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Design and Analysis of Energy-based Controller for 3-Link Robots with a Single Actuator ⋆

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Abstract: For a 3-link planar robot moving in the vertical plane with a single actuator, this paper studies the effect of its actuator configuration from the perspective of the energy-based control. For the control objective of simultaneous stabilization of the actuated variable and of the desired level of the total mechanical energy of the robot corresponding to those of the upright equilibrium point, where all three links of the robot are in the upright position, this paper presents an energy-based controller for three configurations of the single actuator in a unified way. Moreover, this paper carries out a global motion analysis of the robot with a single actuator at joint 1 or 2 under the presented controller. Specifically, this paper shows that the control objective is achievable for the robot with the actuator at joint 1 for almost all initial conditions of angular displacements and velocities without any condition on the mechanical parameters of the robot, and the control objective is achievable for the robot with the actuator at joint 2 for almost all initial conditions provided that another condition on the mechanical parameters of the robot is satisfied. The numerical simulation shows that the presented control can be applied to the swing-up and stabilizing control for a physical 3-link robot with a single actuator at joint 1 or 2.

Keywords: Underactuated robotic systems, two passive joints, robot control, nonlinear control, controllability, motion analysis, Lyapunov stability.

1. INTRODUCTION

The last two decades have witnessed considerable progresses in the study of underactuated robotic systems, see, e.g., Spong [1995], Reyhanoglu et al. [1999], Jiang [2002], Fang et al. [2012]. One of the most important control problems for underactuated robots with passive (unactuated) joint(s) is the set-point control (regulation or stabilization) of a desired equilibrium point of the robots, that is, to find a feedback control law that makes the desired equilibrium point asymptotically stable. Many researchers studied a particular problem of the set-point control called the swing-up and stabilizing control (Furuta et al. [1991], Spong [1995], Fantoni et al. [2000], Kolesnichenko and Shiriaev [2002], Ma and Su [2002]). Indeed, the swing-up and stabilizing control is to swing a planar robot to a small neighborhood of the UEP (upright equilibrium point) and then balance it about that point, where all links are in the upright position. The swing-up and stabilizing control for various kinds of 2-DOF (degree-of-freedom) systems has been solved by the energy-based control, see, e.g., Fantoni et al. [2000], Kolesnichenko and Shiriaev [2002] regarding the Pendubot, and Xin and Kaneda [2007] regarding the Acrobot.

Although there are many results for mechanical systems with underactuation degree one (Grizzle et al. [2005], Acosta et al. [2005]), that is, the number of control inputs is one less than that of degrees of freedom, designing and analyzing controllers for mechanical systems with underactuation degree greater than one is still a challenging problem. In this paper, we study a 3-link planar robot moving in the vertical plane with a single actuator, that is, a robot with underactuation degree two. Specifically, we investigate the effect of its actuator configuration from the perspective of the energy-based control. We say a joint of a robot is active (A) if it is actuated and is passive (P) if it is unactuated. The APP robot, PAP robot, and PPA robot denote the robot with the actuator at the joints 1, 2, and 3, respectively. These robots are three cases of the failure of two actuators of the 3-link planar robot.

For the 3-link planar robot with a single actuator, we study whether we can achieve the control objective of simultaneous stabilization of the actuated variable and of the desired level of the total mechanical energy of the robot corresponding to those of the UEP by using the energy-based control proposed in Fantoni et al. [2000], Kolesnichenko

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and Shiriaev [2002]. We present an energy-based controller for three configurations of the single actuator in a unified way. Moreover, we present a global motion analysis of the APP and PAP robots under the controller and present some conditions on control gains for achieving the control objective. For the APP robot, we show that the control objective is achievable for almost all initial conditions without any condition on the mechanical parameters of the robot. However, for the PAP robot, we show that it is achievable for almost all initial conditions provided that another condition on the mechanical parameters of the robot is satisfied. In this way, we show two differences between the APP and PAP robots from the perspective of the energy-based control.

Finally, when the control objective of the energy-based control is achieved, it has been proved that the underactuated robots with underactuation degree one can be swung-up to any arbitrarily small neighborhood of the UEP due to tracking homoclinic orbits of these robots, see, e.g., Fantoni et al. [2000], Kolesnichenko and Shiriaev [2002], Xin and Kaneda [2007]. However, different from this fact, it is unclear theoretically whether there exists time such that one of the APP and PAP robots can be swung up close to the UEP. This shows a limitation of using the energy-based control to solve the swing-up and stabilizing control for underactuated robots with underactuation degree two. From numerical simulation investigations to the physical 3-link robot in Nishimura and Funaki [1998], we find that by choosing control gains appropriately each of the APP and PAP robots can be swung up close to the UEP for achieving a successful switch to a local stabilizing controller for balancing at the UEP.

2. PRELIMINARY KNOWLEDGE

Consider a 3-link planar robot with a single actuator shown in Fig. 1. For link i ($i = 1, 2, 3$), m_i is its mass, l_i is its length, l_{ci} is the distance from joint i to its center of mass (COM), and J_i is the moment of inertia around its COM, θ_i is the angle measured counter-clockwise from the positive Y-axis.

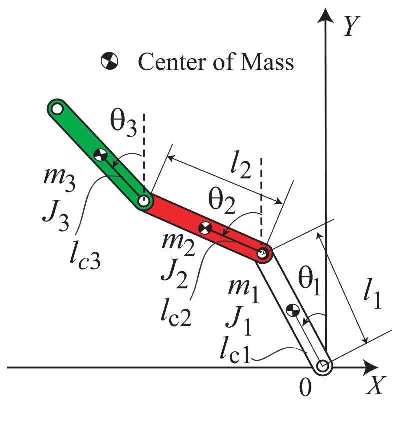

Fig. 1. 3-link planar robot.

In this paper, to simplify notations, for $i, j = 1, 2, 3$, we use $S_i = \sin\theta_i$, $C_i = \cos\theta_i$, $S_{ij} = \sin(\theta_i - \theta_j)$, and $C_{ij} = \cos(\theta_i - \theta_j)$. The motion equation of the robot is

$$M(\theta)\ddot{\theta} + H(\theta, \dot{\theta}) + G(\theta) = W\tau, \tag{1}$$

where $\theta = [\,\theta_1,\ \theta_2,\ \theta_3\,]^{\mathrm{T}}$ and

$$M = \begin{bmatrix} \alpha_{11} & \alpha_{12}C_{21} & \alpha_{13}C_{31} \\ \alpha_{12}C_{21} & \alpha_{22} & \alpha_{23}C_{32} \\ \alpha_{13}C_{31} & \alpha_{23}C_{32} & \alpha_{33} \end{bmatrix}, \tag{2}$$

$$H = \begin{bmatrix} 0 & -\alpha_{12}\dot{\theta}_2 S_{21} & -\alpha_{13}\dot{\theta}_3 S_{31} \\ \alpha_{12}\dot{\theta}_1 S_{21} & 0 & -\alpha_{23}\dot{\theta}_3 S_{32} \\ \alpha_{13}\dot{\theta}_1 S_{31} & \alpha_{23}\dot{\theta}_2 S_{32} & 0 \end{bmatrix}\dot{\theta}, \tag{3}$$

$$G = \begin{bmatrix} -\beta_1 S_1 \\ -\beta_2 S_2 \\ -\beta_3 S_3 \end{bmatrix}, \tag{4}$$

with

$$W = \begin{cases} [\,1,\ 0,\ 0\,]^{\mathrm{T}}, \text{ the APP robot} \\ [\,0,\ 1,\ 0\,]^{\mathrm{T}}, \text{ the PAP robot} \\ [\,0,\ 0,\ 1\,]^{\mathrm{T}}, \text{ the PPA robot} \end{cases}, \tag{5}$$

$$\begin{cases} \alpha_{11} = J_1 + m_1 l_{c1}^2 + (m_2 + m_3) l_1^2, \\ \alpha_{22} = J_2 + m_2 l_{c2}^2 + m_3 l_2^2, \\ \alpha_{33} = J_3 + m_3 l_{c3}^2, \\ \alpha_{12} = (m_2 l_{c2} + m_3 l_2) l_1, \\ \alpha_{13} = m_3 l_1 l_{c3}, \\ \alpha_{23} = m_3 l_2 l_{c3}, \end{cases} \tag{6}$$

$$\begin{cases} \beta_1 = (m_1 l_{c1} + m_2 l_1 + m_3 l_1) g, \\ \beta_2 = (m_2 l_{c2} + m_3 l_2) g, \\ \beta_3 = m_3 l_{c3} g, \end{cases} \tag{7}$$

where g is the acceleration of gravity.

The total mechanical energy of the robot is

$$E(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^{\mathrm{T}} M(\theta)\dot{\theta} + P(\theta), \tag{8}$$

where $P(\theta)$ is its potential energy defined as

$$P(\theta) = \beta_1 C_1 + \beta_2 C_2 + \beta_3 C_3. \tag{9}$$

3. DESIGN OF ENERGY-BASED CONTROLLER FOR 3-LINK ROBOT WITH A SINGLE ACTUATOR

In this section, we apply the energy-based control design procedure in Fantoni et al. [2000], Kolesnichenko and Shiriaev [2002] to design a controller for three different configurations of a single actuator of the 3-link robot. Consider the following UEP:

$$\theta = 0, \quad \dot{\theta} = 0. \tag{10}$$

Let E_r be the potential energy of the robot at the UEP.

$$E_r = E(\theta, \dot{\theta})\Big|_{\theta=0,\ \dot{\theta}=0} = \beta_1 + \beta_2 + \beta_3. \tag{11}$$

Since the actuated angular displacement can be expressed as $W^{\mathrm{T}}\theta$ (which is a scalar), the control objective is to design τ such that

$$\lim_{t\to\infty} E = E_r, \quad \lim_{t\to\infty} W^{\mathrm{T}}\dot{\theta} = 0, \quad \lim_{t\to\infty} W^{\mathrm{T}}\theta = 0. \tag{12}$$

The Lyapunov function candidate is

$$V = \frac{1}{2}(E - E_r)^2 + \frac{k_D}{2}(W^{\mathrm{T}}\dot{\theta})^2 + \frac{k_P}{2}(W^{\mathrm{T}}\theta)^2, \tag{13}$$

where scalars $k_D > 0$ and $k_P > 0$ are control parameters.

Using $\dot{E} = W^{\mathrm{T}}\dot{\theta}\tau$, we obtain the time-derivative of V along the trajectories of (1) as follows:

$$\dot{V} = W^{\mathrm{T}}\dot{\theta}\left((E-E_r)\tau + k_D W^{\mathrm{T}}\ddot{\theta} + k_P W^{\mathrm{T}}\theta\right). \quad (14)$$

If we can find τ such that

$$(E-E_r)\tau + k_D W^{\mathrm{T}}\ddot{\theta} + k_P W^{\mathrm{T}}\theta = -k_V W^{\mathrm{T}}\dot{\theta}, \quad (15)$$

for $k_V > 0$, then

$$\dot{V} = -k_V (W^{\mathrm{T}}\dot{\theta})^2 \leq 0. \quad (16)$$

To investigate under what condition (15) is solvable with respect to τ for any $(\theta,\ \dot{\theta})$, we obtain $\ddot{\theta}$ from (1) and substitute it into (15). This yields $\Lambda(\theta,\dot{\theta})\tau = k_D W^{\mathrm{T}} M^{-1}(\theta)(H(\theta,\dot{\theta})+G(\theta)) - k_V W^{\mathrm{T}}\dot{\theta} - k_P W^{\mathrm{T}}\theta$, where

$$\Lambda(\theta,\dot{\theta}) = E(\theta,\dot{\theta}) - E_r + k_D W^{\mathrm{T}} M^{-1}(\theta) W. \quad (17)$$

Hence, when

$$\Lambda(\theta,\dot{\theta}) \neq 0, \quad \text{for } \forall(\theta,\dot{\theta}), \quad (18)$$

we obtain

$$\tau = \Lambda^{-1}\left(k_D W^{\mathrm{T}} M^{-1}(H+G) - k_V W^{\mathrm{T}}\dot{\theta} - k_P W^{\mathrm{T}}\theta\right). (19)$$

By using LaSalle's invariant principle Khalil [2002], we can obtain the following lemma.

Lemma 1. Consider the closed-loop system consisting of the robot (1) and the controller (19). Suppose that $k_D > 0$, $k_P > 0$, and $k_V > 0$. Then, the controller (19) has no singularities for any $(\theta,\ \dot{\theta})$ if and only if

$$k_D > k_{Dm} = \max_{\theta}\left((E_r - P(\theta))(W^{\mathrm{T}} M^{-1}(\theta) W)^{-1}\right) \quad (20)$$

holds. In this case,

$$\lim_{t\to\infty} V = V^*, \quad \lim_{t\to\infty} E = E^*, \quad (21)$$

$$\lim_{t\to\infty} W^{\mathrm{T}}\theta = \text{a constant}, \quad \lim_{t\to\infty} W^{\mathrm{T}}\dot{\theta} = 0, \quad (22)$$

where V^* and E^* are constant. Moreover, as $t \to \infty$, every closed-loop solution, $(\theta(t),\ \dot{\theta}(t))$, approaches the invariant set

$$\Xi = \left\{(\theta,\dot{\theta}) \;\middle|\; E(\theta,\dot{\theta}) \equiv E^*,\ W^{\mathrm{T}}\theta \equiv \text{a constant}\right\}, \quad (23)$$

where "$\equiv$" denotes that an equality holds for all time t.

4. GLOBAL MOTION ANALYSIS

We analyze the motion of the APP and PAP robots to characterize the invariant set Ξ in (23) by studying two cases of $E^* \neq E_r$ and $E^* = E_r$, separately. Consider an equilibrium configuration $\theta^e = [\,\theta_1^e,\ \theta_2^e,\ \theta_3^e\,]^{\mathrm{T}}$ under equilibrium torque τ^e. From (15) (from which we derived the controller (19)) and (1) with $E(\theta^e, 0) \equiv P(\theta^e)$ and $\theta \equiv \theta^e$, we obtain

$$(P(\theta^e) - E_r)\tau^e + k_P W^{\mathrm{T}}\theta^e = 0, \quad (24)$$

$$G(\theta^e) = W\tau^e. \quad (25)$$

4.1 APP Robot

Regarding the APP robot with $W = [\,1,\ 0,\ 0\,]^{\mathrm{T}}$, from (24) and (25), we obtain

$$\beta_1(E_r - P(\theta^e))\sin\theta_1^e + k_P\theta_1^e = 0, \quad (26)$$

$$\theta_i^e = 0 \text{ or } \pi \pmod{2\pi}, \quad i = 2,3. \quad (27)$$

Define

$$\Omega_{app} = \{(\theta^e, 0) \mid \theta^e \text{ satisfies (26), (27), } P(\theta^e) \neq E_r\} \quad (28)$$

and

$$S_{app} = \left\{(\theta,\dot{\theta}) \;\middle|\; E(\theta,\dot{\theta}) \equiv E_r,\ \theta_1 \equiv 0\right\}. \quad (29)$$

We present the following theorem.

Theorem 1. Consider the closed-loop system consisting the APP robot described by (1) and the controller (19). Suppose that k_D satisfies (20), $k_P > 0$ and $k_V > 0$ hold. Then, as $t \to \infty$, every closed-loop solution, $(\theta(t),\ \dot{\theta}(t))$, approaches the invariant set:

$$\Xi_{app} = \Omega_{app} \cup S_{app}, \quad (30)$$

where Ω_{app} (corresponding to $E^* \neq E_r$ with E^* be the convergent value of E defined in Lemma 1) and S_{app} (corresponding to $E^* = E_r$) are defined in (28) and (29), respectively.

If one of the equilibrium points in Ω_{app} is stable, then the control objective (12) can not be achieved. Regarding Ω_{app} defined in (28), $\theta_1^e = 0$ satisfies (26) for any k_P. We find a condition on k_P such that (26) has a unique solution $\theta_1^e = 0$ for all θ_2^e and θ_3^e in (27). This yields that Ω_{app} in (28) just contains the up–up–down, up–down–up, and up–down–down equilibrium points defined in

$$\Omega_{apps} = \{(\theta^e, 0) \mid \theta^e \equiv (0,0,\pi),\ (0,\pi,0),\ (0,\pi,\pi)\}. \quad (31)$$

We present another main result of this paper.

Theorem 2. Consider the closed-loop system consisting the APP robot described by (1) and the controller (19). Suppose that k_D satisfies (20), $k_P > 0$ and $k_V > 0$ hold. If k_P satisfies

$$k_P > k_{Pm} = \max_{\pi \leq w \leq 2\pi} f_{app}(w), \quad (32)$$

where for $w \neq 0$,

$$f_{app}(w) = \frac{-\beta_1\left(\beta_1(1-\cos w) + 2\beta_2 + 2\beta_3\right)\sin w}{w}. \quad (33)$$

Then Ω_{app} in (28) equals Ω_{apps} in (31). Moreover, the Jacobian matrix evaluated at each equilibrium point of Ω_{apps} has at least one eigenvalue in the open-right-half-plane; that is, each equilibrium point of Ω_{apps} is unstable.

We provide the following remark for Theorems 1 and 2.

Remark 1. Since the Jacobian matrix evaluated at each equilibrium point of Ω_{apps} has at least one eigenvalue in the open-right-half-plane, the set of initial states from which E is not convergent to E_r is of Lebesgue measure zero, see e.g., Ortega et al. [2002] (p. 1225). Thus, under the controller (19), from all initial states with the exception of a set of Lebesgue measure zero of the APP robot, the closed-loop solution $(\theta(t), \dot{\theta}(t))$ approaches S_{app}, and the control objective (12) is achieved.

4.2 PAP Robot

Regarding the PAP robot with $W = [\,0,\ 1,\ 0\,]^{\mathrm{T}}$, from (24) and (25), we obtain

$$\beta_2(E_r - P(\theta^e))\sin\theta_2^e + k_P\theta_2^e = 0, \tag{34}$$

$$\theta_i^e = 0 \ \text{ or } \ \pi \quad (\mathrm{mod}\ \ 2\pi), \quad i = 1, 3. \tag{35}$$

Define

$$\Omega_{pap} = \{(\theta^e, 0) \mid \theta^e \text{ satisfies (34), (35), } P(\theta^e) \neq E_r\} \tag{36}$$

and

$$S_{pap} = \left\{(\theta, \dot{\theta}) \;\middle|\; E(\theta, \dot{\theta}) \equiv E_r,\ \theta_2 \equiv 0\right\}. \tag{37}$$

We present the following theorem.

Theorem 3. Consider the closed-loop system consisting the PAP robot described in (1) and the controller (19). Suppose that the mechanical parameters of the PAP robot satisfy

$$(\alpha_{12}^2\alpha_{13}\alpha_{23} + 2\alpha_{12}\alpha_{23}^2\alpha_{33} - 3\alpha_{13}\alpha_{23}^3)\beta_1 + (\alpha_{12}\alpha_{13}\alpha_{23}^2 + 2\alpha_{11}\alpha_{12}^2\alpha_{23} - 3\alpha_{12}^3\alpha_{13})\beta_3 = 0. \tag{38}$$

Suppose that k_D satisfies (20), $k_P > 0$ and $k_V > 0$ hold. Then, with the quantities defined in Lemma 1, as $t \to \infty$, every closed-loop solution, $(\theta(t),\ \dot{\theta}(t))$, approaches the invariant set:

$$\Xi_{pap} = \Omega_{pap} \cup S_{pap}, \tag{39}$$

where Ω_{pap} (corresponding to $E^* \neq E_r$) and S_{pap} (corresponding to $E^* = E_r$) are defined in (36) and (37), respectively.

We give the following remark about Theorem 3.

Remark 2. It is worth pointing out that the condition (38) is a sufficient condition such that the robot remains at an equilibrium point under the controller (19) for the case $E^* \neq E_r$. Different from the corresponding result for the APP robot, there is no condition on the mechanical parameters of the robot. The physical explanation of the condition (38) is still under investigation.

Similar to Theorem 2, we find a condition on k_P such that (34) has a unique solution $\theta_2^e = 0$ for all θ_1^e and θ_3^e in (35). This yields that Ω_{pap} in (36) just contains the up-up-down, down-up-up, and down-up-down equilibrium points defined in

$$\Omega_{paps} = \{(\theta^e, 0) \mid \theta^e \equiv (0, 0, \pi),\ (\pi, 0, 0),\ (\pi, 0, \pi)\}\,. \tag{40}$$

We present another main result of this paper.

Theorem 4. Consider the closed-loop system consisting the PAP robot (1) and the controller (19). Suppose that the mechanical parameters of the PAP robot satisfy (38). Suppose that k_D satisfies (20), $k_P > 0$ and $k_V > 0$ hold. If k_P satisfies

$$k_P > k_{Pm} = \max_{\pi \leq w \leq 2\pi} f_{pap}(w), \tag{41}$$

where for $w \neq 0$,

$$f_{pap}(w) = \frac{-\beta_2\left(\beta_2(1-\cos w) + 2\beta_1 + 2\beta_3\right)\sin w}{w}. \tag{42}$$

Then Ω_{pap} in (36) equals Ω_{paps} in (40). Moreover, the Jacobian matrix evaluated at each equilibrium point of Ω_{paps} has at least one eigenvalue in the open-right-half-plane; that is, each equilibrium point of Ω_{paps} is unstable.

Finally, although the UEP is included in S_{app} in (29) or S_{pap} in (37), it is unclear whether there exists time for the APP or PAP robot moving close to the point. Through our simulation investigation, we find that the APP or PAP robot can be swung up close to the point by choosing the control parameters k_D, k_P, and k_V. Since the UEP is unstable in the closed-loop system, when the APP or PAP robot is swung up close to the point, we need to switch the controller (19) to a local stabilizing controller (43) to balance the robot about the point, see Section 5 for further reference.

5. SIMULATION RESULTS

We verified the theoretical results via numerical simulation for using the mechanical parameters of the physical 3-link robot in Nishimura and Funaki [1998] which are given in Table 1. We took $g = 9.8$ m/s^2.

Table 1. Mechanical parameters of physical 3-link robot in Nishimura and Funaki [1998].

Link i	Link 1	Link 2	Link 3
m_i [kg]	0.41	4.10	0.41
l_i [m]	0.268	0.258	0.268
l_{ci} [m]	0.134	0.128	0.095
J_i [kg $\cdot$ m^2]	4.52×10^{-3}	6.11×10^{-2}	4.52×10^{-3}

When the robot is linearly controllable at the UEP, the robot can be locally stabilized about the UEP by a state-feedback controller

$$\tau = -Kx, \tag{43}$$

where constant gain K can be obtained by apply the LQR method to the linearized model of the 3-link robot in (1) around the UEP for example.

Took an initial condition $\left(\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3\right) = (-5\pi/6, -6\pi/7, -7\pi/8, 0, 0, 0)$, which is close to the downward equilibrium point $(\theta_1, \theta_2, \theta_3, \dot{\theta}_1,\ \dot{\theta}_2, \dot{\theta}_3) = (-\pi, -\pi, -\pi, 0, 0, 0)$. The condition for switching the controller (19) to the controller (43) was taken as

$$|\theta_1| + |\theta_2| + |\theta_3| + 0.1|\dot{\theta}_1| + 0.1|\dot{\theta}_2| + 0.1|\dot{\theta}_3| < 0.5. \tag{44}$$

5.1 APP Robot

From (20) and (32), we have $k_D > 11.2100$ and $k_P > 81.1069$. We validated that the objective (12) was achieved for the control parameters satisfying the above conditions, and $E^* \neq E_r$ did not occur in our simulation investigation. To swing-up the APP robot close to the UEP, we took $k_D = 12.8495$, $k_P = 101.6513$, and $k_V = 34.9863$. The simulation results are shown in Figs. 2 and 3. Figure 2 shows that V and $E - E_r$ converged to 0. Figure 3 shows that θ_1 converged to 0, and the time responses of θ_2 and θ_3 depicted by modular 2π were complicated. From the time response of three angles, we know that the robot was swung up very quickly close to the UEP during 4 s to 5 s.

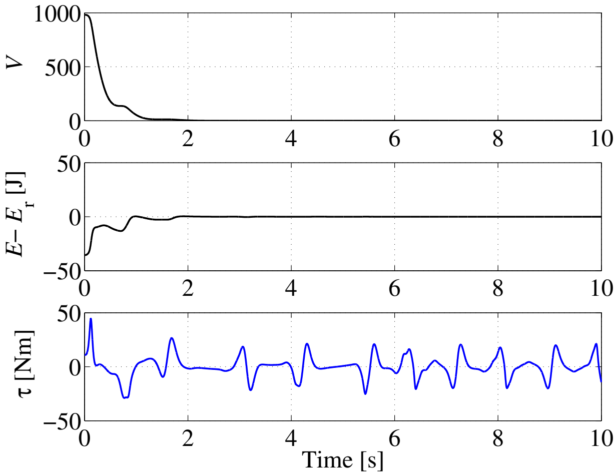

Fig. 2. Time responses of V, $E - E_r$, and τ for the APP robot under the controller (19).

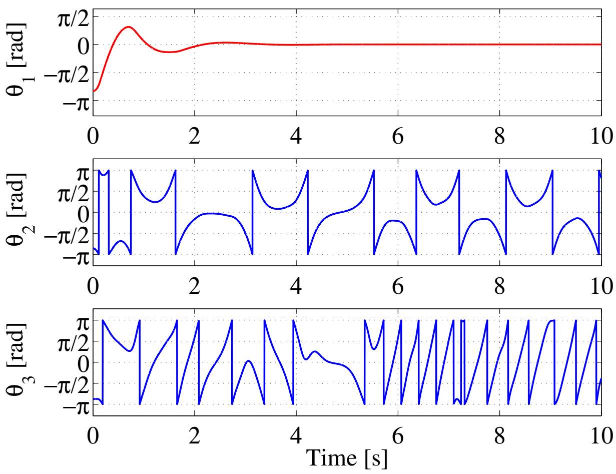

Fig. 3. Time responses of θ_1, θ_2, and θ_3 for the APP robot under the controller (19).

When the APP robot moved close to the UEP, according to (44), we switched the controller (19) to a local stabilizing controller (43), with $K = [24.8073, -130.5994, 285.4057, 11.0839, 5.2043, 40.7782]$, which was computed by using the Matlab function "lqr" with the weight matrix related to state x being a 6×6 identity matrix and the weight related to the torque being 1. The simulation result is given in Fig. 4. The switch was taken at about $t = 4.67$ s. Thus, we showed numerically effectiveness of the proposed control for swinging up and stabilizing control of this APP robot.

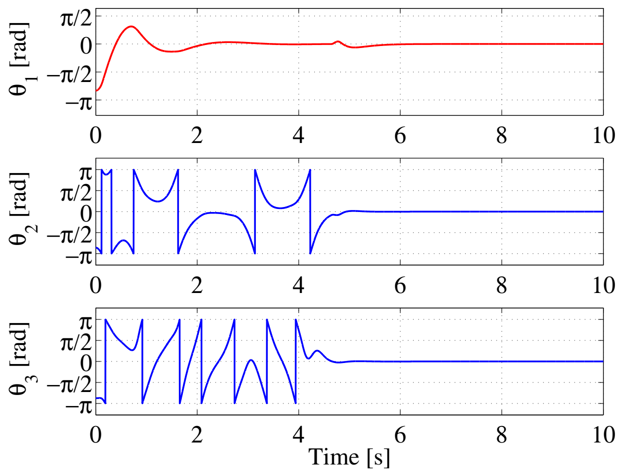

Fig. 4. Time responses of θ_1, θ_2, and θ_3 for the APP robot under the controllers (19) and (43).

5.2 PAP Robot

First, we verify that the condition (38) on the mechanical parameters holds for the above robot. This shows that the control objective for this PAP robot is achievable from all initial states with the exception of a set of Lebesgue measure zero provided the conditions on the control parameters (20) and (41), that is, $k_D > 5.1964$ and $k_P > 44.9673$, are satisfied. Second, to swing-up the PAP robot close to the UEP, we took $k_D = 5.2088$, $k_P = 46.8100$, $k_V = 49.3061$. The simulation results are shown in Figs. 5 and 6. Figure 5 shows that V and $E - E_r$ converged to 0. Figure 6 shows that θ_2 converged to 0, and the time responses of θ_1 and θ_3 depicted by modular 2π. From the time response of three angles, we know that the robot was swung up quickly close to the UEP during 10 s to 12 s.

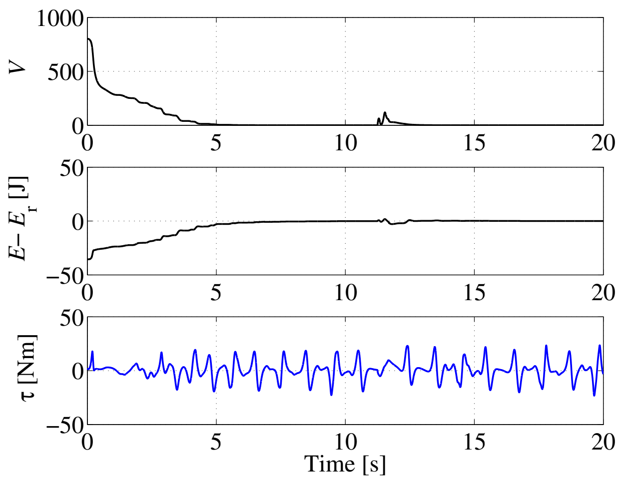

Fig. 5. Time responses of V, $E - E_r$, and τ for the PAP robot under the controller (19).

When the PAP robot moved close to the UEP, according to (44) we switched the controller (19) to a local stabilizing controller (43) with $K = [-752.7714, 12.4238, 480.9975, -79.9495, 5.8035, 68.2008]$. The simulation result is given in Fig. 7. The switch was taken at about $t = 11.2$ s.

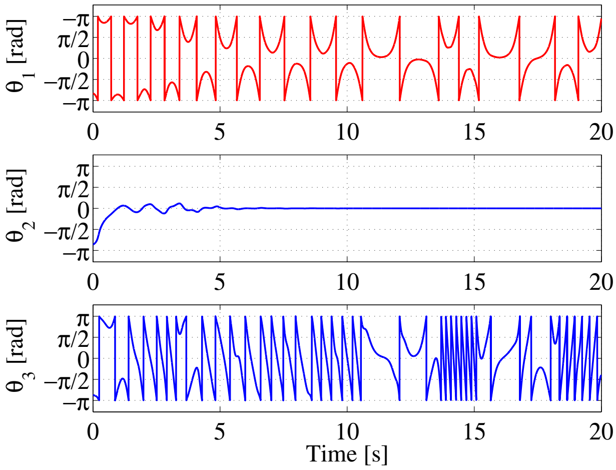

Fig. 6. Time responses of θ_1, θ_2, and θ_3 for the PAP robot under the controller (19).

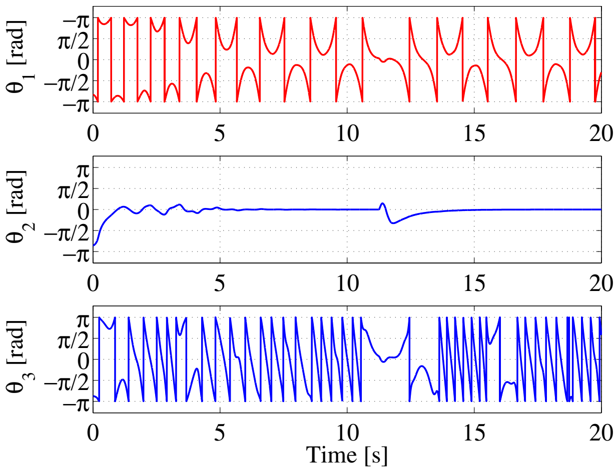

Fig. 7. Time responses of θ_1, θ_2, and θ_3 for the PAP robot under the controllers (19) and (43).

Thus, we showed numerically effectiveness of the proposed control for swinging up and stabilizing control of this PAP robot.

6. CONCLUSION

For the 3-link planar robot moving in the vertical plane with a single actuator, we studied the effect of its actuator configuration from the perspective of the energy-based control. We presented an energy-based control for three configurations of the single actuator in a unified way, and we succeeded in carrying out the global motion analysis of the APP and PAP robots under the presented controller by using properties of the mechanical parameters of the 3-link robot. Specifically, we showed that the control objective (12) for the APP robot is achievable for almost all initial conditions without any condition on the mechanical parameters of the robot. On the other hand, we showed that the control objective (12) for the PAP robot is achievable for almost all initial conditions provided that another condition on the mechanical parameters of the robot is satisfied. We also presented numerical results of the physical 3-link robot to validate the theoretical results and showed a successful application to the swing-up and stabilizing control of the APP and PAP robots.

It is expected that the energy-based controller (19) is effective for the PPA robot with some conditions on control gains and/or on the mechanical parameters of the robot after achieving a global motion analysis of the robot under the controller.

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