s_{k_1} \leq h + \tau_{var}$.

Given k, we define

$$\theta_{p,k}^i = \frac{1}{h+\tau_{var}}\Big(\frac{s_{p+1}-s_p}{2} + \min\{t_{k+1}, t_{\varrho_k^i+1} + \tau_{\varrho_k^i+1}^i\} - s_{p+1}\Big)$$

and suppose that $s_{k'} = t_{k+1}$. Then by equation (7), we have

$$\begin{aligned}
V(t_{k+1}) &= V(t_1) - \sum_{i=1}^m \int_{t_1}^{t_{k+1}} \hat{y}_i(t-\tau(t_{\kappa_i(t)}))y_i(t)dt \\
&\leq V(t_1) - (1-\alpha)\sum_{p=0}^{k'-1}(s_{p+1}-s_p)\sum_{i=1}^m \hat{y}_i(t_{\varrho_p^i})^2
\end{aligned}$$

$$
\begin{aligned}
&+(h+\tau_{var})\sum_{p=0}^{k'-1}(s_{p+1}-s_p)\sum_{i=1}^{m}\sum_{j=1}^{m} l_{ij}^E\theta_{p,k}^i\hat{y}_i(t_{\varrho_p^i})\hat{y}_j(t_{\varrho_p^j}),\\
&+\sum_{p=1}^{k-1}\sum_{i=1}^{m}(h+\tau_{p+1}^i-\tau_p^i)\hat{y}_i(t_p)\\
&\times\sum_{j=1}^{m} l_{ij}^E\Big(d_{ij}^-(t_p)\hat{y}_j(t_{p-1})+d_{ij}(t_p)\hat{y}_j(t_p)\Big)\\
&+\sum_{i=1}^{m}(h-\tau_k^i)\hat{y}_i(t_k)\\
&\times\sum_{j=1}^{m} l_{ij}^E\Big(d_{ij}^-(t_k)\hat{y}_j(t_{k-1})+d_{ij}(t_k)\hat{y}_j(t_k)\Big) \qquad (9)
\end{aligned}
$$

In equation (9), by Lemma 2,

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{m} l_{ij}^E\theta_{p,k}^i\hat{y}_i(t_{\varrho_p^i})\hat{y}_j(t_{\varrho_p^j})\\
&\quad=[\hat{y}_1(t_{\varrho_p^1}),\hat{y}_2(t_{\varrho_p^2}),\ldots,\hat{y}_m(t_{\varrho_p^m})]\\
&\qquad\times\operatorname{diag}([\theta_{p,k}^1,\theta_{p,k}^2,\ldots,\theta_{p,k}^m])L^E\\
&\qquad\times[\hat{y}_1(t_{\varrho_p^1}),\hat{y}_2(t_{\varrho_p^2}),\ldots,\hat{y}_m(t_{\varrho_p^m})]^T\\
&\quad\le\lambda_n\sum_{i=1}^{m}\hat{y}_i(t_{\varrho_p^i})^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(1-\alpha)\sum_{p=0}^{k'-1}(s_{p+1}-s_p)\sum_{i=1}^{m}\hat{y}_i(t_{\varrho_p^i})^2-(h+\tau_{var})\\
&\times\sum_{p=0}^{k'-1}(s_{p+1}-s_p)\sum_{i=1}^{m}\sum_{j=1}^{m} l_{ij}^E\theta_{p,k}^i\hat{y}_i(t_{\varrho_p^i})\hat{y}_j(t_{\varrho_p^j})\\
\ge&\sum_{p=0}^{k'-1}(s_{p+1}-s_p)\Big(1-\alpha-(h+\tau_{var})\lambda_n\Big)\sum_{i=1}^{m}\hat{y}_i(t_{\varrho_p^i})^2\\
=&\Big(1-\alpha-(h+\tau_{var})\lambda_n\Big)\sum_{i=1}^{m}\tau_1^i\hat{y}_i(t_0)^2\\
&+\Big(1-\alpha-(h+\tau_{var})\lambda_n\Big)\sum_{i=1}^{m}\sum_{p=1}^{k-1}(h+\tau_{p+1}^i-\tau_p^i)\hat{y}_i(t_p)^2\\
&+\Big(1-\alpha-(h+\tau_{var})\lambda_n\Big)\sum_{i=1}^{m}(h-\tau_k^i)\hat{y}_i(t_k)^2
\end{aligned}
$$

Next, we study the other quantity in equation (9):

$$
\begin{aligned}
&\sum_{p=1}^{k-1}\sum_{i=1}^{m}(h+\tau_{p+1}^i-\tau_p^i)\hat{y}_i(t_p)\\
&\quad\times\sum_{j=1}^{m} l_{ij}^E\Big(d_{ij}^-(t_p)\hat{y}_j(t_{p-1})+d_{ij}(t_p)\hat{y}_j(t_p)\Big)\\
&\quad+\sum_{i=1}^{m}(h-\tau_k^i)\hat{y}_i(t_k)\\
&\quad\times\sum_{j=1}^{m} l_{ij}^E\Big(d_{ij}^-(t_k)\hat{y}_j(t_{k-1})+d_{ij}(t_k)\hat{y}_j(t_k)\Big)
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{p=1}^{k-1}\sum_{i=1}^{m}\sqrt{h+\tau_{p+1}^i-\tau_p^i}\,\hat{y}_i(t_p)\sum_{j=1}^{m}\sqrt{h+\tau_{p+1}^i-\tau_p^i}\,l_{ij}^E\\
&\times\Bigg(\frac{d_{ij}^-(t_p)}{\sqrt{h+\tau_p^j-\tau_{p-1}^j}}\sqrt{h+\tau_p^j-\tau_{p-1}^j}\,\hat{y}_j(t_{p-1})\\
&+\frac{d_{ij}(t_p)}{\sqrt{h+\tau_{p+1}^j-\tau_p^j}}\sqrt{h+\tau_{p+1}^j-\tau_p^j}\,\hat{y}_j(t_p)\Bigg)\\
&+\sum_{i=1}^{m}\sqrt{h-\tau_k^i}\,\hat{y}_i(t_k)\sum_{j=1}^{m}\sqrt{h-\tau_k^i}\,l_{ij}^E\\
&\times\Bigg(\frac{d_{ij}^-(t_k)}{\sqrt{h+\tau_k^j-\tau_{k-1}^j}}\sqrt{h+\tau_k^j-\tau_{k-1}^j}\,\hat{y}_j(t_{k-1})\\
&+\frac{d_{ij}(t_k)}{\sqrt{h-\tau_k^j}}\sqrt{h-\tau_k^j}\,\hat{y}_j(t_k)\Bigg)\\
\le&\sum_{p=1}^{k-1}\sum_{i=1}^{m}\Bigg(\Big(\sqrt{2}\tau_{asyn}\sigma_\Xi+\frac{\sqrt{2h\tau_{\max}}\sigma_\Xi}{2}\Big)(h+\tau_{p+1}^i-\tau_p^i)\hat{y}_i(t_p)^2\\
&+\frac{\sqrt{2h\tau_{\max}}\sigma_\Xi}{2}(h+\tau_p^i-\tau_{p-1}^i)\hat{y}_i(t_{p-1})^2\Bigg)\\
&+\sum_{i=1}^{m}\Big(\tau_{asyn}\sigma_\Xi+\frac{\sqrt{h\tau_{\max}}\sigma_\Xi}{2}\Big)(h-\tau_k^i)\hat{y}_i(t_k)^2\\
&+\sum_{i=1}^{m}\frac{\sqrt{h\tau_{\max}}\sigma_\Xi}{2}(h+\tau_k^i-\tau_{k-1}^i)\hat{y}_i(t_{k-1})^2
\end{aligned}
$$

where the last inequality follows from Lemma 1 and the following fact:

$$
\begin{cases}
\sqrt{\dfrac{h+\tau_{p+1}^i-\tau_p^i}{h+\tau_p^j-\tau_{p-1}^j}}\,d_{ij}^-(t_p)<\sqrt{2h\tau_{\max}}\\
\sqrt{\dfrac{h+\tau_{p+1}^i-\tau_p^i}{h+\tau_{p+1}^j-\tau_p^j}}\,d_{ij}(t_p)<\sqrt{2}\tau_{asyn}\\
\sqrt{\dfrac{h-\tau_k^i}{h+\tau_k^j-\tau_{k-1}^j}}\,d_{ij}^-(t_k)<\sqrt{h\tau_{\max}}\\
\sqrt{\dfrac{h-\tau_k^i}{h-\tau_k^j}}\,d_{ij}(t_k)\le\tau_{asyn}
\end{cases}
$$

Therefore, we have

$$
\begin{aligned}
V(t_{k+1})\le&V(t_1)-\Big(1-\alpha-(h+\tau_{var})\lambda_n\Big)\sum_{i=1}^{m}\tau_1^i\hat{y}_i(t_0)^2\\
&+\frac{\sqrt{2h\tau_{\max}}\sigma_\Xi}{2}(h+\tau_1^i-\tau_0^i)\hat{y}_i(t_0)^2\\
&-\sum_{i=1}^{m}\sum_{p=1}^{k-2}\Big(1-\alpha-(h+\tau_{var})\lambda_n-\sqrt{2}\tau_{asyn}\sigma_\Xi\\
&-\sqrt{2h\tau_{\max}}\sigma_\Xi\Big)(h+\tau_{p+1}^i-\tau_p^i)\hat{y}_i(t_p)^2\\
&-\Big(1-\alpha-(h+\tau_{var})\lambda_n-\sqrt{2}\tau_{asyn}\sigma_\Xi
\end{aligned}
$$

$$- \frac{(\sqrt{2}+1)\sqrt{h\tau_{\max}}}{2}\sigma_{\Xi}\Big)(h+\tau_k^i-\tau_{k-1}^i)\hat{y}_i(t_{k-1})^2$$
$$- \Big(1-\alpha-(h+\tau_{var})\lambda_n - \tau_{asyn}\sigma_{\Xi}$$
$$- \frac{\sqrt{h\tau_{max}}\sigma_{\Xi}}{2}\Big)\sum_{i=1}^{m}(h-\tau_k^i)\hat{y}_i(t_k)^2$$

Inequality (4) guarantees that the parameters before $\hat{y}_i(t_p)$, $p = 1, 2, \ldots, k$, are all negative and thus

$$\lim_{k\to\infty} \hat{y}(t_k)^T \hat{y}(t_k) = 0.$$

By Lemma 3 and equation (6), we have $\lim_{k\to\infty} y(t_k)^T y(t_k) = 0$ and $\lim_{t\to\infty} \dot{y}(t) = 0$, respectively, which together yield that

$$\lim_{t\to\infty} y(t) = 0;$$

in other words,

$$\lim_{t\to\infty} x(t)^T M_I {M_I}^T x(t) = 0.$$

Since graph $\mathcal{G}$ is connected, we have

$$\lim_{t\to\infty} x_i(t) = \delta, i = 1, 2, \ldots, n.$$

4. SIMULATION

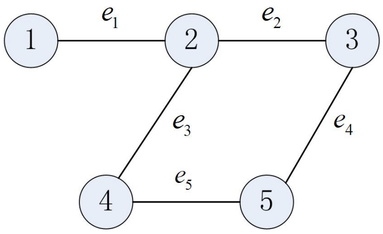

Fig. 1. Interaction Topology

In the simulation, we consider a network of 5 agents with randomly generated initial states 3.9652, 0.6159, 7.8018, 3.3758, 6.0787, under the interaction topology given in Fig. 1. Then $\lambda_n = 4.48$ and by Gershgorin Circle Theorem, σ_{Ξ} is upper bounded by 5. By assuming that $\tau_{\max} = 0.8h$, $\tau_{asyn} \le 0.3h$, and $\tau_{var} \le 0.3h$, we get that $h = 0.021$ is a sufficient condition for solving the average consensus problem. In the simulation, $h = 0.0208$ and the time-delays are randomly generated between $\tau_{\max} - \tau_{asyn}$ and $\tau_{\max}$. The state trajectories and the edge-event number in each period of h are shown in Fig. 2.

5. CONCLUSIONS

In this paper, by Lyapunov methods, we studied the delay robustness of a class of time-event hybrid-driven sampled-data consensus protocols and examined the relationship between the maximum event-detecting periods and transmission time-delays. We showed that they are all determined by the algebraic structure of interaction topologies. However, we didn't solve the problem on how to compute σ_{Ξ}, which is used for estimating allowable time-delays. Our future work will be aimed at extending the present results in more general settings, such as fixed or time-varying directed networks and asynchronous periodic or aperiodic event-detections with multiple time-varying delays.

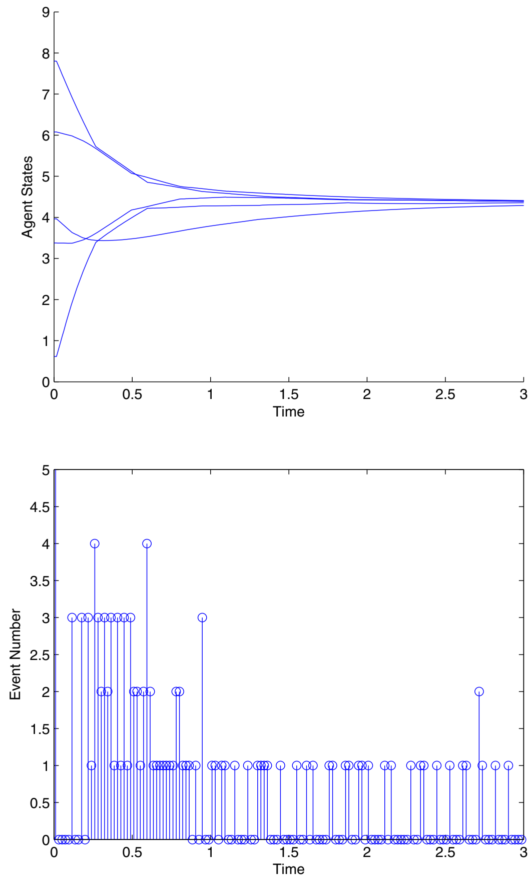

Fig. 2. State trajectories and edge-event numbers in each period of h

APPENDIX

Lemma 1. For any column vectors ξ and ζ and real matrix A with compatible dimensions,

$$\xi^T A\zeta \le \frac{1}{2}\sigma_{\max}(A)(\xi^T\xi + \zeta^T\zeta),$$

where $\sigma_{\max}(A)$ is the largest singular value of A.

Proof. This lemma is obvious by observing that

$$\xi^T A\zeta = \frac{1}{2}[\xi^T, \zeta^T]\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} \xi \\ \zeta \end{bmatrix}.$$

■

Lemma 2. For any column vectors ξ, real positive-semidefinite matrix A, and nonnegative diagonal matrix Θ with compatible dimensions, if the diagonal entries of Θ are all not larger than 1, then

$$\xi^T \Theta A\xi \le \lambda_{\max}(A)\xi^T\xi,$$

$\lambda_{\max}(A)$ is the largest eigenvalue of A.

Proof. By Lemma 1, $\xi^T\Theta A\xi \le \frac{1}{2}\sigma_{\max}(A)(\xi^T\Theta^2\xi + \xi^T\xi) \le \sigma_{\max}(A)\xi^T\xi$. Note that $\sigma_{\max}(A) = \lambda_{\max}(A)$ by the positive-semidefinite property of A. ■

Lemma 3. For any pair of adjacent agents i, j at time t_k, $\hat{x}_{ij}(t_k)$ and $x_i(t_k) - x_j(t_k)$ share the same sign and

$$(1-\alpha)|\hat{x}_{ij}(t_k)| \le |x_i(t_k) - x_j(t_k)| \le (1+\alpha)|\hat{x}_{ij}(t_k)|.$$

Proof. We only prove the second part. If the mutual data-sampling between agents i and j is triggered, then $\kappa_{ij}(t_k) = k$ and thus $\hat{x}_{ij}(t_k) = x_i(t_k) - x_j(t_k)$; otherwise, $\kappa_{ij}(t_k) = \kappa_{ij}(t_{k-1})$. In the latter case, by equation (2), we have

$$\begin{aligned}|x_i(t_k) - x_j(t_k)| \leq &|\hat{x}_{ij}(t_k)| + |x_i(t_k) - x_i(t_{\kappa_{ij}(t_k)})| \\ &+ |x_j(t_k) - x_j(t_{\kappa_{ji}(t_k)})| \\ \leq &(1+\alpha)|\hat{x}_{ij}(t_k)|\end{aligned}$$

and

$$\begin{aligned}|x_i(t_k) - x_j(t_k)| \geq &|\hat{x}_{ij}(t_k)| - |x_i(t_k) - x_i(t_{\kappa_{ij}(t_k)})| \\ &- |x_j(t_k) - x_j(t_{\kappa_{ji}(t_k)})| \\ \geq &(1-\alpha)|\hat{x}_{ij}(t_k)|.\end{aligned}$$

■

REFERENCES

K. J. Åström (2008). Event based control. *Analysis and Design of Nonlinear Control Systems*, 127-147.

K. J. Åström and B. M. Bernhardsson(2002). Comparison of Riemann and Lebesgue sampling for first order stochastic systems. in *Proceedings of the 41st IEEE Conference on Decision and Control* (pp. 2011-2016).

M. Cao, A. S. Morse, and B. D. O. Anderson (2008). Agreeing asynchronously. *IEEE Transactions on Automatic Control*, 53(8), 1826-1838.

Y. Cao and W. Ren (2010). Sampled-data discrete-time coordination algorithms for double-integrator dynamics under dynamic directed interaction. *International Journal of Control*, 83(3), 506-515.

D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson (2012). Distributed event-triggered control for multi-agent systems, *IEEE Transactions on Automatic Control*, 57(5), 1291-1297.

Y. Fan, G. Feng, Y.Wang, and C. Song(2013). Distributed event-triggered control of multi-agent systems with combinational measurements. *Automatica*, 49, 671-675.

Y. Gao and L. Wang (2010). Consensus of multiple dynamic agents with sampled information. *IET Control Theory and Applications*, 4(6), 945-956.

Y. Gao and L. Wang (2011). Sampled-data based consensus of continuous-time multi-agent systems with time-varying topology. *IEEE Transactions on Automatic Control*, 56(5), 1226-1231.

Z. Ji, H. Lin, and H. Yu(2012). Leaders in multi-agent controllability under consensus algorithm and tree topology. *Systems & Control Letters*, 61(9), 918-925.

M. D. Lemmon (2010). Event-triggered feedback in control, estimation, and optimization, in networked control systems. *Lecture Notes in Control and Information Sciences*, Springer-Verlag Berlin Heidelburg, 405, 293-358.

J. Lin, A. S. Morse, and B. D. O. Anderson (2004). The multiagent rendezvous problem – the asynchronous case. in *Proceedings of the 43rd IEEE Conference on Decision and Control* (pp. 1926-1931).

Z. W. Liu, Z. H. Guan, X. Shen, and G. Feng (2012). Consensus of multi-agent networks with aperiodic sampled communication via impulsive algorithms using position-only measurements. *IEEE Transactions on Automatic Control*, 57(10), 2639-2643.

M. Mesbahi and M. Egerstedt (2010). *Graph Theoretic Methods in Multiagent Networks*, Princeton University Press, Princeton and Oxford.

R. Olfati-Saber, J. A. Fax, and R. M. Murray (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1), 215-233.

J. Qin and H. Gao (2012). A sufficient condition for convergence of sampled-data consensus for double-integrator dynamics with nonuniform and time-varying communication delays. *IEEE Transactions on Automatic Control*, 57(9), 2417-2422.

W. Ren and R. W. Beard (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5), 655-661.

G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson (2013). Event-based broadcasting for multi-agent average consensus. *Automatica*, 49, 245-252.

F. Xiao, X. Meng, and T. Chen (2012). Average sampled-data consensus driven by edge events. in *Proceedings of the 31st Chinese Control Conference* (pp. 6239-6244).

F. Xiao and L. Wang (2008). Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays. *IEEE Transactions on Automatic Control*, 53(8), 1804-1816.

G. Xie, H. Liu, L. Wang, and Y. Jia (2009a). Consensus in networked multi-agent systems via sampled control: fixed topology case. in *Proceedings of the 2009 American Control Conference* (pp. 3902-3907).

G. Xie, H. Liu, L. Wang, and Y. Jia (2009b). Consensus in networked multi-agent systems via sampled control: switching topology case. in *Proceedings of the 2009 American Control Conference* (pp. 4525-4530).