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Networked Control with a Stochastic Scheduling: a Time-Delay Approach ⋆

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Abstract: This paper is concerned with the stability analysis of networked control systems with communication constraints, variable delays and variable sampling intervals. The scheduling of sensor communication is defined by a stochastic protocol. The activation probability of each sensor node is a given constant, whereas it is assumed that collisions occur with a certain probability. The resulting closed-loop system is a stochastic impulsive system with delays in the continuous dynamics and in the reset equations. The system matrices have stochastic parameters with Bernoulli distributions. Sufficient conditions for the exponential mean-square stability are derived via a Lyapunov-Krasovskii-based method. The efficiency of the method is illustrated on the example of a batch reactor. It is demonstrated how our time-delay approach allows treating network-induced delays larger than the sampling intervals.

Keywords: networked control systems, Lyapunov functional, stochastic protocol, stochastic impulsive system

1. INTRODUCTION

Networked Control Systems (NCSs) are systems with the control loops closed over networks, where sensors, controllers and actuators can be distributed over a wide geographical area (see e.g., Antsaklis and Baillieul [2007]). In many such systems, only one node is allowed to use the communication channel at once. The communication along the data channel is then orchestrated by a scheduling rule called protocol. Three recent approaches for NCSs are based on discrete-time systems (Fujioka [2009]), impulsive/hybrid systems (Heemels et al. [2010])and time-delay systems (Fridman et al. [2004], Gao and Chen [2008]).

The time-delay approach has been introduced recently for the stabilization of NCSs under the Round-Robin (RR) protocol in Liu et al. [2012b] and under a weighted Try-Once-Discard (TOD) protocol in Liu et al. [2012a]. The closed-loop system is modeled as a switched system with multiple and ordered time-varying delays under RR scheduling or as a hybrid system with time-varying delays in the dynamics and in the reset equations under the TOD scheduling. Differently from the existing results on NCSs in the presence of scheduling protocols (in the frameworks of hybrid and discrete-time systems), the transmission delay is allowed to be larger than the sampling interval, but a crucial point is that data packet dropout is not allowed for large delays in Liu et al. [2012b] and Liu et al. [2012a].

A stochastic protocol was introduced in Tabbara and Nesic [2008] and analyzed for the input-output stability of NCSs in the presence of data packet dropouts or collisions based on hybrid systems. An i.i.d (independent and identically-distributed) sequence of Bernoulli random variables is applied to describe the stochastic protocol. Communication delays, however, are not included in the analysis. The stability of NCSs under a stochastic protocol, where the activated node is modeled by a Markov chain, has been studied in Donkers et al. [2012] by applying the discrete-time modeling framework. In Donkers et al. [2012], data packet dropouts can be regarded as prolongations of the sampling interval for small delays.

In the present paper, to overcome the gap between the data packet dropouts phenomenon and the large communication delays for scheduling protocols (e.g., deterministic protocols: RR and TOD), we develop the time-delay approach to the stability analysis of NCSs under a stochastic protocol. As in Liu et al. [2012b] and Liu et al. [2012a], differently from the hybrid and discrete-time approaches, we allow the transmission delays to be larger than the sampling intervals in the presence of scheduling protocols. For the simplicity of technical development, we consider a system with only two sensor nodes. A Remark 4 is given for extension to the case of more sensors nodes. The activation probability of each node is a given constant, whereas it is assumed that the collisions occur with a certain probability. The resulting closed-loop system is a stochastic impulsive system with delays in the continuous dynamics and in the reset equations. The system matrices have stochastic parameters with Bernoulli distributions. By developing appropriate Lyapunov-Krasovskii techniques, we derive Linear Matrix Inequalities (LMIs) conditions for the exponential mean-square stability of closed-loop system.

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The efficiency and advantages of the presented approach are illustrated by one example.

Notation: Throughout the paper, the space of functions $\phi : [-\tau_M, 0] \to \mathcal{R}^n$, which are absolutely continuous on $[-\tau_M, 0]$, and have square integrable first-order derivatives is denoted by $W[-\tau_M, 0]$ with the norm $\|\phi\|_W = \max_{\theta\in[-\tau_M,0]} |\phi(\theta)| + \left[\int_{-\tau_M}^{0} |\dot{\phi}(s)|^2 ds\right]^{\frac{1}{2}}$. $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers.

2. SYSTEM MODEL

2.1 NCS model

Consider the system architecture in Fig.1 with plant

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^m$ the control input and A, B are system matrices of appropriate dimensions. The initial condition is given by $x(0) = x_0$.

The NCS has two sensors, a controller and an actuator connected via two networks. For the sake of simplicity, we limit the presentation to two sensor nodes. Their measurements are given by $y_i(t) = C_i x(t), i = 1, 2$. Let $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathcal{R}^{n_y}$. We denote by s_k the unbounded and monotonously increasing sequence of sampling instants

$$0 = s_0 < s_1 < \ldots < s_k < \ldots, \quad k \in \mathbb{Z}_{\geq 0}, \ \lim_{k\to\infty} s_k = \infty.$$

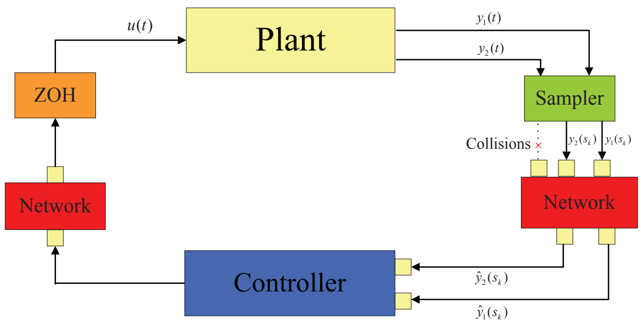

Fig. 1. NCS architecture

At each sampling instant s_k, at most one of the outputs $y_i(s_k) \in \mathcal{R}^{n_i}(n_1 + n_2 = n_y)$ is transmitted over the network. We suppose that the transmissions over the two networks are subject to a variable total delay η_k. Denote $s_k + \eta_k$ by t_k.

Differently from Donkers et al. [2011], Heemels et al. [2010], we do not restrict the network delays to be small. Following Liu et al. [2012a] and Liu et al. [2012b], we allow the delay to be large provided that the previous sample cannot reach the destination (to the controller or to the actuator) after the current one. Assume that the network-induced delay η_k and the time span between the next updating instant t_{k+1} and the current sampling instant s_k are bounded:

$$t_{k+1} - t_k + \eta_k \leq \tau_M, \ 0 \leq \eta_m \leq \eta_k \leq \eta_M, \ k \in \mathbb{Z}_{\geq 0}, \tag{2}$$

where η_m and η_M are known bounds and $\tau_M = MATI + \eta_M$, where $MATI$ denotes the Maximum Allowable Transmission Interval. Since $MATI = \tau_M - \eta_M \leq \tau_M - \eta_m$, $\eta_m > \frac{\tau_M}{2}$ implies that the network delays are non-small due to $\eta_k \geq \eta_m > \tau_M - \eta_m$. For the given examples in Section 4, we will show that when $\eta_m > \frac{\tau_M}{2}$ our method is feasible.

Remark 1. Differently from Naghshtabrizi et al. [2010], where subscript k in t_k corresponds to the measurements that are not lost, in our paper k corresponds to the sampling time. This is because we consider the probability of collisions (see further details below).

Remark 2. We follow a commonly used assumption on the boundedness of the network-induced delays, e.g., Donkers et al. [2011], Gao and Chen [2008]. Another possibility is the Markov chain model of the network-induced delays in NCSs e.g., Huang and Nguang [2008], Nilsson [1998].

2.2 Stochastic scheduling protocol

Denote by $\hat{y}(s_k) = \begin{bmatrix} \hat{y}_1(s_k) \\ \hat{y}_2(s_k) \end{bmatrix} \in \mathcal{R}^{n_y}$ the sensor output submitted to the scheduling protocol. At each sampling instant at most one of the system nodes $i \in \{1, 2\}$ is active, that is at most one of $\hat{y}_i(s_k)$ values is updated with the recent output $y_i(s_k)$. The choice of the active output node is defined by a stochastic scheduling protocol. In some cases, collisions may occur when nodes access the network (Tabbara and Nesic [2008]). If this happens, then packet with sensor data is dropped. The updating process $\hat{y}(s_k)$ is assumed to be i.i.d with the probabilities given by

$$\text{Prob}\left\{\hat{y}(s_k) = \begin{bmatrix} \hat{y}_1(s_{k-1}) \\ \hat{y}_2(s_{k-1}) \end{bmatrix}\right\} = \beta_0, \tag{3}$$

$$\text{Prob}\left\{\hat{y}(s_k) = \begin{bmatrix} y_1(s_k) \\ \hat{y}_2(s_{k-1}) \end{bmatrix}\right\} = \beta_1, \tag{4}$$

$$\text{Prob}\left\{\hat{y}(s_k) = \begin{bmatrix} \hat{y}_1(s_{k-1}) \\ y_2(s_k) \end{bmatrix}\right\} = \beta_2, \tag{5}$$

where β_i, $i = 0, 1, 2$ are non-negative scalars and $\sum_{i=0}^{2} \beta_i = 1$.

2.3 Stochastic impulsive time-delay model

Following Liu et al. [2012a], we consider the error between the system output $y(s_k)$ and the last available information $\hat{y}(s_{k-1})$:

$$\begin{aligned} &e(t) = col\{e_1(t), e_2(t)\} \equiv \hat{y}(s_{k-1}) - y(s_k), \\ &\quad t \in [t_k, t_{k+1}), \ k \in \mathbb{Z}_{\geq 0}, \ \hat{y}(s_{-1}) \triangleq 0, \ e(t) \in \mathcal{R}^{n_y}. \end{aligned} \tag{6}$$

We suppose that the controller and the actuator are event-driven. $\hat{y}(s_k) = \begin{bmatrix} \hat{y}_1(s_k) \\ \hat{y}_2(s_k) \end{bmatrix} \in \mathcal{R}^{n_y}(k \in \mathbb{Z}_{\geq 0})$ is the current output information at the controller.

Static output feedback control Assume that there exists a matrix $K = [\,K_1 \ K_2\,]$, $K_1 \in \mathcal{R}^{m\times n_1}, K_2 \in \mathcal{R}^{m\times n_2}$ such that $A + BKC$ is Hurwitz. The static output feedback controller has the form

$$u(t) = u_k = K\hat{y}(t_k - \eta_k), \quad t \in [t_k, t_{k+1}), \ k \in \mathbb{Z}_{\geq 0}. \tag{7}$$

Thus, (3), (4) and (5) are equivalent to

$$\text{Prob}\left\{u_k = K_1\hat{y}_1(s_{k-1}) + K_2\hat{y}_2(s_{k-1}) \triangleq u^0\right\} = \beta_0, \tag{8}$$

$$\text{Prob}\Big\{u_k = K_1 y_1(s_k) + K_2 \hat{y}_2(s_{k-1}) \triangleq u^1\Big\} = \beta_1, \quad (9)$$

$$\text{Prob}\Big\{u_k = K_1 \hat{y}_1(s_{k-1}) + K_2 y_2(s_k) \triangleq u^2\Big\} = \beta_2, \quad (10)$$

respectively. Following Yue et al. [2009], we introduce the indicator functions

$$\pi_{\{u_k=u^i\}} = \begin{cases} 1, & u_k = u^i \\ 0, & u_k \neq u^i, \end{cases} \quad i = 0,1,2, \ k \in \mathbb{Z}_{\geq 0}. \quad (11)$$

Thus, we have

$$\begin{aligned} &\mathbb{E}\{\pi_{\{u_k=u^i\}}\} = \mathbb{E}\{[\pi_{\{u_k=u^i\}}]^2\} = \text{Prob}\{u_k = u^i\} = \beta_i, \\ &\mathbb{E}\{[\pi_{\{u_k=u^i\}} - \beta_i][\pi_{\{u_k=u^j\}} - \beta_j]\} = \begin{cases} -\beta_i\beta_j, & i \neq j, \\ \beta_i(1-\beta_i), & i = j. \end{cases} \end{aligned} \quad (12)$$

Therefore, we obtain the stochastic impulsive closed-loop system:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_1 x(t_k - \eta_k) + \sum_{i=1}^{2}(1 - \pi_{\{u_k=u^i\}})B_i e_i(t), \\ &\dot{e}(t) = 0, \quad t \in [t_k, t_{k+1}) \end{aligned} \quad (13)$$

where $A_1 = BKC, \ B_i = BK_i, \ i = 1, 2.$

Next, taking into account (4), (5) and (11), we obtain

$$\begin{aligned} \hat{y}(s_k) &= \begin{bmatrix} \hat{y}_1(s_k) \\ \hat{y}_2(s_k) \end{bmatrix} \\ &= \begin{bmatrix} \pi_{\{u_k=u^1\}} y_1(s_k) + (1 - \pi_{\{u_k=u^1\}})\hat{y}_1(s_{k-1}) \\ \pi_{\{u_k=u^2\}} y_2(s_k) + (1 - \pi_{\{u_k=u^2\}})\hat{y}_2(s_{k-1}) \end{bmatrix} \\ &= y(s_k) + \begin{bmatrix} (1 - \pi_{\{u_k=u^1\}})e_1(t_k) \\ (1 - \pi_{\{u_k=u^2\}})e_2(t_k) \end{bmatrix}, \end{aligned}$$

which gives

$$\begin{aligned} e(t_{k+1}) &= \hat{y}(s_k) - y(s_{k+1}) \\ &= \begin{bmatrix} (1-\pi_{\{u_k=u^1\}})e_1(t_{k+1}^-) \\ (1-\pi_{\{u_k=u^2\}})e_2(t_{k+1}^-) \end{bmatrix} + y(s_k) - y(s_{k+1}). \end{aligned}$$

Thus, the delayed reset system is given by

$$\begin{aligned} x(t_{k+1}) &= x(t_{k+1}^-), \\ e(t_{k+1}) &= \begin{bmatrix} (1 - \pi_{\{u_k=u^1\}})e_1(t_{k+1}^-) \\ (1 - \pi_{\{u_k=u^2\}})e_2(t_{k+1}^-) \end{bmatrix} \\ &\quad + Cx(t_k - \eta_k) - Cx(t_{k+1} - \eta_{k+1}). \end{aligned} \quad (14)$$

Since $x(t_k - \eta_k) = x(t - \tau(t))$ for $t \in [t_k, t_{k+1})$ with $\tau(t) = t - t_k + \eta_k \in [\eta_m, \tau_M]$ (cf. (2)), the impulsive system model (13)-(14) contains the piecewise-continuous delay $\tau(t)$ in the continuous-time dynamics (13). Even for $\eta_k = 0$, we have the delayed state $x(t_k) = x(t - \tau(t))$ with $\tau(t) = t - t_k$. The initial condition for (13)-(14) has the form of $x(t) = \phi(t), \ t \in [t_0 - \tau_M, t_0]$ and $e(t_0) = -Cx(t_0 - \eta_0) = -Cx_0$, where $\phi(t)$ is a continuous function on $[t_0 - \tau_M, t_0]$.

Remark 3. Applying the Bernoulli-distributed stochastic variables $\pi_{\{u_k=u^i\}}, \ i = 0, 1, 2$, the closed-loop system is given by an impulsive time-delay system with stochastic parameters in the system matrices. Note that the Bernoulli distribution has been applied to NCS with probabilistic measurements missing (Wang et al. [2006]), random packet losses (Wang et al. [2007]), stochastic sampling intervals (Gao et al. [2009]) and time-delay system with stochastic interval delays (Yue et al. [2009]).

Remark 4. Our results can be extended to the case of any $N \geq 3$ sensor nodes $y_i(t) = C_i x(t) \in \mathcal{R}^{n_i}, i = 1, \ldots, N, \ \sum_{i=1}^{N} n_i = n_y$. We denote $\tilde{C}^T = \left[\, C_1^T \ \ldots \ C_N^T \,\right]$, $\tilde{K} = \left[\, K_1 \ \ldots \ K_N \,\right]$ and $\tilde{e}(t) = col\{e_1(t), \ldots, e_N(t)\} \equiv \hat{y}(s_{k-1}) - y(s_k), \ t \in [t_k, t_{k+1}), \ k \in \mathbb{Z}_{\geq 0}$. At each sampling instant s_k, the probabilities that $\hat{y}_i(s_k)$ is updated with the recent output $y_i(s_k)$ and collisions happen are denoted by $\beta_i \geq 0, i = 1, \ldots, N$ and $\beta_0 \geq 0$, respectively, where $\sum_{i=0}^{N} \beta_i = 1$. Then the static output feedback controller has the form

$$\begin{aligned} u(t) &= u_k = K\hat{y}(t_k - \eta_k) \\ &= \begin{cases} \displaystyle\sum_{j=1}^{N} K_j \hat{y}_j(s_{k-1}) \triangleq u^0, \\ \displaystyle\sum_{j=1, j\neq i}^{N} K_j \hat{y}_j(s_{k-1}) + K_i y_i(s_k) \triangleq u^i, \\ i = 1, \ldots, N, t \in [t_k, t_{k+1}), \ k \in \mathbb{Z}_{\geq 0}, \end{cases} \end{aligned} \quad (15)$$

with $\text{Prob}\{u_k = u^i\} = \beta_i, i = 0, 1, \ldots, N$. Thus, the continuous dynamics of the stochastic impulsive closed-loop model (1), (15) with N sensor nodes can be expressed as

$$\begin{aligned} &\dot{x}(t) = Ax(t) + \tilde{A}_1 x(t_k - \eta_k) + \sum_{i=1}^{N}(1 - \tilde{\pi}_{\{u_k=u^i\}})\tilde{B}_i \tilde{e}_i(t), \\ &\dot{\tilde{e}}(t) = 0, \quad t \in [t_k, t_{k+1}) \end{aligned}$$

where $\tilde{A}_1 = B\tilde{K}\tilde{C}, \ \tilde{B}_i = BK_i, \ i = 1, \ldots, N$ and $\tilde{\pi}_{\{u_k=u^i\}} = \begin{cases} 1, & u_k = u^i \\ 0, & u_k \neq u^i, \end{cases} \ i = 0, 1, \ldots, N, \ k \in \mathbb{Z}_{\geq 0}$. The delayed reset system is given by

$$\begin{aligned} x(t_{k+1}) &= x(t_{k+1}^-), \\ e_i(t_{k+1}) &= (1 - \tilde{\pi}_{\{u_k=u^i\}})e_i(t_{k+1}^-) + C_i x(t_k - \eta_k) \\ &\quad - C_i x(t_{k+1} - \eta_{k+1}), \ i = 1, \ldots, N. \end{aligned}$$

Dynamic output feedback We consider the dynamic output feedback of (1) under the assumption that the controller is directly connected to the actuator. The controller is assumed to be given by

$$\begin{aligned} &\dot{x}_c(t) = A_c x_c(t) + B_c \hat{y}(s_k), \\ &u(t) = C_c x_c(t) + D_c \hat{y}(s_k), \quad t \in [t_k, t_{k+1}), \ k \in \mathbb{Z}_{\geq 0}, \end{aligned} \quad (16)$$

where $x_c(t) \in \mathcal{R}^{n_c}$ is the state of the controller, A_c, B_c, C_c and D_c are matrices of appropriate dimensions. Let $e_i(t)(i = 1, 2)$ be defined by (6). The closed-loop system (1), (16) with (3)-(5) can be presented in the form of (13)-(14), where x, e and matrices are replaced by the ones with bars as follows:

$$\bar{x} = [x^T \ x_c^T]^T, \quad \bar{A} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} BD_c \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & 0 \\ B_c \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & 0 \end{bmatrix},$$

$$\bar{C} = \left[\frac{\bar{C}_1}{\bar{C}_2}\right] = \left[\begin{array}{cc} C_1 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ C_2 & 0 \end{array}\right], \ \bar{B}_1 = \bar{B}_2 = \begin{bmatrix} BD_c \\ B_c \end{bmatrix},$$

$$\bar{e}(t) = \begin{bmatrix} \bar{e}_1(t) \\ \bar{e}_2(t) \end{bmatrix}, \bar{e}_1(t) = \begin{bmatrix} e_1(t) \\ 0 \end{bmatrix}, \ \bar{e}_2(t) = \begin{bmatrix} 0 \\ e_2(t) \end{bmatrix}.$$

3. EXPONENTIAL MEAN-SQUARE STABILITY OF STOCHASTIC IMPULSIVE DELAYED SYSTEM

Definition 1. The impulsive system (13)-(14) is said to be exponentially mean-square stable with respect to x if there exist constants $b > 0, \kappa > 0$ such that

$$\mathbb{E}\{|x(t)|^2\} \le be^{-2\kappa(t-t_0)}\mathbb{E}\{\|x_{t_0}\|_W^2 + |e(t_0)|^2\},\ t \ge t_0$$

for the solutions of the stochastic impulsive system (13)-(14) initialized with $e(t_0) \in \mathcal{R}^{n_y}$ and $x(t) = \phi(t),\ t \in [t_0 - \tau_M, t_0]$.

Our objective of this section is to derive LMI conditions for the exponential mean-square stability of the impulsive system (13)-(14) with respect to x. We extend the Lyapunov-Krasovskii Functional (LKF) introduced in Liu et al. [2012a] to the following form:

$$\begin{aligned}
&V_e(t) = V(t, x_t, \dot{x}_t) + \sum_{i=1}^{2} e_i^T(t)Q_i e_i(t),\\
&V(t, x_t, \dot{x}_t) = \tilde{V}(t, x_t, \dot{x}_t)\\
&\quad +(\tau_M - \eta_m)\int_{t_k-\eta_k}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)Q\dot{x}(s)ds,\\
&\tilde{V}(t, x_t, \dot{x}_t) = x^T(t)Px(t)\\
&\quad + \int_{t-\eta_m}^{t} e^{2\alpha(s-t)}x^T(s)S_0x(s)ds\\
&\quad + \int_{t-\tau_M}^{t-\eta_m} e^{2\alpha(s-t)}x^T(s)S_1x(s)ds\\
&\quad +\eta_m \int_{-\eta_m}^{0}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)R_0\dot{x}(s)dsd\theta\\
&\quad +(\tau_M - \eta_m)\int_{-\tau_M}^{-\eta_m}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)R_1\dot{x}(s)dsd\theta,\\
&P > 0, S_i > 0, R_i > 0, Q > 0, Q_j > 0, \alpha > 0,\\
&i = 0, 1, j = 1, 2,\ t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_{\ge 0},
\end{aligned} \tag{17}$$

where $x_t(\theta) \stackrel{\Delta}{=} x(t + \theta),\ \theta \in [-\tau_M, 0]$. Here the term

$$e_i^T(t)Q_ie_i(t) \equiv e_i^T(t_k)Q_ie_i(t_k),\ t \in [t_k, t_{k+1}), i = 1, 2 \tag{18}$$

is piecewise-constant. The term $\tilde{V}(t, x_t, \dot{x}_t)$ represents the standard Lyapunov functional for systems with a time-varying delay $\tau(t) \in [\eta_m, \tau_M]$. The term $(\tau_M - \eta_m)\int_{t_k-\eta_k}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)Q\dot{x}(s)ds$ is borrowed from Liu et al. [2012a] and is inserted to cope with the delays in the reset conditions. It is continuous on $[t_k, t_{k+1})$ and does not grow at the jumps $t = t_{k+1}$, since

$$\begin{aligned}
&(\tau_M - \eta_m)\int_{t_{k+1}-\eta_{k+1}}^{t_{k+1}} e^{2\alpha(s-t_{k+1})}\mathbb{E}\{|\sqrt{Q}\dot{x}(s)|^2\}ds\\
&-(\tau_M - \eta_m)\int_{t_k-\eta_k}^{t_{k+1}^-} e^{2\alpha(s-t_{k+1})}\mathbb{E}\{|\sqrt{Q}\dot{x}(s)|^2\}ds\\
&\le -(\tau_M - \eta_m)e^{-2\alpha\tau_M}\int_{t_k-\eta_k}^{t_{k+1}-\eta_{k+1}} \mathbb{E}\{|\sqrt{Q}\dot{x}(s)|^2\}ds\\
&\le -e^{-2\alpha\tau_M}\mathbb{E}\{|\sqrt{Q}[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2\},
\end{aligned} \tag{19}$$

where we applied Jensen's inequality (Gu et al. [2003]). The infinitesimal operator $\mathcal{L}$ of $V_e(t)$ is defined as

$$\mathcal{L}V_e(t) = \lim_{\Delta\to 0^+}\frac{1}{\Delta}\{\mathbb{E}\{V_e(t + \Delta)|t\} - V_e(t)\}. \tag{20}$$

The following lemma gives sufficient conditions for the exponential stability of (13)-(14) with respect to x in the mean-square sense:

Lemma 1. If there exist positive constant α, matrices $U_i > 0,\ i = 1, 2$ and $V_e(t)$ of (17) such that along (13)-(14) for $t \in [t_k, t_{k+1})$

$$\mathbb{E}\{\mathcal{L}V_e(t) + 2\alpha V_e(t) - \frac{1}{\tau_M - \eta_m}\sum_{i=1}^{2} e_i^T(t)(Q_i - U_i)e_i(t)\} \le 0, \tag{21}$$

with

$$\Omega = \begin{bmatrix} \Omega_{11} & (\beta_0 + \beta_2)C_1^TQ_1 & (\beta_0 + \beta_1)C_2^TQ_2\\ * & \Omega_{22} & 0\\ * & * & \Omega_{33}\end{bmatrix} \le 0 \tag{22}$$

holds, where

$$\begin{aligned}
\Omega_{11} &= \sum_{i=1}^{2} C_i^TQ_iC_i - Qe^{-2\alpha\tau_M},\\
\Omega_{22} &= (\beta_0 + \beta_2)Q_1 - U_1,\\
\Omega_{33} &= (\beta_0 + \beta_1)Q_2 - U_2.
\end{aligned}$$

Then $V_e(t)$ satisfies

$$\mathbb{E}\{V_e(t_{k+1}) - V_e(t_{k+1}^-) + \sum_{i=1}^{2} e_i^T(t_k)(Q_i - U_i)e_i(t_k)\} \le 0. \tag{23}$$

Moreover, the following bound is valid for the solution of (13)-(14) with the initial condition $x_{t_0}, e(t_0)$:

$$\begin{aligned}
&\mathbb{E}\{V(t, x_t, \dot{x}_t)\} \le e^{-2\alpha(t-t_0)}\mathbb{E}\{V_e(t_0)\},\ t \ge t_0,\\
&V_e(t_0) = V(t_0, x_{t_0}, \dot{x}_{t_0}) + \sum_{i=1}^{2} e_i^T(t_0)Q_ie_i(t_0),
\end{aligned} \tag{24}$$

implying the exponential mean-square stability of (13)-(14) with respect to x.

Proof: Since $\int_{t_k}^{t} e^{-2\alpha(t-s)}ds \le \tau_M - \eta_m,\ t \in [t_k, t_{k+1})$ and $\mathcal{L}[e^{2\alpha t}V_e(t)] = e^{2\alpha t}[2\alpha V_e(t) + \mathcal{L}V_e(t)], \alpha > 0$, (21) implies

$$\begin{aligned}
&\mathbb{E}\{V_e(t)\} \le e^{-2\alpha(t-t_k)}\mathbb{E}\{V_e(t_k)\}\\
&\quad + \sum_{i=1}^{2}\mathbb{E}\{e_i^T(t_k)(Q_i - U_i)e_i(t_k)\},\ t \in [t_k, t_{k+1}).
\end{aligned} \tag{25}$$

Therefore,

$$\mathbb{E}\{V(t, x_t, \dot{x}_t)\} \le e^{-2\alpha(t-t_k)}\mathbb{E}\{V_e(t_k)\},\ t \in [t_k, t_{k+1}). \tag{26}$$

Note that

$$\begin{aligned}
\mathbb{E}\{V_e(t_{k+1})\} &= \mathbb{E}\{\tilde{V}_{|t=t_{k+1}} + \sum_{i=1}^{2} e_i^T(t_{k+1})Q_ie_i(t_{k+1})\\
&\quad +(\tau_M - \eta_m)\int_{t_{k+1}-\eta_{k+1}}^{t_{k+1}} e^{2\alpha(s-t_{k+1})}\dot{x}^T(s)Q\dot{x}(s)ds\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\{e_i^T(t_{k+1})Q_ie_i(t_{k+1})\}\\
&= \mathbb{E}\{(1 - \beta_i)e_i^T(t_k)Q_ie_i(t_k)\\
&\quad +2(1 - \beta_i)e_i^T(t_k)Q_iC_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]\\
&\quad +|\sqrt{Q_i}C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2\},\ i = 1, 2.
\end{aligned}$$

Taking (19) and (22) into account, we obtain

$$\begin{aligned}
&\mathbb{E}\{V_e(t_{k+1}) - V_e(t_{k+1}^-) + \sum_{i=1}^{2} e_i^T(t_k)(Q_i - U_i)e_i(t_k)\} \\
&\le \mathbb{E}\Big\{\sum_{i=1}^{2} e_i^T(t_{k+1})Q_i e_i(t_{k+1}) - \sum_{i=1}^{2} e_i^T(t_k)Q_i e_i(t_k) \\
&\quad + \sum_{i=1}^{2} e_i^T(t_k)(Q_i - U_i)e_i(t_k) \\
&\quad - e^{-2\alpha\tau_M}|\sqrt{Q}[x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1})]|^2\Big\} \\
&= \mathbb{E}\Big\{\sum_{i=1}^{2}\{e_i^T(t_k)[(1-\beta_i)Q_i - U_i]e_i(t_k) \\
&\quad + 2(1-\beta_i)e_i^T(t_k)Q_iC_i[x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1})] \\
&\quad + |\sqrt{Q_i}C_i[x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1})]|^2\} \\
&\quad - e^{-2\alpha\tau_M}|\sqrt{Q}[x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1})]|^2\Big\} \\
&= \mathbb{E}\{\zeta(t)^T\Omega\zeta(t)\} \le 0,
\end{aligned}$$

where $\zeta(t) = col\{x(t_k-\eta_k) - x(t_{k+1}-\eta_{k+1}), e_1(t_k), e_2(t_k)\}$ and Ω is given by (22).

Therefore, (23) is valid and, together with (25) for $t = t_{k+1}^-$, implies

$$\begin{aligned}
\mathbb{E}\{V_e(t_{k+1})\} &\le e^{-2\alpha(t_{k+1}-t_k)}\mathbb{E}\{V_e(t_k)\} \\
&\le e^{-2\alpha(t_{k+1}-t_{k-1})}\mathbb{E}\{V_e(t_{k-1})\} \\
&\le e^{-2\alpha(t_{k+1}-t_0)}\mathbb{E}\{V_e(t_0)\}.
\end{aligned}$$

The latter inequality, with $k+1$ replaced by k and (26) give (24). The inequality (24) implies exponential mean-square stability of (13)-(14) with respect to x because $\lambda_{min}(P)\mathbb{E}\{|x(t)|^2\} \le \mathbb{E}\{V(t, x_t, \dot{x}_t)\}, \mathbb{E}\{V(t_0, x_{t_0}, \dot{x}_{t_0})\} \le v\mathbb{E}\{\|x_{t_0}\|_W^2\}$ for some scalar $v > 0$. □

By using Lemma 1 and the standard arguments for the delay-dependent analysis, we derive LMI conditions for the exponential mean-square stability of (13)-(14) with respect to x:

Theorem 1. Given $0 \le \eta_m < \tau_M, \alpha > 0, \beta_i \ge 0, i = 0,1,2,$ $\sum_{i=0}^{2}\beta_i = 1$ and K_1, K_2. Suppose there exist $n \times n$ matrices $P > 0$, $Q > 0$, $S_j > 0$, $R_j > 0, j = 0,1$, S_{12} and $n_l \times n_l$ matrices $Q_l > 0$, $U_l > 0$, $l = 1,2$ such that (22) and

$$\begin{bmatrix} R_1 & S_{12} \\ * & R_1 \end{bmatrix} \ge 0, \quad \begin{bmatrix} \Sigma_{11} & \Sigma_{12}^T \\ * & \Sigma_{22} \end{bmatrix} < 0,$$

are feasible, where

$$\Sigma_{11} = \begin{bmatrix} \varphi_{11} & R_0e^{-2\alpha\eta_m} & PA_1 & 0 & \varphi_{51} & \varphi_{52} \\ * & \varphi_{22} & \varphi_{23} & S_{12}e^{-2\alpha\tau_M} & 0 & 0 \\ * & * & \varphi_{33} & \varphi_{23} & 0 & 0 \\ * & * & * & \varphi_{44} & 0 & 0 \\ * & * & * & * & \psi_1 & 0 \\ * & * & * & * & * & \psi_2 \end{bmatrix},$$

$$\Sigma_{12} = \begin{bmatrix} H^TA & 0 & H^TA_1 & 0 & (\beta_0+\beta_2)H^TB_1 & (\beta_0+\beta_1)H^TB_2 \\ 0 & 0 & 0 & 0 & -\beta_1H^TB_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_2H^TB_2 \end{bmatrix},$$

$$\begin{aligned}
&\Sigma_{22} = \mathrm{diag}\{-H, -\beta_1H, -\beta_2H\}, \\
&\varphi_{11} = PA + A^TP + S_0 - R_0e^{-2\alpha\eta_m} + 2\alpha P, \\
&\varphi_{22} = (-S_0 + S_1 - R_0)e^{-2\alpha\eta_m} - R_1e^{-2\alpha\tau_M}, \\
&\varphi_{23} = (R_1 - S_{12})e^{-2\alpha\tau_M}, \\
&\varphi_{33} = (-2R_1 + S_{12} + S_{12}^T)e^{-2\alpha\tau_M}, \\
&\varphi_{44} = -(S_1 + R_1)e^{-2\alpha\tau_M}, \\
&\varphi_{5i} = (\beta_0 + \beta_{3-i})PB_i, \\
&\psi_i = -\frac{1}{\tau_M - \eta_m}(Q_i - U_i) + 2\alpha Q_i, \\
&H = \eta_m^2R_0 + (\tau_M - \eta_m)^2R_1 + (\tau_M - \eta_m)Q, \ i = 1,2.
\end{aligned}$$

Then, the solutions of (13)-(14) satisfy the bound (24). Hence, the closed-loop system with initial condition $x_{t_0}, e(t_0)$ is exponentially mean-square stable with respect to x. If the above LMIs are feasible with $\alpha = 0$, then the bound (24) holds with a small enough $\alpha_0 > 0$.

Remark 5. Compare the number of scalar decision variables in the LMIs under the RR protocol in Liu et al. [2012b] and under the stochastic protocol in Theorem 1. The LMIs (two of $6n\times6n$ and two of $3n\times3n$) under the RR protocol have $8.5n^2 + 2.5n$ variables, which is bigger than the number $4.5n^2 + 4n$ of variables (for $y_i(s_k) \in \mathcal{R}^{n/2}(i = 1,2)$) in the LMIs (one of $8n \times 8n$ and two of $2n \times 2n$) under the stochastic protocol in Theorem 1. Under the TOD protocol in Liu et al. [2012a], the number of variables is $4.25n^2 + 3.5n$, but in two of $5.5n \times 5.5n$ LMIs and one of $2n \times 2n$ LMI.

4. EXAMPLE: BATCH REACTOR

We illustrate the efficiency of the given conditions on a benchmark example of a batch reactor under the dynamic output feedback (Donkers et al. [2011], Heemels et al. [2010]), where

$$A = \begin{bmatrix} 1.380 & -0.208 & 6.715 & -5.676 \\ -0.581 & -4.2902 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \ C = \begin{bmatrix} C_1 \\ \hline C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ \hline 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline -2 & 0 & 0 & -2 \\ 0 & 8 & 5 & 0 \end{array}\right].$$

Assume that $\beta_0 = 0$, which means that the event of collision does not occur. Let $\beta_1 = 0.6$. For the values of η_m given in Table 1, we apply Theorem 1 with $\alpha = 0$ and find the maximum values of $\tau_M = MATI + \eta_M$ that preserve the mean-square stability of the impulsive system (13)-(14) with respect to x (see Table 1). From Table 1 it is seen that the results of our method essentially improve the results in Heemels et al. [2010] in terms of both TOD and RR protocols. Moreover, we can find that the introduced stochastic protocol stabilizes the system for smaller τ_M than the RR and TOD, which have been developed based on the time-delay approach in Liu et al. [2012b] and Liu et al. [2012a], respectively. When $\eta_m > \frac{\tau_M}{2}(\eta_m = 0.03, 0.04)$, our method is applicable. Choosing $\eta_m = 0.02$, by Theorem 1 with $\alpha = 0$, we obtain the corresponding maximum values of τ_M shown in Fig. 2 for different β_1.

Choosing $\eta_m = 0.02, \tau_M = 0.025$, by applying Theorem 1, we obtain the maximum values of the decay rate α shown in Fig. 3 for different β_1.

Remark 6. Simulations in the example above show that our stochastic protocol is conservative compared to the TOD protocol in Liu et al. [2012a] and RR protocol in Liu

Table 1. Example: max. value of $\tau_M = MATI + \eta_M$ for different η_m

$\tau_M \setminus \eta_m$	0	0.004	0.02	0.03	0.04
Heemels et al. [2010] ($\eta_M = 0.004$, TOD)	0.0108	0.0133	-	-	-
Heemels et al. [2010] ($\eta_M = 0.004$, RR)	0.0088	0.0088	-	-	-
Donkers et al. [2011] ($\eta_M = 0.03$, TOD)	0.069	0.069	0.069	0.069	-
Donkers et al. [2011] ($\eta_M = 0.03$, RR)	0.068	0.068	0.068	0.068	-
Liu et al. [2012a](TOD)	0.035	0.037	0.047	0.053	0.059
Liu et al. [2012b] (RR)	0.042	0.044	0.053	0.058	0.063
Theorem 1 ($\beta_1 = 0.6$)	0.025	0.028	0.041	0.050	0.058

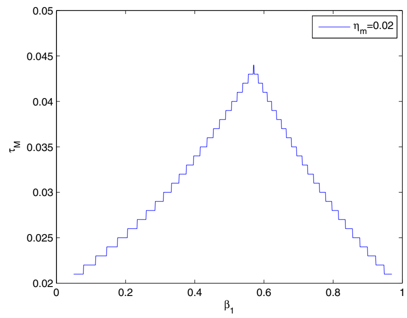

Fig. 2. Example: effect of β_1 on the τ_M for $\beta_0 = 0$ and $\eta_m = 0.02$

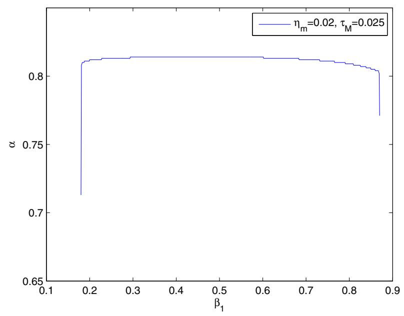

Fig. 3. Example: effect of β_1 on the decay rate α for $\beta_0 = 0$, $\eta_m = 0.02$ and $\tau_M = 0.025$

et al. [2012b]. However, we cannot immediately conclude that among the scheduling protocols based on time-delay approach, TOD and RR are superior to stochastic protocol. The results in this paper could be further improved in terms of other stochastic protocols.

5. CONCLUSIONS

In this paper, a time-delay approach was developed for the stabilization of NCSs under a stochastic protocol, variable delays and variable sampling intervals. By developing appropriate Lyapunov-Krasovskii-based method, the exponential mean-square stability conditions for the delayed impulsive system were derived in terms of LMIs. Future work will involve consideration of more general NCS models, including stochastic communication delays and scheduling protocols for the actuator nodes.

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