An $\mathcal{O}\left(\log N\right)$ Parallel Algorithm for Newton Step Computation in Model Predictive Control

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Abstract: The use of Model Predictive Control is steadily increasing in industry as more complicated problems can be addressed. Due to that online optimization is usually performed, the main bottleneck with Model Predictive Control is the relatively high computational complexity. Hence, much research has been performed to find efficient algorithms that solve the optimization problem. As parallel hardware is becoming more commonly available, the demand of efficient parallel solvers for Model Predictive Control has increased. In this paper, a tailored parallel algorithm that can adopt different levels of parallelism for solving the Newton step is presented. With sufficiently many processing units, it is capable of reducing the computational growth to logarithmic in the prediction horizon. Since the Newton step computation is where most computational effort is spent in both interior-point and active-set solvers, this new algorithm can significantly reduce the computational complexity of highly relevant solvers for Model Predictive Control.

Keywords: Model Predictive Control, Parallel Computation, Optimization

1. INTRODUCTION

Model Predictive Control (MPC) is one of the most commonly used control strategies in industry. Some important reasons for its success include that it can handle multivariable systems and constraints on control signals and state variables in a structured way, Maciejowski [2002]. In each sample an optimization problem is solved and in the methods considered in this paper, the optimization problem is assumed to be solved on-line. Depending on which type of system and problem formulation that is used the optimization problem can be of different types, and the most common variants are linear MPC, nonlinear MPC and hybrid MPC. In most cases, the effort spent in the optimization problems boils down to solving Newton-system-like equations. Hence, lots of research has been done in the area of solving this type of system of equations efficiently when it has the special form from MPC, see *e.g.* Jonson [1983], Rao et al. [1998], Hansson [2000], Bartlett et al. [2002], Vandenberghe et al. [2002], Åkerblad and Hansson [2004], Axehill and Hansson [2006], Axehill [2008], Axehill and Hansson [2008], Diehl et al. [2009], Nielsen et al. [2013].

In recent years much effort has been spent on efficient parallel solutions, Constantinides [2009]. In Soudbakhsh and Annaswamy [2013] an extended Parallel Cyclic Reduction algorithm is used to reduce the computation to smaller systems of equations that are solved in parallel. The computational complexity of this algorithm is reported to be $\mathcal{O}\left(\log N\right)$, where N is the prediction horizon. Laird et al. [2011], Zhu and Laird [2008] and Reuterswärd [2012] adopt a time-splitting approach to split the prediction horizon into blocks. The subproblems in the blocks are connected through common variables and are solved in parallel using Schur complements. The common variables are computed via a consensus step where a dense system of equations involving all common variables has to be solved sequentially.

In O'Donoghue et al. [2013] a splitting method based on Alternating Direction Method of Multipliers (ADMM) is used, where some steps of the algorithm can be computed in parallel. Stathopoulos et al. [2013] develop an iterative three-set splitting QP solver. In this method the prediction horizon is split into smaller subproblems that can be solved in parallel and a consensus step using ADMM is performed to obtain the final solution.

In this paper there are two main contributions. First, it is shown that an equality constrained MPC problem of prediction horizon N can be reduced in parallel to a new, smaller MPC problem on the same form but with prediction horizon $p < N$. Since the new problem also has the structure of an MPC problem, it can be solved in $\mathcal{O}(p)$. Second, by repeating the reduction procedure it can be shown that an equality constrained MPC problem corresponding to the Newton step can be solved non-iteratively in parallel, giving a computational complexity growth as low as $\mathcal{O}\left(\log N\right)$. The major computational effort when solving an MPC problem is often spent on computing the Newton step, and doing this in parallel as proposed in this paper significantly reduces the overall computational effort of the solver.

In this article, $\mathbb{S}_{++}^n$ ($\mathbb{S}_{+}^n$) denotes symmetric positive (semi) definite matrices with n columns. Furthermore, let $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}_{i,j} = \{i, i+1, \ldots, j\}$. Symbols in sans-serif font (*e.g.* x) denote vectors of stacked element. The product operator is defined as

$$\prod_{t=t_0}^{t_1} A_t \triangleq A_{t_1} \cdots A_{t_0}. \tag{1}$$

Definition 1. For a set of linear constraints $Ax = b$, the linear independence constraint qualification (LICQ) holds if the constraint gradients are linearly independent, *i.e.* if A has full row rank. When LICQ is violated it is referred to as primal degeneracy.

2. PROBLEM FORMULATION

The optimization problem that is solved at each sample in linear MPC is a convex QP problem in the form

$$\min_{\mathsf{x},\mathsf{u}} \quad \sum_{t=0}^{N-1} \left(\frac{1}{2} \begin{bmatrix} x_t^T & u_t^T \end{bmatrix} Q_t \begin{bmatrix} x_t \\ u_t \end{bmatrix} + l_t^T \begin{bmatrix} x_t \\ u_t \end{bmatrix} + c_t\right)$$
$$+ \frac{1}{2} x_N^T Q_N x_N + l_N^T x_N + c_N \tag{2}$$
$$\text{s.t.} \quad x_0 = \bar{x}_0$$
$$x_{t+1} = A_t x_t + B_t u_t + a_t, \ t \in \mathbb{Z}_{0,N-1}$$
$$u_t \in \mathcal{U}_t, \ t \in \mathbb{Z}_{0,N-1},$$

where the equality constraints are the dynamics equations of the system, and $\mathcal{U}_t$ is the set of feasible control signals. In this paper, let the following assumptions hold for all t

Assumption 1. $\mathcal{U}_t$ consists of constraints of the form $u_{t,\min} \leq u_t \leq u_{t,\max}$, *i.e.* upper and lower bounds on the control signal.

Assumption 2.
$$Q_t = \begin{bmatrix} Q_{x,t} & Q_{xu,t} \\ Q_{xu,t}^T & Q_{u,t} \end{bmatrix} \in \mathbb{S}_+^{n_x+n_u}, \ Q_{u,t} \in \mathbb{S}_{++}^{n_u}, \ Q_N \in \mathbb{S}_+^{n_x}. \tag{3}$$

The problem (2) can be solved using different methods, see *e.g.* Nocedal and Wright [2006]. Two common methods are interior-point (IP) methods and active-set (AS) methods. IP methods approximate the inequality constraints with barrier functions, whereas the AS methods iteratively changes the set of inequality constraints that hold with equality until the optimal active set has been found. In both types, the main computational effort is spent while solving Newton-system-like equations often corresponding to an equality constrained MPC problem with prediction horizon N (or to a problem with similar structure)

$$\mathcal{P}(N): \quad \begin{aligned} \min_{\mathsf{x},\mathsf{u}} \quad & \sum_{t=0}^{N-1} \left(\frac{1}{2} \begin{bmatrix} x_t^T & u_t^T \end{bmatrix} Q_t \begin{bmatrix} x_t \\ u_t \end{bmatrix} + l_t^T \begin{bmatrix} x_t \\ u_t \end{bmatrix} + c_t\right) \\ & + \frac{1}{2} x_N^T Q_N x_N + l_N^T x_N + c_N \\ \text{s.t.} \quad & x_0 = \bar{x}_0 \\ & x_{t+1} = A_t x_t + B_t u_t + a_t, \ t \in \mathbb{Z}_{0,N-1}. \end{aligned} \tag{4}$$

Even though this problem might look simple and irrelevant it is the workhorse of many optimization routines for linear, nonlinear and hybrid MPC. $\mathcal{P}(N)$ is the resulting problem after the equality constraints corresponding to active control signal constraints have been eliminated as in an AS method (only control signal constraints are considered). Note that u_t and the corresponding matrices have potentially changed dimensions from (2). Further, let the following assumption hold (without loss of generality)

Assumption 3. LICQ holds for (4).

3. PROBLEM DECOMPOSITION

The structure of the equality constrained MPC problem (4) can be exploited by splitting it into smaller subproblems that only share a small number of common variables. Given the value of the common variables, the subproblems can be solved individually. These smaller subproblems are obtained by splitting the prediction horizon in $p+1$ intervals $i \in \mathbb{Z}_{0,p}$ (each of length N_i) and introducing initial constraints $x_{0,i} = \hat{x}_i$ for each subproblem and terminal constraints $x_{N_i,i} = d_i$ for $i \in \mathbb{Z}_{0,p-1}$. The connections between the subproblems are then given by the coupling constraints $\hat{x}_{i+1} = d_i$ for $i \in \mathbb{Z}_{0,p-1}$. Let $x_{t,i}$ and $u_{t,i}$ denote the states and control signals in subproblem i and let the indices of the matrices be defined analogously. For notational aspects, and without loss of generality, the terminal state d_i is parametrized as $d_i = \hat{A}_i \hat{x}_i + \hat{B}_i \hat{u}_i + \hat{a}_i$, where $\hat{x}_i \in \mathbb{R}^{n_x}$ and $\hat{u}_i \in \mathbb{R}^{n_{\hat{u}}}$ with $n_{\hat{u}} \leq n_x$ are the common variables. The choice of this notation will soon become clear. Then, the MPC problem (4) can be cast in the equivalent form

$$\min_{\mathsf{x},\mathsf{u}} \quad \sum_{i=0}^{p} \sum_{t=0}^{N_i-1} \left(\frac{1}{2} \begin{bmatrix} x_{t,i}^T & u_{t,i}^T \end{bmatrix} Q_{t,i} \begin{bmatrix} x_{t,i} \\ u_{t,i} \end{bmatrix} + l_{t,i}^T \begin{bmatrix} x_{t,i} \\ u_{t,i} \end{bmatrix} + c_{t,i}\right)$$
$$+ \frac{1}{2} x_{N_p,p}^T Q_{N_p,p} x_{N_p,p} + l_{N_p,p}^T x_{N_p,p} + c_{N_p,p}$$
$$\text{s.t.} \quad \hat{x}_0 = \bar{x}_0$$
$$\forall i \in \mathbb{Z}_{0,p} \begin{cases} x_{0,i} = \hat{x}_i \\ x_{t+1,i} = A_{t,i} x_{t,i} + B_{t,i} u_{t,i} + a_{t,i}, \\ \qquad\qquad\qquad t \in \mathbb{Z}_{0,N_i-1} \\ x_{N_i,i} = d_i = \hat{A}_i \hat{x}_i + \hat{B}_i \hat{u}_i + \hat{a}_i, \ i \neq p \end{cases}$$
$$\hat{x}_{i+1} = d_i = \hat{A}_i \hat{x}_i + \hat{B}_i \hat{u}_i + \hat{a}_i, \ i \in \mathbb{Z}_{0,p-1}. \tag{5}$$

Note that the first initial state $\hat{x}_0$ is equal to the initial state of the original problem (4). For $i \in \mathbb{Z}_{0,p-1}$ the individual subproblems in (5) are given by

$$\begin{aligned} \min_{\mathsf{x},\mathsf{u}} \quad & \sum_{t=0}^{N_i-1} \left(\frac{1}{2} \begin{bmatrix} x_{t,i}^T & u_{t,i}^T \end{bmatrix} Q_{t,i} \begin{bmatrix} x_{t,i} \\ u_{t,i} \end{bmatrix} + l_{t,i}^T \begin{bmatrix} x_{t,i} \\ u_{t,i} \end{bmatrix} + c_{t,i}\right) \\ \text{s.t.} \quad & x_{0,i} = \hat{x}_i \\ & x_{t+1,i} = A_{t,i} x_{t,i} + B_{t,i} u_{t,i} + a_{t,i}, \ t \in \mathbb{Z}_{0,N_i-1} \\ & x_{N_i,i} = \hat{A}_i \hat{x}_i + \hat{B}_i \hat{u}_i + \hat{a}_i. \end{aligned} \tag{6}$$

Here i is the index of the subproblem. The last subproblem $i = p$ is on the same form (6) but with the additional term

$$\frac{1}{2} x_{N_p,p}^T Q_{N_p,p} x_{N_p,p} + l_{N_p,p}^T x_{N_p,p} + c_{N_p,p} \tag{7}$$

in the objective function, and no constraint on $x_{N_p,p}$.

Temporarily excluding details, each subproblem (6) can be solved parametrically and the solution to each subproblem is a function of the common variables $\hat{x}_i$ and $\hat{u}_i$ ($\hat{x}_p$ for $i = p$). By inserting these parametric solutions of all subproblems in (5) and using the coupling constraints between the subproblems, problem (5) can be reduced to an equivalent master problem

$$\mathcal{P}(p): \quad \begin{aligned} \min_{\hat{\mathsf{x}},\hat{\mathsf{u}}} \quad & \sum_{i=0}^{p-1} \left(\frac{1}{2} \begin{bmatrix} \hat{x}_i^T & \hat{u}_i^T \end{bmatrix} \hat{Q}_i \begin{bmatrix} \hat{x}_i \\ \hat{u}_i \end{bmatrix} + \tilde{l}_i^T \begin{bmatrix} \hat{x}_i \\ \hat{u}_i \end{bmatrix} + \hat{c}_i\right) \\ & + \frac{1}{2} \hat{x}_p^T \hat{Q}_p \hat{x}_p + \tilde{l}_p^T \hat{x}_p + \hat{c}_p \\ \text{s.t.} \quad & \hat{x}_0 = \bar{x}_0 \\ & \hat{x}_{i+1} = \hat{A}_i \hat{x}_i + \hat{B}_i \hat{u}_i + \hat{a}_i, \ i \in \mathbb{Z}_{0,p-1}. \end{aligned} \tag{8}$$

Here $\hat{Q}_i$, $\tilde{l}_i$ and $\hat{c}_i$ are computed in each subproblem and represent the value function. The dynamics constraints in the master problem are given by the coupling constraints between the subproblems in (5). This new MPC problem is on the same form as the original equality constrained problem (4), but with prediction horizon $p < N$. The reduction of the problem is summarized in Theorem 1 and is graphically depicted in Fig. 1, where the dotted lines represents repetition of the structure. This approach
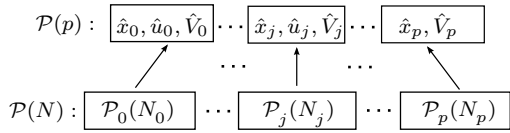
Fig. 1. The parameters $\hat{x}_i$ and $\hat{u}_i$ in each subproblem i can be interpreted as new state and control variables in the reduced MPC problem with prediction horizon p. The value functions $\hat{V}_i(\hat{x}_i, \hat{u}_i)$ are the terms in the new objective function.

is similar to primal decomposition, Lasdon [1970], Boyd et al. [2008], where the $p+1$ subproblems share common variables $\hat{x}_i$ and $\hat{u}_i$ that are computed iteratively. In the work presented in this paper the common variables are however not computed iteratively but instead determined directly by solving the new, reduced MPC problem at the upper level in Fig. 1. Inserting the optimal $\hat{x}_i$ and $\hat{u}_i$ ($\hat{x}_p$ for $i = p$) into the subproblems given by (6) gives the solution to (4).

Theorem 1. Consider an optimization problem $\mathcal{P}(N)$ defined in (4) where Assumption 3 holds. Then $\mathcal{P}(N)$ can be reduced to $\mathcal{P}(p)$ in parallel, where $1 \leq p < N$. The optimal solution X^* and λ^* to $\mathcal{P}(N)$ can be computed in parallel from the solution $\hat{X}^*$ and $\hat{\lambda}^*$ to $\mathcal{P}(p)$.

Proof. For the proof of Theorem 1, see Appendix A.1.

In the remainder of this section it will be shown how the subproblems (6) are solved parametrically and how the matrices needed in (8) are computed.

3.1 Solution of the subproblems

In this section, it will be shown that each subproblem $i \in \mathbb{Z}_{0,p}$ given by (6) can be solved parametrically and that the solution can be expressed as a function of the common variables $\hat{x}_i$ and $\hat{u}_i$ for $i \in \mathbb{Z}_{0,p-1}$ and $\hat{x}_p$ for $i = p$.

For now it is assumed that LICQ holds for (6). The optimization problem can be cast in a more compact form

$$\min_{X_i} \quad \frac{1}{2} X_i^T \mathbf{Q}_i X_i + \mathbf{l}_i^T X_i + \mathbf{c}_i \qquad (9)$$
$$\text{s.t.} \quad \mathbf{A}_i X_i = \mathbf{b}_i + \mathbf{G}_i \theta_i,$$

by defining $\theta_i \triangleq [\hat{x}_i^T \ \hat{u}_i^T]^T$ and

$$X_i \triangleq \begin{bmatrix} x_{0,i}^T, u_{0,i}^T \ \cdots \ x_{N_i,i}^T \end{bmatrix}^T, \ \lambda_i \triangleq \begin{bmatrix} \lambda_{0,i}^T \ \cdots \ \lambda_{N_i,i}^T \ \lambda_{tc,i}^T \end{bmatrix}^T, \quad (10)$$

$$\mathbf{Q}_i \triangleq \text{diag}\left(Q_{0,i}, ..., Q_{N_i-1,i}, 0\right), \ \mathbf{l}_i \triangleq \begin{bmatrix} l_{0,i}^T \ \cdots \ l_{N_i-1,i}^T \ 0 \end{bmatrix}^T, \quad (11)$$

$$\mathbf{A}_i \triangleq \begin{bmatrix} -I & 0 & \cdots & & & \cdots & 0 \\ A_{0,i} & B_{0,i} & -I & 0 & \cdots & & \vdots \\ 0 & 0 & A_{1,i} & \cdots & & & \\ \vdots & \vdots & & \ddots & & & 0 \\ \vdots & \vdots & & & A_{N_i-1,i} & B_{N_i-1,i} & -I \\ 0 & \cdots & & & \cdots & 0 & -I \end{bmatrix}, \ \mathbf{c}_i \triangleq \sum_{t=0}^{N_i-1} c_{t,i}, \quad (12)$$

$$\mathbf{b}_i \triangleq -\begin{bmatrix} 0 & a_{0,i}^T & \cdots & a_{N_i-1,i}^T & \hat{a}_i^T \end{bmatrix}, \ \mathbf{G}_i \triangleq -\begin{bmatrix} I & 0 & \cdots & 0 & \hat{A}_i^T \\ 0 & 0 & \cdots & 0 & \hat{B}_i^T \end{bmatrix}^T \quad (13)$$

The dual variables λ_i in the subproblem are introduced as

$$\lambda_{0,i} \leftrightarrow x_{0,i} = \hat{x}_i \qquad (14)$$
$$\lambda_{t+1,i} \leftrightarrow x_{t+1,i} = A_{t,i}x_{t,i} + B_{t,i}u_{t,i} + a_{t,i}, \ t \in \mathbb{Z}_{0,N_i-1} \qquad (15)$$
$$\lambda_{tc,i} \leftrightarrow x_{N_i,i} = \hat{A}_i\hat{x}_i + \hat{B}_i\hat{u}_i + \hat{a}_i. \qquad (16)$$

The symbol $\leftrightarrow$ should be interpreted as λ being the dual variable corresponding to the respective equality constraint.

Note that (9) is a very simple multiparametric quadratic programming problem with parameters θ_i and only equality constraints. Hence the optimal primal and dual solution to this problem are both affine functions of the parameters θ_i, Tøndel et al. [2013].

Remark 1. Since the simple parametric programming problem (9) is subject to equality constraints only it is not *piecewise* affine in the parameters. Hence, the solution to the equality constrained problem can be computed cheaply and it does not suffer from the complexity issues of a general multiparametric programming problem.

Since LICQ is assumed to hold, the unique optimal primal and dual solution can be expressed as

$$X_i^*(\theta_i) = K_i^x \theta_i + k_i^x, \qquad (17)$$
$$\lambda_i^*(\theta_i) = K_i^\lambda \theta_i + k_i^\lambda, \qquad (18)$$

for some K_i^x, k_i^x, K_i^λ and k_i^λ, and where i denotes the index of the subproblem. The value function of (6) is obtained by inserting the parametric primal optimal solution (17) into the objective function in (9), resulting in

$$\hat{V}_i(\theta_i) = \frac{1}{2} \theta_i^T \hat{Q}_i \theta_i + \hat{l}_i^T \theta_i + \hat{c}_i, \qquad (19)$$

where $\hat{Q}_i = (K_i^x)^T \mathbf{Q}_i K_i^x$, $\hat{l}_i^T = \mathbf{l}_i^T K_i^x + (k_i^x)^T \mathbf{Q}_i K_i^x$ and $\hat{c}_i = \mathbf{c}_i + \frac{1}{2}(k_i^x)^T \mathbf{Q}_i k_i^x + \mathbf{l}_i^T k_i^x$.

The last subproblem given by (6) with the additional term (7) in the objective function is different from the p first subproblems since there is no terminal constraint on $x_{N_p,p}$. Hence the parametric solution of this subproblem only depends on the initial state $\hat{x}_p$, and λ_p, $\mathbf{Q}_p$, $\mathbf{l}_p$, $\mathbf{c}_p$, $\mathbf{A}_p$, $\mathbf{b}_p$ and $\mathbf{G}_p$ in (10) to (13) are modified accordingly. The derivation of the solution is analogous to the one for the subproblems $i \in \mathbb{Z}_{0,p-1}$, but with $\theta_p = \hat{x}_p$. The unique optimal primal and dual solution is hence given by (17) and (18), and the value function is given by (19) (all with $i = p$).

3.2 Solution of a primal degenerate subproblem

The terminal constraint in a subproblem given by (6) introduces n_x new constraints, which might result in an infeasible subproblem or that LICQ is violated for the subproblem even though this is not the case in the original problem (4). According to Definition 1, violation of LICQ is known as primal degeneracy and the dual variables for a primal degenerate problem are non-unique, Tøndel et al. [2013]. In this section it will be shown how to choose the parameter in the terminal constraint to obtain a feasible subproblem and also how to choose dual variables of subproblem i that satisfy the optimality conditions of the original problem (4).

Since the subproblem is feasible only if there exists a solution to $\mathbf{A}_i X_i = \mathbf{b}_i + \mathbf{G}_i \theta_i$ it is required that $\mathbf{b}_i + \mathbf{G}_i \theta_i \in \mathcal{R}(\mathbf{A}_i)$. This is satisfied if the terminal constraint is chosen carefully, which means that it has to be known which θ_i that will give a feasible solution. To do this, the dynamics constraints in subproblem i can be used to compute the final state in subproblem i given the control signals $\mathbf{u}_i$ and the initial state $\hat{x}_i$ as

$$x_{N_i,i} = \mathcal{A}_i \hat{x}_i + \mathcal{S}_i \mathbf{u}_i + \mathcal{D}_i \mathbf{a}_i, \qquad (20)$$

where

$$\mathcal{A}_i \triangleq \prod_{t=0}^{N_i-1} A_{t,i}, \ \mathcal{D}_i \triangleq \left[\prod_{t=1}^{N_i-1} A_{t,i} \ \cdots \ A_{N_i-1,i} \ I\right] \quad (21)$$

$$\mathcal{S}_i \triangleq \left[\prod_{t=1}^{N_i-1} A_{t,i} B_{0,i} \ \cdots \ A_{N_i-1,i} B_{N_i-2,i} \ B_{N_i-1,i}\right] \quad (22)$$

and a_i and u_i are the stacked $a_{t,i}$ and $u_{t,i}$ for $t \in \mathbb{Z}_{0,N_i-1}$. The feasibility of the subproblem can be ensured by a careful selection of the parametrization of the problem. In this work this is performed by requiring that the final state satisfies the terminal constraint $x_{N_i,i} = d_i = \hat{A}_i\hat{x}_i + \hat{B}_i\hat{u}_i + \hat{a}_i$, where d_i is within the controllable subspace given by $\mathcal{A}_i$, $\mathcal{S}_i$ and $\mathcal{D}_i\mathsf{a}_i$. This can be assured by requiring

$$\hat{A}_i = \mathcal{A}_i, \quad \hat{B}_i = \mathcal{T}_i, \quad \hat{a}_i = \mathcal{D}_i\mathsf{a}_i, \quad (23)$$

where the columns of $\mathcal{T}_i$ form a basis for the range space of $\mathcal{S}_i$. (Note that for a non-degenerate problem, $\hat{A}_i = 0$, $\hat{B}_i = I$ and $\hat{a}_i = 0$ are valid choices since $\mathcal{S}_i$ has full row rank.) By using this parametrization, the master problem can only use parameters that will result in a feasible subproblem.

The optimal parametric primal and dual solution to a primal degenerate problem on the form (9) are given by (17) and

$$\lambda_i^*(\theta_i) = K_i^\lambda\theta_i + k_i^\lambda + \lambda_i^\mathcal{N}, \quad (24)$$

where $\lambda_i^\mathcal{N} \in \mathcal{N}\left(\mathbf{A}_i^T\right)$, Tøndel et al. [2013]. The null space $\mathcal{N}\left(\mathbf{A}_i^T\right)$ is given by Lemma 2.

Lemma 2. The null space of $\mathbf{A}_i^T$ is given by

$$\mathcal{N}\left(\mathbf{A}_i^T\right) = \{z \mid z = Z_i w_i, \ \forall w_i \in \mathcal{N}\left(\mathcal{S}_i^T\right)\}, \quad (25)$$

where

$$Z_i \triangleq \left[-\hat{A}_i \ -\mathcal{D}_i \ I\right]^T, \quad (26)$$

and $\mathcal{S}_i$ is the controllability matrix.

Proof. For the proof of Lemma 2, see Appendix A.2.

Remark 2. Note that Z_i is computed cheaply since the matrices $\hat{A}_i$ and $\mathcal{D}_i$ are already computed.

The dual variables of (5) are introduced by (14)-(16) for each subproblem, and by

$$\hat{\lambda}_{-1} \leftrightarrow \hat{x}_0 = \bar{x}_0 \quad (27)$$

$$\hat{\lambda}_i \leftrightarrow \hat{x}_{i+1} = \hat{A}_i\hat{x}_i + \hat{B}_i\hat{x}_i + \hat{a}_i, \ i \in \mathbb{Z}_{0,p-1}, \quad (28)$$

for the coupling constraints that connect the subproblems in (5). Note that $\lambda_{tc,i}$ in (16) is the dual variable corresponding to the terminal constraint in each subproblem, whereas (28) are the dual variables corresponding to the coupling constraints between the subproblems (interpreted as the dynamics constraints in the reduced MPC problem (8)). Hence, $\lambda_{tc,i}$ is computed in the subproblem, and $\hat{\lambda}_i$ is computed when (8) is solved. This is depicted in Fig. 2 where the upper level corresponds to problem (8) and the lower level to problem (5). For primal degenerate subproblems, the dual solution is non-unique. In order to choose dual solutions to the subproblems that satisfy the optimality conditions of the original problem (4), the relations between the dual variables of different subproblems are studied. These relations are given by Theorem 3 and Corollary 4.

Theorem 3. Consider an MPC problem on the form (5) where Assumption 3 holds. Let the dual variables be defined by (14), (15), (16), (27) and (28). Then the relations between the optimal dual solutions in different subproblems are given by
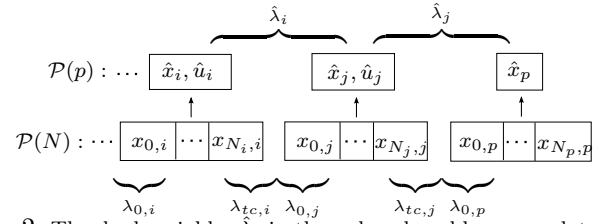

Fig. 2. The dual variables $\hat{\lambda}_i$ in the reduced problem are related to the dual variables $\lambda_{tc,i}$ and $\lambda_{0,i+1}$ in the subproblems.

$$\lambda_{0,p} = \hat{\lambda}_{p-1} \quad (29)$$

$$\lambda_{0,i} = \hat{\lambda}_{i-1} - \hat{A}_i^T\left(\lambda_{tc,i} + \hat{\lambda}_i\right), \ i \in \mathbb{Z}_{0,p-1} \quad (30)$$

$$\hat{B}_i^T\left(\lambda_{tc,i} + \hat{\lambda}_i\right) = 0, \ i \in \mathbb{Z}_{0,p-1} \quad (31)$$

$$\lambda_{N_i,i} = -\lambda_{tc,i}, \ i \in \mathbb{Z}_{0,p-1}, \quad (32)$$

where $\hat{A}_i$ and $\hat{B}_i$ are defined by (23).

Proof. For the proof of Theorem 3, see Appendix A.3.

Corollary 4. Let the assumptions in Theorem 3 be satisfied, and let LICQ hold for all subproblems $i \in \mathbb{Z}_{0,p}$. Then the optimal dual variables in the subproblems are unique and the relations between the dual solutions in the subproblems are given by

$$\lambda_{0,i} = \hat{\lambda}_{i-1}, \ i \in \mathbb{Z}_{0,p} \quad (33)$$

$$\lambda_{tc,i} = -\hat{\lambda}_i = -\lambda_{0,i+1}, \ i \in \mathbb{Z}_{0,p-1} \quad (34)$$

$$\lambda_{N_i,i} = -\lambda_{tc,i} = \lambda_{0,i+1}, \ i \in \mathbb{Z}_{0,p-1}. \quad (35)$$

Proof. Let LICQ hold for all subproblems $i \in \mathbb{Z}_{0,p}$ in (5). Then $\mathcal{N}\left(\mathbf{A}_i^T\right) = \emptyset$, $i \in \mathbb{Z}_{0,p}$ and the dual solution is unique. Furthermore,

$$\text{rank}(\mathcal{S}_i) = n_x \Rightarrow \mathcal{T}_i = \hat{B}_i \text{ non-singular} \Rightarrow \quad (36)$$

$$\{\text{Using (31) in Theorem 3}\} \Rightarrow \lambda_{tc,i} = -\hat{\lambda}_i. \quad (37)$$

Inserting (37) into (30) and (32) gives $\lambda_{N_i,i} = \lambda_{0,i+1}$, $i \in \mathbb{Z}_{0,p-1}$, and $\lambda_{0,i} = \hat{\lambda}_{i-1}$, $i \in \mathbb{Z}_{0,p}$ by also using (29).

Lemma 2 is used to choose the null space element $\lambda_i^\mathcal{N}$ in (24) to obtain the correct dual solution for subproblem i. According to the lemma $\lambda_i^\mathcal{N}$ can be computed as $\lambda_i^\mathcal{N} = Z_i w_i$, $w_i \in \mathcal{N}\left(\mathcal{S}_i^T\right)$, giving the optimal dual variables for subproblem i as

$$\lambda_i^*(\theta_i, w_i) = K_i^\lambda\theta_i + k_i^\lambda + Z_i w_i, \ w_i \in \mathcal{N}\left(\mathcal{S}_i^T\right). \quad (38)$$

Let $\gamma_i = K_i^\lambda\theta_i + k_i^\lambda$ be the dual solution when the minimum norm null space element is selected, and let $\hat{\lambda}_i$ be given by the dual solution to problem (8). Then it follows from Theorem 3 that

$$\gamma_{0,i} = \hat{\lambda}_{i-1} - \hat{A}_i^T(\gamma_{tc,i} + \hat{\lambda}_i), \ i \in \mathbb{Z}_{0,p-1} \quad (39)$$

$$\hat{B}_i^T(\gamma_{tc,i} + \hat{\lambda}_i) = 0, \ i \in \mathbb{Z}_{0,p-1} \quad (40)$$

$$\gamma_{N_i,i} = -\gamma_{tc,i}, \ i \in \mathbb{Z}_{0,p-1}. \quad (41)$$

To obtain the same optimal dual solution λ_i as for the original problem, the freedom in the choice of the dual variables from (38) is exploited, *i.e.*,

$$\lambda_i = \gamma_i + Z_i w_i. \quad (42)$$

In order to obtain the relation $\lambda_{N_i,i} = \hat{\lambda}_i = \lambda_{0,i+1}$ as in the non-degenerate case, (30)-(32) give that $\lambda_{tc,i} = -\hat{\lambda}_i$ must hold. The last block in (26) and (38) gives $\lambda_{tc,i} = \gamma_{tc,i} + w_i$, and based on this w_i is chosen as

$$w_i = -(\gamma_{tc,i} + \hat{\lambda}_i) \in \mathcal{N}\left(\mathcal{S}_i^T\right) \Rightarrow \lambda_{tc,i} = -\hat{\lambda}_i. \quad (43)$$
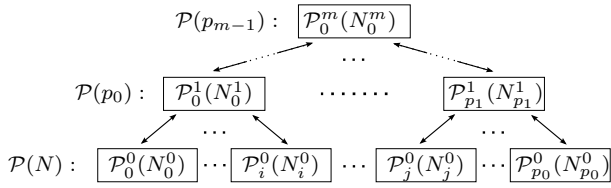
Fig. 3. The tree structure obtained when the MPC problems are reduced in several steps. Each level in the tree forms an MPC problem that is again split into several smaller problems.

Note that (40) gives that $w_i \in \mathcal{N}\left(\hat{B}_i^T\right) = \mathcal{N}\left(\mathcal{S}_i^T\right)$. By using this choice of w_i in the optimal dual solution (38) together with (26), (39) and (41) the following hold

$$\lambda_{0,i} = \gamma_{0,i} + \lambda_{0,i}^{\mathcal{N}} = \gamma_{0,i} - \hat{A}_i^T w_i = \hat{\lambda}_{i-1}, \ i \in \mathbb{Z}_{0,p-1} \quad (44)$$

$$\lambda_{N_i,i} = \gamma_{N_i,i} + \lambda_{N_i,i}^{\mathcal{N}} = -\gamma_{tc,i} - w_i = \hat{\lambda}_i, \ i \in \mathbb{Z}_{0,p-1} \quad (45)$$

Hence, the chosen dual solution of subproblem i satisfies the optimality conditions for (4) if it is computed as

$$\lambda_i^*(\theta_i, \hat{\lambda}_i) = K_i^\lambda \theta_i + k_i^\lambda - Z_i(\gamma_{tc,i} + \hat{\lambda}_i) = \gamma_i - Z_i(\gamma_{tc,i} + \hat{\lambda}_i). \quad (46)$$

The dual solution to the original problem can be retrieved from (46) for $i \in \mathbb{Z}_{0,p-1}$ and (18) for $i = p$.

4. PROBLEM REDUCTION IN PARALLEL

Theorem 1 states that the original problem $\mathcal{P}(N)$ can be solved by first reducing it to $\mathcal{P}(p)$ with $p < N$, and then solve the smaller $\mathcal{P}(p)$ to determine the optimal parameters of the subproblems. However, $\mathcal{P}(p)$ can instead be reduced again to an even smaller MPC problem, and in this section Theorem 1 will be used repeatedly to obtain a problem structure that can be solved in parallel. This can be summarized in a tree structure, see Fig. 3. Let the MPC problem at level k be denoted $\mathcal{P}(p_{k-1})$, and let $\mathcal{P}_i^k(N_i^k)$ be the i:th subproblem with prediction horizon N_i^k at level k. The problem $\mathcal{P}(p_{k-1})$ is reduced to the equivalent $\mathcal{P}(p_k)$ by solving all subproblems $\mathcal{P}_i^k(N_i^k)$, $i \in \mathbb{Z}_{0,p_k}$ parametrically according to Section 3. Since all of the subproblems $\mathcal{P}_i^k(N_i^k)$ are independent, this can be done in parallel. The reduction of the MPC problem is continued until a problem with the minimal desired prediction horizon $p_{m-1} = N_0^m$ is obtained.

The original problem $\mathcal{P}(N)$ is solved by first reducing the problem in m steps until $\mathcal{P}(p_{m-1})$ is obtained, and thereafter propagating the solution of $\mathcal{P}(p_{m-1})$ down to level $k = 0$. Since information is exchanged between parents and children only, the optimal solution to each $\mathcal{P}_i^k(N_i^k)$ can be computed individually from the other subproblems at level k. Hence, this can be performed in parallel.

Remark 3. Note that at each level k in the tree in Fig. 3, the common variables for level $k-1$ are computed. Hence, the consensus step to decide the common variables are done in one iteration and it is not necessary to iterate to get consensus between the subproblems as in many other methods.

5. PARALLEL COMPUTATION OF NEWTON STEP

The theory presented in this paper is summarized in Algorithms 1 and 2. The algorithms can be used to compute the Newton step which is defined by the solution to (4). This is where most computational effort is needed when solving (2). The computations can be performed

using several processors, and the level of parallelism can be tuned to fit the hardware, *i.e.* the number of processing units, memory capacity, bus speed and more. The level of parallelism is decided by adjusting the number of subproblems at each level in the tree in Fig. 3.

5.1 Algorithms for parallel Newton step computation

The algorithm for solving $\mathcal{P}(N)$ in parallel is based on two major steps; reduce the MPC problem in several steps while building the tree and propagate the solution from the top level downwards to the bottom level. In both steps standard parallel numerical linear algebra could be used to parallelize further, *e.g.* matrix multiplications, backward and forward substitutions and factorizations. This paper focuses on parallelization using the inherent structure of the MPC problem, and the discussion about possibilities to parallelize the computations will be limited to this scope.

The first step, to construct the tree in Fig. 3, is summarized in Algorithm 1. Since all subproblems are independent of each other, the parfor-loop on Line 8 to 12 in Algorithm 1 can be performed in parallel on different processors. Let $p_{\max}$ be the maximum number of subproblems at any level in the tree. Then, if there are $p_{\max}$ processors available, all subproblems $\mathcal{P}_i^k(N_i^k)$ at level k can be solved simultaneously.

Algorithm 1 Parallel reduction of MPC problem

1: Initiate level counter $k := 0$
2: Initiate the first number of subsystems $p_{-1} = N$
3: Set the minimal number of subproblems $p_{\min}$
4: **while** $p_k > p_{\min}$ **do**
5:     Compute desired p_k to define the number of subproblems (with $p_k < p_{k-1}$)
6:     Split the prediction horizon $0, \ldots, p_{k-1}$ in $p_k + 1$ segments $0, \ldots, N_0^k$ up to $0, \ldots, N_{p_k}^k$
7:     Create subproblems $i = 0, \ldots, p_k$ for each time block
8:     **parfor** $i = 0, \ldots, p_k$ **do**
9:         Solve subproblem i parametrically and store K_i^x, k_i^x, K_i^λ and k_i^λ
10:         Compute $\hat{A}_i, \hat{B}_i, \hat{a}_i, \hat{Q}_i, \hat{l}_i$ and $\hat{c}_i$ for the next level
11:         Compute and store Z_i
12:     **end parfor**
13:     Update level counter $k := k + 1$
14: **end while**
15: Compute maximum level number $k := k - 1$

The second step is to propagate the solution from the top down in the tree until the bottom level is reached. This is summarized in Algorithm 2. Since all subproblems in the tree only use information from their parents, the parfor-loop at Line 4 to Line 10 can be computed in parallel. As for the first step, if there is one processor for each subproblem, all problems at each level in the tree can be solved simultaneously.

The equality constrained problem (4) was formed by eliminating the inequality constraints in (2) that hold with equality. The dual variables ν corresponding to these eliminated constraints are important in *e.g.* AS methods and can be computed as

$$\nu_{i,t} = Q_{xv,t,i}^T x_{t,i} + Q_{uv,t,i}^T u_{t,i} + B_{v,t,i}^T \lambda_{t,i} + l_{v,t,i} + Q_{v,t,i}^T v_{t,i}, \quad (47)$$

for $t \in \mathbb{Z}_{0,N_i-1}$ for each subproblem $i = \mathbb{Z}_{0,p}$. Here $v_{t,i}$ are the values of the eliminated control signals in (2). For the derivation of this expression, see *e.g.* Axehill [2008]. The computation of ν is described in Algorithm 3, which can be performed in parallel if $p_{\max}$ processors are available.

Algorithm 2 Parallel propagation of solution

1: Initialize the first parameter as $\bar{x}_0$
2: Get level counter k from Algorithm 1
3: **while** $k \geq 0$ **do**
4:     **parfor** $i = 0, \ldots, p_k$ **do**
5:       Compute primal solution given by (17)
6:       Compute dual solution given by (18)
7:       **if** Primal degenerate subproblem
8:         Select the dual solution according to (46)
9:       **end if**
10:     **end parfor**
11:     **if** k==0 **then**
12:       Compute ν_i according to Algorithm 3
13:     **end if**
14:     Update level counter $k := k - 1$
15: **end while**

Algorithm 3 Compute eliminated dual variables

1: **parfor** $i = 0, \ldots, p_0$ **do**
2:     **parfor** $t = 0, \ldots, N_i - 1$ **do**
3:       Compute ν_i according to (47).
4:     **end parfor**
5: **end parfor**

Note that each $\nu_{t,i}$ in each subproblem can be computed in parallel if even more processors are available.

So far no assumptions on the length of the prediction horizon of each subproblem has been made. If however the lengths of each subproblem is fixed to N_s, and the prediction horizon of the original problem is chosen as $N = N_s^{m+1}$ for simplicity, then the tree will get $m + 1$ levels. Furthermore, assume that N_s^m processors are available. Then, since $m = \log_{N_s}(N) - 1$, the computational complexity grows logarithmically in the prediction horizon, *i.e.* as $\mathcal{O}(\log N)$. The optimal length N_s of the subproblems can be adjusted to fit the hardware which the algorithms are implemented on. Depending on the number of processors, the available memory and the communication delays between processors, the size of N_s might be adjusted.

5.2 Numerical results

The proposed algorithm for computing the Newton step has been implemented in MATLAB and used to solve random stable MPC problems in the form (4). The algorithm has been implemented serially, and the parallel computation times are simulated by summing over the maximum solution time at each level in the tree. Hence, memory and communication delays have not been addressed but are assumed small in comparison to the cost of the computations. In the implemented algorithm K_i^x, k_i^x, K_i^λ and k_i^λ are computed using the methods proposed in Tøndel et al. [2013]. The implementation faced some numerical issues when unstable LTI systems where used, and these issues have not yet been addressed. Hence, stable systems have been used to evaluate performance.

The numerical results for the algorithm when solving Newton steps for problems with $n_x = 15$, $n_u = 10$ and $N_s = 2$ are seen in Fig. 4. The computation times are averaged over several runs. Here, the proposed algorithm has been compared to a well known state-of-the-art serial algorithm based on the Riccati factorization from *e.g.* Axehill [2008] which is known to have $\mathcal{O}(N)$ complexity growth. From the figure, the linear complexity of the Riccati based algorithm is evident. It is not obvious from this plot that the complexity grows logarithmically for this implementation of the proposed parallel algorithm. However, it can be observed that the computational time required by the
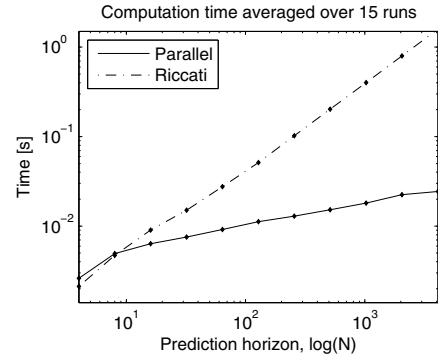

Fig. 4. Averaged computation times for the parallel solution of $\mathcal{P}(N)$. It is compared to a serial Riccati algorithm with $\mathcal{O}(N)$ complexity.

parallel algorithm is significantly less and the growth of the computational complexity is much lower.

The simulations were performed on an Intel Core i7-3517U CPU @ 1.9GHz running Windows 7 (version 6.1, build 7601: Service Pack 1) and MATLAB (8.0.0.783, R2012b).

6. CONCLUSIONS

In this paper a new algorithm for computing Newton steps for MPC problems in parallel has been presented. It has been shown that the corresponding equality constrained MPC problem can be reduced in parallel to a new problem on the same form but with shorter prediction horizon. By repeating this in several steps, a tree structure of small MPC problems with short prediction horizons is obtained and can efficiently be solved in parallel. The proposed algorithm computes the Newton step arising in MPC problems in $\mathcal{O}(\log N)$ complexity growth. In numerical experiments it has been shown that the proposed parallel algorithm outperforms an existing well known state-of-the-art serial algorithm. For future work, MPC problems with general linear constraints will be addressed, and if the stability assumption can be removed if for example a pre-stabilization technique is employed.

Appendix A. PROOFS

The original equality constrained MPC problem is given by (4), where

$$l_t = \begin{bmatrix} l_{x,t}^T & l_{u,t}^T \end{bmatrix}^T, \tag{A.1}$$

and λ_{t+1} is the dual variable corresponding to the equality constraint $x_{t+1} = A_t x_t + B_t u_t + a_t$. Then the KKT system gives the following equations for $t \in \mathbb{Z}_{0,N-1}$

$$Q_{x,t} x_t + Q_{xu,t} u_t + l_{x,t} - \lambda_t + A_t^T \lambda_{t+1} = 0 \tag{A.2}$$

$$Q_{xu,t}^T x_t + Q_{u,t} u_t + l_{u,t} + B_t^T \lambda_{t+1} = 0 \tag{A.3}$$

$$x_{t+1} = A_t x_t + B_t u_t + a_t \tag{A.4}$$

$$x_0 = \bar{x}_0, \quad Q_N x_N + l_N - \lambda_N = 0 \tag{A.5}$$

The extended problem that is composed of $p + 1$ subproblems that share the common variables is given by (5). The common variables $\hat{x}_i$ and $\hat{u}_i$ are introduced as optimization variables in the extended problem. Let the dual variables for the subproblems $i \in \mathbb{Z}_{0,p}$ be defined by (14)-(16), (27) and (28). Then the corresponding KKT system of this extended problem consists of the following equations (for all subproblems $i \in \mathbb{Z}_{0,p}$)

$$Q_{x,t,i} x_{t,i} + Q_{xu,t,i} u_{t,i} + l_{x,t,i} - \lambda_{t,i} + A_{t,i}^T \lambda_{t+1,i} = 0 \tag{A.6}$$

$$Q_{xu,t,i}^T x_{t,i} + Q_{u,t,i} u_{t,i} + l_{u,t,i} + B_{t,i}^T \lambda_{t+1,i} = 0 \tag{A.7}$$

for $t \in \mathbb{Z}_{0,N_i-1}$. For the last subproblem there is also an equation corresponding to the last term in the objective function

$$Q_{N_p,p}x_{N_p,p} + l_{N_p,p} - \lambda_{N_p,p} = 0. \quad (A.8)$$

Furthermore, the relation between the dual variables $\lambda_{N_i,i}$, $\lambda_{0,i}$, $\lambda_{tc,i}$ and $\hat{\lambda}_i$ for $i = 0,\ldots,p-1$ are given directly by the KKT system

$$\lambda_{0,p} = \hat{\lambda}_{p-1} \quad (A.9)$$
$$\lambda_{0,i} = \hat{\lambda}_{i-1} - \hat{A}_i^T(\lambda_{tc,i} + \hat{\lambda}_i), \ t \in \mathbb{Z}_{0,p-1} \quad (A.10)$$
$$\hat{B}_i^T(\lambda_{tc,i} + \hat{\lambda}_i) = 0, \ t \in \mathbb{Z}_{0,p-1} \quad (A.11)$$
$$\lambda_{N_i,i} = -\lambda_{tc,i}, \ t \in \mathbb{Z}_{0,p-1}. \quad (A.12)$$

The primal feasibility constraints that must be satisfied in the KKT system are given by the equality constraints in (5).

A.1 Proof of Theorem 1

The reduction of $\mathcal{P}(N)$ to $\mathcal{P}(p)$ with $p < N$ follows directly from the theory presented in Section 3.

The optimal primal variables in subproblem i and $i+1$ are related as $x_{0,i+1}^* = x_{N_i,i}^*$, whereas the dual variables given by (46) are related according to (33)-(35). The primal variables in the subproblems satisfy the equality constraints in (5), and hence, by using $x_{0,i+1}^* = x_{N_i,i}^*$, also the equations (A.4). By inserting (33)-(35) into (A.6) and (A.7) and using $x_{0,i+1}^* = x_{N_i,i}^*$, the resulting equations are identical to (A.2) and (A.3). Hence, the solution to the system of equations defined by (33)-(35) and (A.6)-(A.8) is a solution to the original KKT system of the problem in (4). Assumption 3 gives uniqueness of the solution and the unique optimal solution to (4) can hence be computed from X_i^* and λ_i^* for $i \in \mathbb{Z}_{0,p}$.

A.2 Proof of Lemma 2

The null space of $\mathbf{A}_i^T$ is given by all $\lambda_i^{\mathcal{N}}$ such that $\mathbf{A}_i^T \lambda_i^{\mathcal{N}} = 0$, which can be expressed as

$$-\lambda_{t,i}^{\mathcal{N}} + A_{t,i}^T \lambda_{t+1,i}^{\mathcal{N}} = 0, \ t \in \mathbb{Z}_{0,N_i-1} \quad (A.13)$$
$$B_{t,i}^T \lambda_{t+1,i}^{\mathcal{N}} = 0, \ t \in \mathbb{Z}_{0,N_i-1} \quad (A.14)$$
$$\lambda_{N_i,i}^{\mathcal{N}} = -\lambda_{tc,i}^{\mathcal{N}}. \quad (A.15)$$

Equation (A.13) and (A.15) can be combined into

$$\lambda_i^{\mathcal{N}} = \begin{bmatrix} \lambda_{0,i}^{\mathcal{N},T} & \ldots & \lambda_{tc,i}^{\mathcal{N},T} \end{bmatrix}^T = \begin{bmatrix} -\hat{A}_i & -\mathcal{D}_i & I \end{bmatrix}^T \lambda_{tc,i}^{\mathcal{N}}, \quad (A.16)$$

where $\hat{A}_i$ and $\mathcal{D}_i$ are defined as in (21). By using (A.14), $\lambda_{tc,i}^{\mathcal{N}}$ has to satisfy $\lambda_{tc,i}^{\mathcal{N}} \in \mathcal{N}(\mathcal{S}_i^T)$. For notational convenience, let $w_i = \lambda_{tc,i}^{\mathcal{N}}$ and define Z_i as

$$Z_i \triangleq \begin{bmatrix} -\hat{A}_i & -\mathcal{D}_i & I \end{bmatrix}^T. \quad (A.17)$$

Then the null space element $\lambda_i^{\mathcal{N}}$ is computed as

$$\lambda_i^{\mathcal{N}} = Z_i w_i, \quad w_i \in \mathcal{N}(\mathcal{S}_i^T). \quad (A.18)$$

A.3 Proof of Theorem 3

Consider an MPC problem (5). Then the relations between the optimal dual variables in different subproblems are directly given by (A.9)-(A.12) by writing down the KKT system.

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