Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Port Hamiltonian System in Descriptor Form for Balanced Reduction: Application to a Nanotweezer ⋆

Yongxin Wu * **Boussad Hamroun** * **Yann Le Gorrec** ** **Bernhard Maschke** *

* *LAGEP, Université de Lyon, Lyon, F-69003, France; Université Lyon 1, CNRS, UMR 5007, LAGEP, Villeurbanne, F-69622, France (e-mail: wu, maschke, hamroun@lagep.univ-lyon1.fr).*
** *FEMTO-ST CNRS UMR 6174, ENSMM, 26 chemin de d'épitaphe, F-25030 Besançon, France (e-mail: yann.le.gorrec@ens2m.fr).*

Abstract: This paper proposes a method of balanced model reduction for constrained linear port Hamiltonian systems. Constrained linear port Hamiltonian systems are first written in a canonical descriptor form such that the Hamiltonian structure is preserved. The computations of the controllability and observability Gramians are then used to derive the balanced port Hamiltonian representation of the system. The method of flow constraint is applied to reduce the system. Finally, numerical simulations for the reduction of a micro mechanical actuator model is given to illustrate the effectiveness of the proposed method.

Keywords: Port Hamiltonian system, Descriptor system, Balanced reduction, Gramian, flow constraint method

1. INTRODUCTION

Port Hamiltonian systems are an ideal frame for the compositional modeling of finite- and infinite-dimensional physical systems [Duindam et al. 2009, van der Schaft and Maschke 2013] which might leads to high-order control systems. Different structure preserving reduction methods have been suggested for linear Port Hamiltonian systems [Polyuga and van der Schaft 2010, Polyuga and van der Schaft 2011, Polyuga and van der Schaft 2012, Gugercin et al. 2012, Gentili et al. 2011]

In this paper we shall consider the structure preserving reduction of linear constrained Hamiltonian systems with ports. We give at first a coordinate transform for the constrained port Hamiltonian system to a particular port Hamiltonian descriptor system and we keep the constraint of the system along the reduction procedure in contrast to the other port Hamiltonian system reduction methods. Secondly we shall suggest an alternative to the positive real balancing methods, based on the the available storage and the required supply, for the reduction of bounded real and positive real systems [Antoulas 2005, Gugercin and Antoulas 2004]; [Reis and Stykel 2010] or port Hamiltonian systems [Polyuga and van der Schaft 2012]. This alternative consists in adapting the Lyapounov balancing method, based on the controllability and observability Gramians, of descriptor systems suggested by [Stykel 2004] by using, instead of the truncation, the flow constraint method of [Polyuga and van der Schaft 2012] for the reduction of the balanced system which preserves the port Hamiltonian structure.

⋆ This work was supported by ANR sponsored project HAMECMOPSYS under Reference Code ANR-11-BS03-0002.

This paper is organized as follows. First, we show how to transform constrained port Hamiltonian systems into a canonical descriptor form (Section 2) preserving its structure. In Section 3 we give a balanced realization of port Hamiltonian descriptor systems and then use the flow-constraint method to reduce the order of the balanced port Hamiltonian descriptor systems. In Section 4 we apply the reduction method to a micro-mechanical manipulator, called nanotweezer.

2. PORT HAMILTONIAN SYSTEM AND DESCRIPTOR FORM

In this section we shall consider *constrained* linear port Hamiltonian systems and transform them into the descriptor form [Dai 1989]. Port Hamiltonian systems with constraints are a particular representation of implicit port Hamiltonian systems defined on Dirac structures [van der Schaft and Maschke 1995, Dalsmo and van der Schaft 1999, Duindam et al. 2009, chap.2] which makes explicitly appear constraint equations as well as the associated Lagrangian multipliers.

A constrained linear port Hamiltonian system may be defined as follows

$$\left\{\begin{aligned} \dot{x} &= JQx + g_c\lambda + gu + g_R e_R \\ 0 &= g_c^T Qx \\ y &= g^T Qx \\ f_R &= g_R^T Qx \end{aligned}\right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $\lambda \in \mathbb{R}^k$ is the vector of Lagrangian multipliers, $H(x) = \frac{1}{2}x^T Qx$ is the Hamiltonian function, $Q \in \mathbb{R}^{n\times n}$ is a positive definite matrix (i.e. $Q = Q^T > 0$) which will be called energy matrix (as an allusion to models of physical systems), $J \in \mathbb{R}^{n\times n}$

is the skew-symmetric the Poisson structure matrix, (i.e. $J = -J^T$), the matrices $g_c \in \mathbb{R}^{n\times k}$ and $g \in \mathbb{R}^{n\times m}$, $g_R \in \mathbb{R}^{n\times m_R}$ describes the input relations of the control ports and the resistive ports respectively. This constrained Port Hamiltonian system will be completed with a linear resistive relation between the port variables (e_R, f_R) such as $e_R = -Df_R$, with $D \in \mathbb{R}^{m_R\times m_R}$ being a symmetric positive matrix (i.e. $D = D^T \geq 0$).

In this way one obtains a dissipative linear Port Hamiltonian system with constraints. Note that the vector $\lambda \in \mathbb{R}^k$ of Lagrangian multipliers is associated with the constraints given by the second equation of (1) and that in a mechanical context $g_c\lambda$ may be interpreted as the constraint force associated with constraints in [van der Schaft and Maschke 1994].

2.1 Elimination of the Lagrangian multipliers

In this section we propose at first to reduce the system by eliminating the Lagrange multipliers. Therefore we define the following coordinate transformation

$$z = \begin{bmatrix} s \\ (g_c^T g_c)^{-1} g_c^T \end{bmatrix} x = Mx$$

where s is a $\mathbb{R}^{n-k\times n}$ matrix such that

$$sg_c = 0 \text{ and } \operatorname{rank}(s) = n - k$$

Note that this coordinate transformation is inspired from [van der Schaft and Maschke 1994]) and differs by the multiplication by $(g_c^T g_c)^{-1}$. If, which is often the case in network models, the matrix $(g_c^T g_c)$ is sparse then the inverse may be computed efficiently. Else it is preferable to use the transformation in [van der Schaft and Maschke 1994]; the results of this paper remain then unchanged when replacing the matrix I_k by the matrix $(g_c^T g_c)$.

In the new coordinates the system (1) becomes

$$\begin{cases} \dot{z} = \bar{J}\bar{Q}z + \bar{g}_c\lambda + \bar{g}u + \bar{g}_R e_R \\ 0 = \bar{g}_c^T \bar{Q}z \\ y = \bar{g}^T \bar{Q}z \\ f_R = \bar{g}_R^T \bar{Q}z \end{cases} \quad (2)$$

with: $\bar{J} = MJM^T = -\bar{J}^T$, $\bar{Q} = M^{-T}QM^{-1} > 0$, $\bar{g}_c = Mg_c$, $\bar{g} = Mg$, $\bar{g}_R = Mg_R$ Note that

$$\bar{g}_c = Mg_c = \begin{bmatrix} s \\ (g_c^T gc)^{-1} g_c^T \end{bmatrix} g_c = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$$

which implies that, decomposing the state vector as follows: $z = [z_1, z_2]^T$, $z_1 \in \mathbb{R}^{n-k}$ and $z_2 \in \mathbb{R}^k$, the system can be written as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{J}_{11} & \bar{J}_{12} \\ \bar{J}_{21} & \bar{J}_{22} \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I_k \end{bmatrix} \lambda + \begin{bmatrix} \bar{g}_1 \\ \bar{g}_2 \end{bmatrix} u + \begin{bmatrix} \bar{g}_{R_1} \\ \bar{g}_{R_2} \end{bmatrix} e_R \quad (3)$$

$$0 = \begin{bmatrix} 0 & I_k \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (4)$$

$$y = \begin{bmatrix} \bar{g}_1^T & \bar{g}_2^T \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (5)$$

$$f_R = \begin{bmatrix} \bar{g}_{R_1}^T & \bar{g}_{R_2}^T \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (6)$$

decomposing the energy matrix into blocks accordingly with the decomposition of the state vector into vectors of size $(n - k)$ and k

$$\bar{Q} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \quad (7)$$

One may observe by considering the first line of (3), that $\dot{z}_1$ is independent of λ and that the constraint equations (4) reduce to

$$\bar{Q}_{21}z_1 + \bar{Q}_{22}z_2 = \frac{\partial \bar{H}}{\partial z_2} = 0 \quad (8)$$

with $\bar{H}(z) = \frac{1}{2} z^T \bar{Q} z$.

Since $\bar{Q}$ is positive definite, $\bar{Q}_{22}$ is invertible and then one eliminate the z_2 component of the state vector and obtain an explicit port Hamiltonian system. The second line of (3) may be used to compute, if needed, the Lagrangian multipliers λ can be computed by $\lambda = \dot{z}_2 - \bar{J}_2\bar{Q}z - \bar{g}_2 u - \bar{g}_{R_2} e_R$ [van der Schaft and Maschke 1994].

However we shall not follow this route which might lead to cumbersome calculations and destroying the sparcity of the systems' matrices. In the sequel we shall eliminate the Lagrangian multipler but retain the full state $z \in \mathbb{R}^n$ with the contraint (8) and treat it as a descriptor system. The procedure is the same as suggested in [Dalsmo and van der Schaft 1999, p.66] for implicit Hamiltonian systems but detailed for systems with ports and expressed in the coordinates z adapted to the constraints.

2.2 Descriptor form of the port Hamiltonian system with constraints

In the system written in the new coordinates, we shall eliminate the second line of equation (3) and combine the first line of equation (3) with the second line of equation (4). Doing so we eliminate the Lagrangian multiplier λ, and the system will be written in the following descriptor form [Dai 1989, chap.1]

$$\begin{cases} \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{J}_{11} & \bar{J}_{12} \\ 0 & I_k \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{g}_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \bar{g}_{R_1} \\ 0 \end{bmatrix} e_R \\ y = \begin{bmatrix} \bar{g}_1^T & \bar{g}_2^T \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ f_R = \begin{bmatrix} \bar{g}_{R_1}^T & \bar{g}_{R_2}^T \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{cases} \quad (9)$$

Note that with the assumption that $\bar{Q}$ is positive definite, the constraint (8) is of index 1 and hence the diagonal block multiplying $\dot{z}_2$ is 0 with nilpotency index 1 .

Now let us prove that the descriptor system (9) is a port Hamiltonian system defined with respect to a Dirac structure according to [van der Schaft and Maschke 1995, Dalsmo and van der Schaft 1999, Duindam et al. 2009, chap.2] . By taking the following notations $f_z = -\dot{z}$, $e_z = \bar{Q}z$, $y = f_p$, $u = e_p$, the system can be formulated as:

$$\underbrace{\begin{bmatrix} I_{n-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix}}_{F} \begin{bmatrix} f_{z_1} \\ f_{z_2} \\ f_p \\ f_R \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{J}_{11} & \bar{J}_{12} & \bar{g}_1 & \bar{g}_{R_1} \\ 0 & I_k & 0 & 0 \\ \bar{g}_1^T & \bar{g}_2^T & 0 & 0 \\ \bar{g}_{R_1}^T & \bar{g}_{R_2}^T & 0 & 0 \end{bmatrix}}_{E} \begin{bmatrix} e_{z_1} \\ e_{z_2} \\ e_p \\ e_R \end{bmatrix} = 0 \quad (10)$$

where $F, E \in \mathbb{R}^{(n+mm_R)\times(n+m+m_R)}$.

Proposition 1. Define the vector of flow variables:

$$f^T := (f_z, f_p, f_R)^T$$

and the vector of effort variables

$$e^T := (e_z, e_p, e_R)^T$$

in the bond space $\mathcal{F} \times \mathcal{E} = \mathbb{R}^N \times \mathbb{R}^N$, where $N = n + m + m_R$, and the structure matrices F and E as in (10). Then the linear subspace $\mathcal{D}$ of $\mathcal{F} \times \mathcal{E}$ defined by:

$$\mathcal{D} = \{f \in \mathcal{F}, e \in \mathcal{E} | Ff + Ee = 0\} \quad (11)$$

is a Dirac structure.

Proof. The proof is given in two steps. First; we have to show that $FE^T + EF^T = 0$ and then that rank$[F|E] = n + m + m_R$.

(1) From (10) and the expressions of F and E one can compute:

$$FE^T = \begin{bmatrix} \bar{J}_{11}^T & 0 & \bar{g}_1 & \bar{g}_{R_1} \\ 0 & 0 & 0 & 0 \\ -\bar{g}_1^T & 0 & 0 & 0 \\ -\bar{g}_{R_1}^T & 0 & 0 & 0 \end{bmatrix} \quad (12)$$

Since $\bar{J}$ is a skew-symmetric matrix, that is $\bar{J}_{11}^T = -\bar{J}_{11}$, the matrix FE^T is skew-symmetric, hence the condition $FE^T + EF^T = 0$ is verified.

(2) One can define a sub-matrix of $[F|E]$ by the first, third and fourth columns of the matrix F, and the second column of the matrix E. This sub-matrix is rank $n+m+m_R$, consequently the rank of the matrix $[F|E]$ is $n + m + m_R$.

As a conclusion the system (9) defines a port Hamiltonian system defined with respect to the Dirac structure $\mathcal{D}$ in (10) and generated by the Hamiltonian function $\bar{H}(z) = \frac{1}{2} z^T \bar{Q} z$. In the sequel we shall call this system a *port Hamiltonian descriptor system.*

2.3 Descriptor form of the dissipative port Hamiltonian system with constraints

Consider the port Hamiltonian descriptor system (9) together with the resistive relation $e_R = -Df_R$, where $D \in \mathbb{R}^{m_R \times m_R}$ is a symmetric positive matrix. However as only the dynamics $\dot{z}_1$ is retained in the descriptor formulation, the dissipative relation is restriced to $e_R = -Df_R$

$$\bar{D} = \begin{bmatrix} \bar{g}_{R_1} \\ 0 \end{bmatrix} D \begin{bmatrix} \bar{g}_{R_1}^T & \bar{g}_{R_2}^T \end{bmatrix} = \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} \\ 0 & 0 \end{bmatrix}$$

system (9) can be written under its descriptor form:

$$\begin{cases} S\dot{z} = \bar{E}\,\bar{Q}\,z + \begin{bmatrix} \bar{g}_1 \\ 0 \end{bmatrix} u \\ y = \bar{g}^T \bar{Q}\,z \end{cases} \quad (13)$$

with the energy matrix $\bar{Q} \in \mathbb{R}^{n \times n}$ defined in (7) , $\bar{g} \in \mathbb{R}^{n \times m}$ and

$$\begin{cases} S = \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix}; \quad \begin{bmatrix} \bar{g}_1 \\ 0 \end{bmatrix} = S\,\bar{g} \\ \bar{E} = \begin{bmatrix} \bar{J}_{11} - \bar{D}_{11} & \bar{J}_{12} - \bar{D}_{12} \\ 0 & I_k \end{bmatrix} = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} \\ 0 & I_k \end{bmatrix}; \end{cases} \quad (14)$$

For an easy interpretation as a descriptor system [Dai 1989] we shall also use the following notations defining input matrix B , the output matrix C as well as the state matrix A

$$B = \begin{bmatrix} \bar{g}_1 \\ 0 \end{bmatrix}; \quad C = \bar{g}^T \bar{Q} = \begin{bmatrix} \bar{g}_1^T & \bar{g}_2^T \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix}$$

$$A = \bar{E}\bar{Q} = \begin{bmatrix} \bar{E}_{11}\bar{Q}_{11} + \bar{E}_{12}\bar{Q}_{21} & \bar{E}_{11}\bar{Q}_{12} + \bar{E}_{12}\bar{Q}_{22} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix}$$

where $\alpha = \bar{E}_{11}\bar{Q}_{11} + \bar{E}_{12}\bar{Q}_{21}$ and $\beta = \bar{E}_{11}\bar{Q}_{12} + \bar{E}_{12}\bar{Q}_{22}$.

2.4 Canonical form for port Hamiltonian descriptor system

As a complement we shall give the transformation of the descriptor dissipative port Hamiltonian system (13) into the canonical Weierstrass form where the state matrix take a canonical form [Dai 1989].

Proposition 2. Considering the equivalency transformation defined by the a (right) transform corresponding to the change of coordinates $z = R\hat{z}$ and the left transform defined by the matrix L as a (left) multiplier or combination matrix with

$$L = \begin{bmatrix} I_{n-k} & -\beta\bar{Q}_{22}^{-1} \\ 0 & L_2^{-1} \end{bmatrix}, \quad R = \begin{bmatrix} I_{n-k} & 0 \\ -\bar{Q}_{22}^{-1}\bar{Q}_{21} & L_2^{-T} \end{bmatrix}$$

where L_2 is a invertible triangular matrix corresponding to the Cholesky factorization of $\bar{Q}_{22}$

$$\bar{Q}_{22} = L_2 L_2^T \quad (15)$$

the port Hamiltonian descriptor system (13) is equivalent to the following descriptor Hamiltonian system :

$$\begin{cases} \hat{S}\dot{z} = \hat{E}\,\hat{Q}\,z + \begin{bmatrix} \hat{g}_1 \\ 0 \end{bmatrix} u \\ y = \hat{g}^T \hat{Q}\,z \end{cases} \quad (16)$$

where

$$\hat{S} = S; \quad \hat{E} = \begin{bmatrix} \bar{J}_{11} - \bar{D}_{11} & 0 \\ 0 & I_k \end{bmatrix}; \quad \hat{Q} = \begin{bmatrix} \bar{Q}_s & 0 \\ 0 & I_k \end{bmatrix} \quad (17)$$

$\bar{Q}_s = \bar{Q}_{11} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{21}$ is the Schur complement of the matrix $\bar{Q}$ and

$$\hat{g}^T = [\bar{g}_1^T \quad \bar{g}_1^T \bar{Q}_{12} L_2^{-T} + \bar{g}_2^T L_2]$$

The descriptor port Hamiltonian system (16) is a canonical Weierstrass form of the system (13).

The expression of the equivalent descriptor system (16) may be checked by direct computation. The fact that it is a canonical Weierstrass form of the descriptor system may be proven by noticing firstly that the matrix S is in canonical form and secondly by computing the state matrix of the equivalent system (16)

$$\hat{A} = \hat{E}\hat{Q} = \begin{bmatrix} \bar{J}_{11} - \bar{D}_{11} & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} \bar{Q}_s & 0 \\ 0 & I_k \end{bmatrix} = \begin{bmatrix} \left(\bar{J}_{11} - \bar{D}_{11}\right)\bar{Q}_s & 0 \\ 0 & I_k \end{bmatrix}$$

which is indeed the canonical form of the state matrix.

3. GEOMETRIC MODEL REDUCTION

In this section we suggest a procedure for a structure preserving reduction of the constrained port Hamiltonian system (1) using the procedure to compute a balanced realization of descriptor systems suggested in [Stykel 2004] and then instead of reduction by truncation, adapt the flow constraint method suggested in [Polyuga and van der Schaft 2012] for the effective calculation of a reduced Port Hamiltonian system in descriptor form (9).

3.1 Controllability and observability

Let us first characterize the observability and controllability properties of the system according to [Stykel 2004, Dai 1989]. Observe by inspection of the Weierstrass canonical form (16) of the system, that it is not Completely controllable as rank $\left[S \vdots \begin{bmatrix} \hat{g}_1 \\ 0 \end{bmatrix} \right] = n - k < n$, it is Impulse controllable as rank $\left[S \vdots \hat{A} \begin{bmatrix} 0 \\ I_k \end{bmatrix} \vdots \begin{bmatrix} \hat{g}_1 \\ 0 \end{bmatrix} \right] = n$ and R-controllable if and only if rank $\left[\lambda I_{n-k} - \left(\bar{J}_{11} - \bar{D}_{11}\right)\bar{Q}_s \vdots \bar{g}_1\bar{Q}_s\right] = n$. In the sequel, for the sake of simplicity we shall assume that the system is R-controllable. Concerning the observability, it is seen immediately that the system is Impulse observable as rank $\begin{bmatrix} S \\ K'\hat{A} \\ \hat{C} \end{bmatrix} = n$ by choosing the left kernel of S as $K' = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$. The Complete observability condition rank $\begin{bmatrix} S \\ \hat{C} \end{bmatrix} = n$ is equivalent with rank $\left(\bar{g}_1^T\bar{Q}_{12}L_2^{-T} + \bar{g}_2^T L_2\right) = k$. The R-observability condition rank $\begin{bmatrix} \lambda S - \hat{A} \\ \hat{C} \end{bmatrix} = n$ reduces to the same condition as R-controllability. Note that as by hypothesis the matrix $\bar{Q}_s$ is positive definite, then the assumption of controllability of the proper subsystem implies also the observability using the port conjugated variable defined by the output matrix $C_c = \bar{g}_1^\top\bar{Q}_s$ [van der Schaft and Maschke 1995], hence the system is also R-observable and R-minimal.

3.2 Balanced realization

The descriptor system is balanced if and only if:

$$G_{1c} = G_{1o} = \Sigma \text{ with } \Sigma = \mathrm{diag}(\varsigma_1, \varsigma_2, \ldots, \varsigma_{n-k}) \qquad (18)$$

where G_{1c} and G_{1o} are the proper controllability and observability Gramians [Stykel 2004], ς_i are the Hankel singular values. One can define two transformation matrices W_b and T_b where $(T_b)^T W_b = I$ such that by transforming the coordinate by $\hat{z} = T_b x_b$, and premultiply by matrix W_b on the two sides of the system, we obtain a balanced system from the system (16) with these new matrices: $\bar{E}_b = W_b^T\hat{E}T_b^{-T} = W_b^T\hat{E}W_b$, $Q_b = T_b^T\hat{Q}T_b > 0$ and $g_b^T = \hat{g}^T T_b^{-T} = \hat{g}^T W_b$.

Then the balanced system can also be written as the same port-Hamiltonian descriptor with (16) and (17) in which the sub-matrices of the balanced system are $J_b = W_{b1}^T\bar{J}_{11}W_{b1}$, $D_b = W_{b1}^T\bar{D}_{11}W_{b1}$, $Q_{b1} = T_{b1}^T\hat{Q}T_{b1} > 0$.

3.3 Reduction by the flow constraint method

In this section we shall reduce the initial system of order n to a system of order $r < n$ by using the flow constraint method proposed by [Polyuga and van der Schaft 2012] instead of the truncation method which proposed by [Stykel 2004]. Using the flow constraint method ensures the conservation of the port Hamiltonian structure of the reduced system. The Dirac structure associated with the balanced descriptor Hamiltonian system is given by:

$$\begin{bmatrix} I_r & 0 & 0 & 0 & 0 \\ 0 & I_{n-k-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & 0 & -I_{m_R} \end{bmatrix}\begin{bmatrix} f_{x_{b_{11}}} \\ f_{x_{b_{12}}} \\ f_{x_{b_2}} \\ f_p \\ f_R \end{bmatrix} + \begin{bmatrix} J_{b_{11}} & J_{b_{12}} & 0 & g_{b_{11}} & g_{R_{b1}} \\ J_{b_{21}} & J_{b_{22}} & 0 & g_{b_{12}} & g_{R_{b2}} \\ 0 & 0 & I_k & 0 & 0 \\ g_{b_{11}}^T & g_{b_{12}}^T & g_{b_2}^T & 0 & 0 \\ g_{R_{b1}}^T & g_{R_{b2}}^T & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e_{x_{b_{11}}} \\ e_{x_{b_{12}}} \\ e_{x_{b_2}} \\ e_p \\ e_R \end{bmatrix} = 0 \qquad (19)$$

where $f_{x_b} = \begin{bmatrix} f_{x_{b_{11}}} \\ f_{x_{b_{12}}} \\ f_{x_{b_2}} \end{bmatrix} = \dot{x}_b$ and $e_{x_b} = \begin{bmatrix} e_{x_{b_{11}}} \\ e_{x_{b_{12}}} \\ e_{x_{b_2}} \end{bmatrix} = Q_b x_b$.
The variables we want to reduce are $f_{x_{b_{12}}}$ and $e_{x_{b_{12}}}$. By using the flow constraint method (see more details in [Polyuga and van der Schaft 2012]), one obtains the following reduced Dirac structure

$$F_r f_r + E_r e_r = 0 \qquad (20)$$

where:

$$F_r = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix}, E_r = \begin{bmatrix} J_{b_s} & 0 & \alpha & \beta \\ 0 & I_k & 0 & 0 \\ \alpha^T & g_{b_2}^T & \gamma & \mu \\ \beta^T & 0 & -\mu^T & \zeta \end{bmatrix} \qquad (21)$$

and:

$$\begin{aligned} J_{b_s} &= J_{b_{11}} - J_{b_{12}}J_{b_{22}}^{-1}J_{b_{21}} \quad & \alpha &= g_{b_{11}} - J_{b_{12}}J_{b_{22}}^{-1}g_{b_{12}} \\ \beta &= g_{R_{b1}} - J_{b_{12}}J_{b_{22}}^{-1}g_{R_{b2}} & \gamma &= -g_{b_{12}}^T J_{b_{22}}^{-1}g_{b_{12}} \\ \mu &= -g_{b_{12}}^T J_{b_{22}}^{-1}g_{R_{b2}} & \zeta &= -g_{R_{b2}}^T J_{b_{22}}^{-1}g_{R_{b2}} \end{aligned} \qquad (22)$$

$F_r, E_r \in \mathbb{R}^{r+k+m\times r+k+m}$ are square matrices where $\gamma = -\gamma^T$ and $\zeta = -\zeta^T$. With the reduced Hamiltonian is $H_r = \frac{1}{2}x_r^T Q_r x_r$, where

$$Q_r = \begin{bmatrix} Q_{b_{11}} & 0 \\ 0 & I_k \end{bmatrix} \qquad (23)$$

one can formulate the reduced Dirac structure in a explicit descriptor Hamiltonian system.

4. APPLICATION TO A NANOTWEEZER

In this section, we shall apply the proposed model reduction method to a Port Hamiltonian model of the nanotweezer of the FEMTO-ST laboratory [Boudaoud et al. 2013]. Consider the simplified model of a silicon nanotweezer used for DNA manipulation given in Figure (1). The tweezers is made up with a flexible arm that can be modeled as a Timoshenko beam clamped to a transverse suspension system. The trapped DNA bundle is approximated by a spring/damper-mass-spring/damper system attached at the tip of Timoshenko beam.

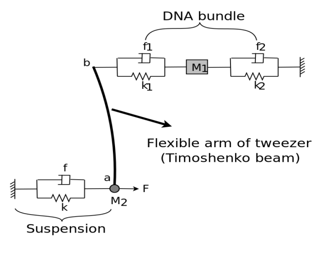

Fig. 1. Suspension-Nanotweezers-DNA

The Timoshenko beam model may be expressed as an infinite dimensional Port Hamiltonian system [Le Gorrec et al. 2005]:

$$\frac{\partial x}{\partial t} = \mathcal{J}\frac{\partial H}{\partial x} \qquad (24)$$

where $\mathcal{J} = P_1\frac{\partial}{\partial z} + P_0$ and

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad (25)$$

The matrices P_1 and P_0 define the skew-symmetric differential operator of order 1 acting on the state space $X = L_2(a; b; \mathbb{R}^4)$. The energy of the beam is expressed in terms of the energy variables,

$$H = \frac{1}{2}\int_a^b (Kx_1^2 + \frac{1}{\rho}x_2^2 + EIx_3^2 + \frac{1}{I_\rho}x_4^2)dz \quad (26)$$

where the state (energy) variables are: the shear displacement x_1 , the transverse momentum distribution x_2, the angular displacement x_3 and the angular momentum distribution x_4. The coefficients ρ, I_ρ, E, I and K are the mass per unit length, the rotary moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively.

Using the mixed finite element semi-discretization method suggested in [Golo et al. 2004], one obtains a finite dimensional explicit Port Hamiltonian system such as:

$$\begin{aligned} \dot{x} &= (J - R)\frac{\partial H}{\partial x} + Bu \\ y &= B^T\frac{\partial H}{\partial x} \end{aligned} \quad (27)$$

where $J = -J^T$, $R = R^T > 0$, H is the Hamiltonian function. According to [Macchelli 2011],[Ramirez and Le Gorrec 2013] the discretization of the Timoshenko beam model, leads to the following structure matrices:

$$J_d = \underbrace{\begin{bmatrix} 0 & M & 0 & 0 \\ M^T & 0 & 0 & 0 \\ 0 & 0 & 0 & M \\ 0 & 0 & M^T & 0 \end{bmatrix}}_{\bar{P}_1} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & -\Phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi^T & 0 & 0 & 0 \end{bmatrix}}_{\bar{P}_2}$$
$$B_d = \begin{bmatrix} 0 & 0 & B_1 & 0 \\ B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \\ 0 & B_2 & 0 & 0 \end{bmatrix} \quad (28)$$

where the sub-matrices are:

$$M = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{bmatrix} \text{ with } M \in \mathbb{R}^{N\times N} \quad (29)$$

$$\Phi = diag(\beta, \cdots, \beta) \text{ with } \Phi \in \mathbb{R}^{N\times N} \quad (30)$$

$$B_1 = \begin{bmatrix} 0 \\ 0_{N-2} \\ 1 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} -1 \\ 0_{N-2} \\ 0 \end{bmatrix} \quad (31)$$

where β is the distance of the infinitesimal section.

The inputs and outputs of the system are the velocities in translation v and rotation ω as well as the forces F and torques T at the boundaries a and b:

$$\begin{aligned} u &= [\, v(b)\ \omega(b)\ F(a)\ T(a)\,]^T = [\, u_1\ u_2\ u_3\ u_4\,]^T \\ y &= [\, F(b)\ T(b)\ -v(a)\ -\omega(a)\,]^T = [\, y_1\ y_2\ y_3\ y_4\,]^T \end{aligned} \quad (32)$$

The DNA bundle and the suspension system can be modeled as two simple finite dimensional port Hamiltonian systems like (27). The matrices of DNA bundle are defined as follow:

$$J_b = -J_b^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix},\ R_b = R_b^T = \begin{bmatrix} \frac{1}{f_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_2 \end{bmatrix}$$
$$g_b^T = \begin{bmatrix} \frac{1}{f_1} & 0 & 0 \\ 0 & 0 & f_\theta \end{bmatrix}, \qquad S_b = \begin{bmatrix} \frac{1}{f_1} & 0 \\ 0 & f_\theta \end{bmatrix} \quad (33)$$

with the energy of the DNA bundle is given as:

$$H_b = \frac{1}{2}\left(k_1(x_{c2} - x_{c1})^2 + k_2x_{c2}^2 + \frac{1}{M}(p_{c2})^2\right) \quad (34)$$

where M is the mass of DNA bundle, x_{c1} and x_{c2} are the relative positions of point b and mass M and $p_{c2} = M\dot{x}_{c2}$ is its momentum. k_1, k_2, f_1 and f_2 represent the constants of the springs and the viscous dampers of the DNA bundle respectively. f_θ is the rotation damper of the DNA bundle in point b. The suspension system is also modeled as system (27), in which the matrices are given as:

$$J_a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ R_a = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix},\ g_a^T = [\,1\ 0\,] \quad (35)$$

with the energy of the suspension system is given as:

$$H_a = \frac{1}{2}(kx_a^2 + \frac{1}{M_2}(p_a)^2) \quad (36)$$

where x_a is the relative position of point a, M_2 is the mass of point a, $p_a = M_2\dot{x}_a$ is its momentum, k and f represent the constant of the springs and the viscous damper of suspension system respectively.

The interconnection relations of the tweezers arm and the suspension system are:

$$u_a = y_3 \text{ and } u_3 = -y_a \quad (37)$$

where

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = y_b \text{ and } u_b = -\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Since the arm of the tweezers is clamped to the suspension system, we consider the additional constraint:

$$y_4 = \omega(a) = 0 \quad (38)$$

With the above interconnections, one can express the total system with the constraint as (1) with the dissipation port is closed and the total energy of the system $H_t = H + H_b + H_a$.

We have chosen the order of the discrete Timoshenko beam model $n_T = 200$, the orders of the DNA bundle and the suspension system are $n_D = 3$ and $n_S = 2$ respectively. The total system order is $n = 205$. The parameters of tweezers arm is given in table (1).

Table 1. The parameters of tweezers arm [Boudaoud et al. 2013]

L	Length	$5150\mu m$
l	Width	$150\mu m$
e	Thickness	$50\mu m$
E	Young's modulus	190 GPa
I	Area moment of inertia	$1.4\times10^{-17}m^4$
G	Shear modulus	80 GPa
ρ	Mass density	$2330\ KG/m^3$

Figure (2) gives the relative H_∞ norms of the input/output systems for the dimensions of the reduced order models r from 0 to 190.

Figure (3) gives the comparative bode plots of discretized systems with 200 or 100 elements, and the reduced system

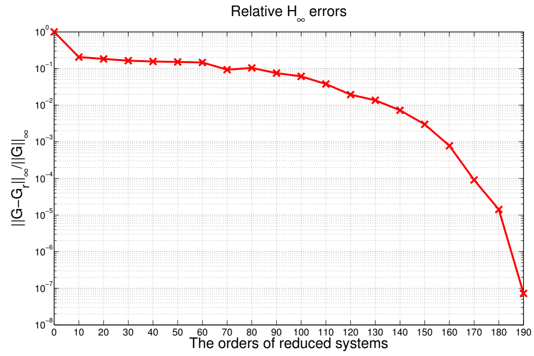

Fig. 2. Relative H_∞ norm errors

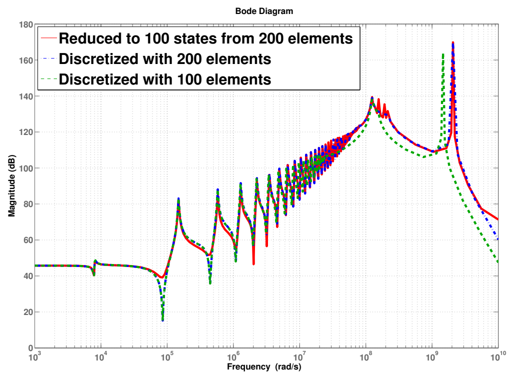

Fig. 3. Bode diagram of discretized systems with 200 or 100 elements and the reduced system with 100 states from 200 elements

with 100 states which is reduced from the discretized system with 200 elements. It shows the intrinsic advantage of balanced reduction method comparing to size equivalent discretization method.

5. CONCLUSION

In this paper, we proposed a model reduction method for constrained port Hamiltonian systems that preserves the Hamiltonian structure and the passivity properties. First of all, we propose a coordinate transformation to express the constrained port Hamiltonian system in its descriptor form [van der Schaft and Maschke 1994]. We then eliminate the Lagrangian multipliers. The balanced realization of this system is achieved by computing the controllability and observability Gramians, and the port Hamiltonian system is reduced by using the flow-constraint reduction method [Polyuga and van der Schaft 2012]. Finally we illustrate the effectiveness of the proposed model reduction method for the constrained port Hamiltonian systems on example of nanotweezers.

REFERENCES

Antoulas, A. (2005). *Approximation of Large-Scale Dynamical Systems*. Philadelphia.

Boudaoud, M., Haddab, Y., and Le Gorrec, Y. (2013). Modeling and optimal force control of a nonlinear electrostatic microgripper. *IEEE/ASME Transactions on Mechatronics*, 18, 1130–1139.

Dai, L. (1989). *Singular Control Systems*. Springer-Verlag New York, Inc., Secaucus, NJ, USA.

Dalsmo, M. and van der Schaft, A. (1999). On representations and integrability of mathematical structures in energy-conserving physical systems. *SIAM Journal of Control and Optimization*, 37(1), 54–91.

Duindam, V., Macchelli, A., Stramigioli, S., and Bruyninckx, H.e. (2009). *Modeling and Control of Complex Physical Systems - The Port-Hamiltonian Approach*. Springer. ISBN 978-3-642-03195-3.

Gentili, L., Macchelli, A., Melchiorri, C., and Mameli, A. (2011). Mastering the complexity of an ultrasonic sealing system: The port-hamiltonian approach. *Mechatronics*, 21(3), 594 – 603.

Golo, G., Talasila, V., van der Schaft, A., and Maschke, B. (2004). Hamiltonian Discretization of Boundary Control Systems. *Automatica*, 40, 757–771.

Gugercin, S. and Antoulas, A. (2004). A survey of model reduction by balanced truncation and some new results. *Int J Control*, 77(8), 748–766.

Gugercin, S., Polyuga, R.V., Beattie, C., and Van Der Schaft, A. (2012). Structure-preserving tangential interpolation for model reduction of port-Hamiltonian systems. *Automatica*.

Le Gorrec, Y., Zwart, H., and Maschke, B. (2005). Dirac Structures and Boundary Control Systems Associated with Skew-symmetric Differential Operators. *SIMA Journal on Control and Optimization*, 44, 1864–1892.

Macchelli, A. (2011). Energy Shaping of Distributed Parameter Port-Hamiltonian Systems Based on Finite Element Approximation. *Systems & Control Letters*, 60, 579–589.

Polyuga, R. and van der Schaft, A. (2010). Structure Preserving Model Reduction of Port-Hamiltonian Systems by Moment Matching at Infinity. *Automatica*, 46, 665–672.

Polyuga, R. and van der Schaft, A. (2011). Structure preserving moment matching for port-Hamiltonian systems: Arnoldi and Lanczos. *Automatic Control, IEEE Transactions on*, 56(6), 1458–1462.

Polyuga, R. and van der Schaft, A. (2012). Effort- and Flow-constraint Reduction Methods for Structure Preserving Model Reduction of Port-Hamiltonian Systems. *Systems & Control Letters*, 61(3), 412 – 421.

Ramirez, H. and Le Gorrec, Y. (2013). Exponential Stability of a Class of PDE's with Dynamic Boundary Control. *In: Proceedings of the 2013 American Control Conference, Washington, USA, June 2013.*

Reis, T. and Stykel, T. (2010). Positive real and bounded real balancing for model reduction of descriptor systems. *International Journal of Control*, 83(1), 74–88.

Stykel, T. (2004). Gramian Based Model Reduction for Descriptor Systems. *Mathematics of Control, Signals, and Systems*, 16, 297–319.

van der Schaft, A. and Maschke, B. (2013). Port-hamiltonian systems on graphs. *SIAM Journal on Control and Optimization*, 51(2), 906–937.

van der Schaft, A. and Maschke, B. (1994). On the Hamiltonian formulation of non-holonomic mechanical systems. *Reports on Mathematical Physics*, 34(2), 225–233.

van der Schaft, A. and Maschke, B. (1995). The Hamiltonian formulation of energy conserving physical systems with external ports. *Archiv für Elektronik und Übertragungstechnik*, 49(5/6), 362–371.