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A hyperbolic view on robust control ⋆

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Abstract: The intention of the paper is to demonstrate the beauty of geometric interpretations in robust control. We emphasize Klein's approach, i.e., the view in which geometry should be defined as the study of transformations and of the objects that transformations leave unchanged, or invariant. We demonstrate through the examples of the basic control tasks that a natural framework to formulate different control problems is the world that contains as points the equivalence classes determined by the stabilizable plants and whose natural motions are the Möbius transforms. Transformations of certain hyperbolic spaces put light into the relations of the different approaches, provide a common background of robust control design techniques and suggest a unified strategy for problem solutions. Besides the educative value a merit of the presentation for control engineers might be a unified view on the robust control problems that reveals the main structure of the problem at hand and give a skeleton for the algorithmic development.

1. INTRODUCTION AND MOTIVATION

Geometry is one of the richest areas for mathematical exploration. The visual aspects of the subject make exploration and experimentation natural and intuitive. At the same time, the abstractions developed to explain geometric patterns and connections make the approach extremely powerful and applicable to a wide variety of situations.

In the nineteenth century development of the Bolyai-Lobachevsky geometry, as the first instance of non-euclidean geometries, had a great impact on the evolution of mathematical thinking. Non-Euclidean geometry has turned out to be more than just a logical curiosity, and many of its basic features continue to play important roles in several branches of mathematics and its applications.

The non-Euclidean world is something that escapes our everyday view and its rules and behaviors are only made accessible by some auxiliary tools, the so called models, that help our imagination and understanding. It happens that the main building blocks of these models has a certain relevance for systems and control theory as well, see, e.g., Helton [1980, 1982], Khargonekar and Tannenbaum [1985], Hassibi et al. [1999], Allen and Healy [2003].

Early approaches of the $\mathcal{H}_\infty$ theory were formulated in the frequency domain, and the solution of multivariate interpolation problems played the central role. The construction amounts to finding solutions to Nevanlinna-Pick interpolation problems, which is related, at a mathematical level, to the Schwarz lemma. The later is a statement on the relationship between the properties of analyticity and a hyperbolic metric on the disk. In these developments rational inner functions (finite Blaschke products) are central objects and, due to the time invariant setting, a characterization of shift invariant subspaces in terms of (rational) inner functions – a Krein space generalization of the Beuerling–Lax theorem – was the main ingredient of the approach, see, e.g., Helton [1978, 1987].

All these topics involves an advanced mathematical machinery in which often the underlying geometrical ideas remain hidden. The aim of the paper is to highlight some of these geometric governing principles that facilitate the solution of these problems. We try to avoid the technical details which can be found in the cited references.

In many of Euclid's theorems, he moves parts of figures on top of other figures. Felix Klein, in the late 1800s, developed an axiomatic basis for Euclidean geometry that started with the notion of an existing set of transformations and he proposed that geometry should be defined as the study of transformations (symmetries) and of the objects that transformations leave unchanged, or invariant. This view has come to be known as the Erlanger Program. The set of symmetries of an object has a very nice algebraic structure: they form a group. By studying this algebraic structure, we can gain deeper insight into the geometry of the figures under consideration. Another advantage of Klein's approach is that it allows us to relate different geometries. In this paper we put an emphasize on this concept of the geometry and its direct applicability to control problems.

Section 2 lists some basic features of hyperbolic geometry relevant to our presentation. We are not going to elaborate the hyperbolic geometry in details, just highlight some facts illustrated through the Poincaré disc model of this ge-

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ometry. For a more detailed elaboration of the subject, see, e.g., Gans [1973], Kelly [1981], Anderson [2005]. Section 3 presents the Klein view through the projective matrix space. The material is based on Schwarz and Zaks [1981, 1985]. The relevance of this model to control problems is detailed in Section 4.

2. ELEMENTS OF HYPERBOLIC GEOMETRY

To visualize hyperbolic geometry, we have to resort to a model. In the Poincaré model the hyperbolic plane is the unit disk, and points are Euclidean points. Lines are portions of circles intersecting the disk and meeting the boundary at right angles. The angles for the model are the same as Euclidean angles. A model with the property that angles are faithfully represented is called a conformal model. A hyperbolic circle is drawn as a Euclidean circle, but its center becomes lopsided toward the outer edge of the unit disk.

Since the reciprocal has a problem at $z = 0$, the complex plane is extended to the Riemann sphere (one-point compactification) denoted by $\hat{\mathbb{C}} = \mathbb{C} + \{\infty\}$. Given a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(M) = ad - bc = 1$ one can define a special fractional-linear transformation of the Riemann sphere, the Möbius transformation, as $\mu_M(z) = \frac{az+b}{cz+d}$. Every Möbius transformation is a bijection of the Riemann sphere and they form a group, i.e., $\mu_M \circ \mu_N = \mu_{MN}$ and $\mu_M^{-1} = \mu_{M^{-1}}$.

The geometrical meaning of the parameters a, b, c, d is not so easy to determine, even if we normalize them to $ad - bc = 1$. Define the cross ratio as

$$\varrho(p; q, r, s) = \frac{(p-q)}{(p-s)} \frac{(r-s)}{(r-q)}.$$

Then the Möbius transformation can be written in the cross ratio form $\mu_M(z) = \varrho(z; z_0, z_1, z_2)$ with

$$z_0 = -b/a, \quad z_1 = (d-b)/(a-c), \quad z_2 = -d/c.$$

Three points determine the Möbius transformation: given (z_0, z_1, z_2) defining a cross ratio, the corresponding Möbius transformation is $\mu_M(z) = m\frac{z-z_0}{z-z_2}$ with

$$M = \begin{pmatrix} m & -mz_0 \\ 1 & -z_2 \end{pmatrix}, \quad m = \frac{z_1 - z_2}{z_1 - z_0}.$$

The cross ratio is invariant under the Möbius transformation: $\varrho(z_0; z_1, z_2, z_3) = \varrho(w_0; w_1, w_2, w_3)$ if $w_i = \mu_M(z_i)$.

Möbius tranformations that map the unit disc $\mathbb{D}$ onto itself form a subgroup, the hyperbolic group. They can be written as $V_{\alpha,\theta}(z) = e^{i\theta} B_\alpha(z)$ where $\alpha \in \mathbb{D}$ and $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ is an elementary Blaschke function. Substitution reveals that $B_\alpha(\alpha) = 0$, $B_\alpha(\alpha_1) = 1$ and $B_\alpha(\alpha_\infty) = \infty$, where $\alpha_1 = \frac{1+\alpha}{1+\bar{\alpha}}$ and $\alpha_\infty = \frac{1}{\bar{\alpha}}$. Note that α_∞ is the point symmetric to α with respect to the unit circle. $|B_\alpha(z)| = 1$ on the unit circle $|z| = 1$ and $|z| < 1 \Rightarrow |B_\alpha(z)| < 1$ and $|z| > 1 \Rightarrow |B_\alpha(z)| > 1$.

Four points lie on the same circle (line) if and only if their cross ratio is a real number. Thus, if the points z_1, z_2 are inside the unit disk, and ω_0, ω_1 (omega points) are on on the unit circle, then the function $d_H(z_1, z_2) = |\log(\varrho(z_1, \omega_0, z_2, \omega_1))|$ is a candidate to measure the distance. Indeed: $\varrho(0, -1, r, 1) = \frac{1-r}{1+r}$, i.e., $d_H(0, r) = \log\frac{1+r}{1-r}$. Observe that the map $r \mapsto \frac{1+r}{1-r}$ sends the interval $[0, 1)$ to $[1, \infty)$. Since the points $0, z$ lie on a diameter, it's omega points are given by $\pm\frac{z}{|z|}$. Thus

$$d_H(0, z) = |\log(\varrho(0, -\frac{z}{|z|}, z, \frac{z}{|z|}))| = \log(\frac{1+|z|}{1-|z|}).$$

The hyperbolic translation $\frac{z-z_1}{1-\bar{z_1}z}$ takes $z_1 \mapsto 0$ and $z_2 \mapsto \frac{z_2-z_1}{1-\bar{z_1}z_2}$. Then the general formula follows as:

$$d_H(z_1, z_2) = 2\,\text{arctanh}\,|\frac{z_2 - z_1}{1 - \bar{z_1}z_2}|. \tag{1}$$

We often prefer to use the pseudo-hyperbolic distance $p_H(u, v) = |\frac{v-u}{1-\bar{u}v}|$ since it is algebraic whereas the expression for d_H is not. The main difference is that the pseudo-hyperbolic distance is not additive along geodesics.

Hyperbolic geometry has no preferred points, but in the Poincaré disk model, however, the origin has a very special role. Similarly for diameters, especially the one on the real axis. Since hyperbolic distance is based on cross ratios, and cross ratios are invariant under Möbius transformations, we can measure distances between points by moving their lines into a special position using a hyperbolic isometry. In general: any geometric property of a configuration of points which is invariant under a hyperbolic isometry, may be reliably investigated after the data has been moved into a convenient position in the model. This is the idea that will be used in the solution of basic control problems, too.

2.1 Extension to operator balls

For the Hilbert spaces $\mathcal{K}, \mathcal{H}$ consider the open unit ball $\mathcal{B}$ of all linear bounded operators from $\mathcal{K}$ to $\mathcal{H}$ ($L(\mathcal{K}, \mathcal{H})$). The map $h : \mathcal{B} \mapsto \mathcal{H}$ is holomorphic on $\mathcal{B}$ if the Fréchet derivative of h at x exists as a bounded complex linear map of $\mathcal{K}$ into $\mathcal{H}$ for each $x \in \mathcal{B}$. Invertible holomorphic maps from $\mathcal{B}$ onto $\mathcal{B}$ are called biholomorphic automorphisms.

In this setting the operator valued Blaschke functions

$$\mathfrak{B}_A(X) = N_A^{-1/2}(A + X)(1 + A^*X)^{-1}N_{A^*}^{1/2},$$

with $N_A = I - AA^*$ are biholomorphic automorphisms of $\mathcal{B}$ ($\mathfrak{B}_A^{-1} = \mathfrak{B}_{-A}$). Then $\mathfrak{B}_A(X)^* = \mathfrak{B}_{A^*}(X^*)$ and $\|\mathfrak{B}_A(X)\| \leq \mathfrak{B}_{\|A\|}(\|X\|)$. Moreover, every biholomorphic mapping h is of the form $h = \mathfrak{B}_{h(0)}(UXV) = U\mathfrak{B}_{-h^{-1}(0)}(X)V$, where U and V are unitary operators. The metric defined as

$$\rho(A, B) = \ln\frac{1 + \|\mathfrak{B}_{-A}(B)\|}{1 - \|\mathfrak{B}_{-A}(B)\|} = \text{arctanh}(\|\mathfrak{B}_{-A}(B)\|)$$

is invariant with respect to biholomorphic automorphisms and provides an extension of the Poincaré disk model of the hyperbolic geometry to the operator ball. For details see, e.g., Harris [1973], Khatskevich [1984], Ostrovskii et al. [2009].

3. THE KLEIN VIEW OF GEOMETRY

We have already described the hyperbolic geometry using the Poincaré disk model defined on $\{z \in \mathbb{C} : |z| < 1\}$ and we can introduce a half plane model on $\{z \in \mathbb{C} : \text{Im}(z) > 0\}$ by using a Möbius transformation, the Cayley map $c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by $c(z) = \dfrac{z-i}{z+i}$, to convert between the two models. Note that lines in the Poincaré disc model passing through 1 are in one-to-one correspondence with the lines that are vertical rays in the upper half plane model.

Another model is the Hyperboloid (Minkowski) model where the points lie on the upper sheet of a hiperboloid, i.e., $\mathbb{H} = \{(x, y, z) \mid z^2 = 1 + x^2 + y^2 \geq 1\}$. We can map $\mathbb{D}$ to $\mathbb{H}$ by stereographic projection from $Z' = (0, 0, -1)$ using the map given by

$$m(x, y) = (\frac{2x}{1 - x^2 - y^2}, \frac{2y}{1 - x^2 - y^2}, \frac{1 + x^2 + y^2}{1 - x^2 - y^2}),$$
$$m^{-1}(x, y, z = \sqrt{1 + x^2 + y^2}) = (\frac{x}{z}, \frac{y}{z}).$$

The lines of the model are the intersections of $\mathbb{H}$ with planes through the origin. This model is related to special relativity and it is also of considerable significance in geometry. The group for the Minkowski model is isomorphic to a subgroup of the projective group $\mathbf{P}(2)$ which shows that hyperbolic geometry is related to projective geometry.

3.1 Projective Matrix Space

Considering the ring $M_n(\mathbb{C})$ of the complex matrices and the set $M^n_{(2n,n)}(\mathbb{C})$ of the pairs $P_1, P_2 \in M_n(\mathbb{C})$ with rank $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = n$, we can define the equivalence relation: $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \cong \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ if and only if there exists $R \in GL_n(\mathbb{C})$ such that $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$. Then the equivalence classes $\underline{P}$ are considered as the points of the projective space $\mathbb{P}$. Accordingly we introduce the map $\underline{\mathrm{i}}$ from $M^n_{(2n,n)}(\mathbb{C})$ to the space $\mathbb{P}$ such that $\underline{P} = \underline{\mathrm{i}}\left\{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\right\} = \underline{\mathrm{i}}^{-1}(\underline{P})$.

$\underline{P}$ is called finite if, for any $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \in \underline{\mathrm{i}}^{-1}(\underline{P}), \quad \det(P_1) \neq 0$ and $\mathbb{P}_f$ denotes the set of finite points. While the points of $\mathbb{P}$ correspond to linear subspaces of a fixed dimension, finite points are related to graph subspaces $\begin{pmatrix} I \\ P \end{pmatrix}$ of the linear operators $P \in M_n(\mathbb{C})$.

We consider the action of nonsingular matrices on $\mathbb{P}$ (projectivity), i.e., for $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2n}(\mathbb{C})$ and $\underline{P} \in \mathbb{P}$ the map defined by $\underline{P}^S = \underline{\mathrm{i}}(P^S)$ where $P^S = S\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$. The projectivities of $\mathbb{P}$ form a group under composition.

Then the matrix Möbius transformation

$$\mathfrak{M}_S(P) = (C + DP)(A + BP)^{-1}$$

is the restriction of the projectivity S to the finite points of the space, i.e., $\mathfrak{M}_S$ operates on $\mathbb{P}_f$.

On the set of the points the Euclidean distance can be defined as $d(\underline{P}, \underline{Q}) = \|P - Q\|$. Then the projectivities that keep this distance invariant are

$$S = \lambda \begin{pmatrix} U_1 & 0 \\ U_2 P_0 & U_2 \end{pmatrix},$$

where P_0 is an arbitrary matrix and U_1, U_2 are unitary.

Using the Hermitian form defined as

$$H_e(P, r) = \begin{pmatrix} I \\ P \end{pmatrix}^* \tilde{H}_e(r) \begin{pmatrix} I \\ P \end{pmatrix}, \quad H_e(r) = \begin{pmatrix} -r^2 I & 0 \\ 0 & I \end{pmatrix},$$

for $r > 0$, we can introduce the Euclidean circle as

$$\gamma_e(\underline{0}, r) = \left\{\underline{P} \in \mathbb{P}_f : \underline{P} = \underline{\mathrm{i}}\begin{pmatrix} I \\ P \end{pmatrix}, H_e(P, r) = 0\right\}.$$

In analogy to the one dimensional case we denote by $\mathbb{D}$ the Euclidean unit disc, i.e., $\mathbb{D} = \{\underline{P} : d(\underline{O}, \underline{P}) < 1\}$ and by $\bar{\mathbb{D}}$ its closure. Note that this is exactly the contractive matrix ball in $M_n(\mathbb{C})$.

Let $\mathcal{J} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$. Then, a non–Euclidean geometry can be defined by using the $\mathcal{J}$-unitary matrices $\Pi \in M_{2n}(\mathbb{C})$, i.e., $\Pi^* \mathcal{J} \Pi = \mathcal{J}$. These matrices form a group.

Setting

$$D^- = \left\{\underline{P} \in \mathbb{P}_f : \underline{P} = \underline{\mathrm{i}}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, -P_1^* P_1 + P_2^* P_2 < 0\right\},$$

note that $D^- = \mathbb{D}$ and every $\underline{P} \in D^-$ has a representant that can be completed to a $\mathcal{J}$–unitary matrix $\tilde{P}$.

If $\underline{P}$ and $\underline{Q}$ are in D^- with $\tilde{P}$ and $\tilde{Q}$ the corresponding $\mathcal{J}$–unitary matrices we can set $\tilde{R} = \begin{pmatrix} R_1 & R_3 \\ R_2 & R_4 \end{pmatrix} = \tilde{P}^* \mathcal{J} \tilde{Q}$. The pseudo–chordal distance is defined as

$$\Psi(\underline{P}, \underline{Q}) = \frac{\varrho(\underline{P}, \underline{Q})}{[1 + \varrho^2(\underline{P}, \underline{Q})]^{1/2}},$$

where $\varrho(\underline{P}, \underline{Q}) = \|R_2\|$, which is a metric [1] on D^-.

The projectivity S maps D^- onto itself and keeps the pseudo–chordal distance invariant (non–Euclidean motions) if and only if the corresponding matrices $\tilde{S}$ are of the form $\tilde{S} = \lambda \tilde{T}$, $\tilde{T}^* \mathcal{J} \tilde{T} = \mathcal{J}$, where $\lambda \neq 0$.

In analogy to the scalar case, for any pair of points in D^-, we define the non–Euclidean distance $E_h(\underline{P}, \underline{Q})$ by

$$E_h(\underline{P}, \underline{Q}) = \frac{1}{2} \log \frac{1 + \Psi(\underline{P}, \underline{Q})}{1 - \Psi(\underline{P}, \underline{Q})}.$$

For arbitrary $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \in M^n_{2n,n}(\mathbb{C})$ we set

$$H_h(P_1, P_2, r) = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}^* \tilde{H}_h(r) \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

with $\tilde{H}_h(r) = \begin{pmatrix} -\sinh^2(r) I & 0 \\ 0 & \cosh^2(r) I \end{pmatrix}, \quad 0 < r < \infty.$

The non–Euclidean circle $\gamma_h(\underline{0}, r)$ and the corresponding non–Euclidean disk $D_h^-(\underline{0}, r)$ are defined as:

$$\gamma_h(\underline{0}, r) = \left\{\underline{P} : \underline{P} = \underline{\mathrm{i}}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, H_h(P_1, P_2, r) = 0\right\},$$

$$D_h^-(\underline{0}, r) = \left\{\underline{P} : \underline{P} = \underline{\mathrm{i}}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, H_h(P_1, P_2, r) < 0\right\}.$$

Both lie in D^-. For $r_e = \tanh(r_h), \quad 0 < r_e < 1$ we obtain $\gamma_e(\underline{0}, r_e) = \gamma_h(\underline{0}, r_h)$ and $D_e^-(\underline{0}, r_e) = D_h^-(\underline{0}, r_h)$.

[1] For $n = 1$ this is the classical pseudo–chordal distance, i.e., $\Psi(z, w) = |w - z|/|1 - \bar{z}w|$.

Let $\underline{M} \in D^-$ and M be the corresponding $\mathcal{J}$–unitary matrix. Then $S = M\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$, with U_1 and U_2 arbitrary unitary matrices, is the non–Euclidean motion that maps $\underline{0}$ into $\underline{M}$. The circle $\gamma_h(\underline{M}, r)$ with radius $0 < r < \infty$, and the disk $D_h^-(\underline{M}, r)$ are defined as maps of $\gamma_h(\underline{0}, r)$ and $\underline{0} \in D_h^-(\underline{0}, r)$ under this non–Euclidean motion.

The Hermitian diameter l_h of D^- is defined by

$$l_h = \left\{ \underline{P} : \underline{P} = \underline{\mathfrak{i}} \begin{pmatrix} P_3 \\ P_4 \end{pmatrix}, P_4^* P_3 - P_3^* P_4 = 0 \right\}.$$

The non–Euclidean straight lines are defined as the maps of l_h under non–Euclidean motions.

4. ROBUST CONTROL AND HYPERBOLIC GEOMETRY

The common tool in formulating robust feedback control problems is to use system interconnections that can be described as linear fractional transforms (LFTs), as a general framework to include the rational dependencies that occur. It is apparent that Möbius transformations are special LFTs. The set of stabilizing controllers for a given plant and the set of all suboptimal $\mathcal{H}_\infty$ can be expressed by using certain Möbius transformations. In what follows the common background of these transformations and their relations with the hyperbolic geometries will be highlighted.

Even the system P is unbounded, through its coprime factorization $P = NM^{-1}$, with M, N suitable bounded causal operators, the associated graph $\mathcal{P} = \Pi\xi = \begin{pmatrix} M \\ N \end{pmatrix}\xi$ is formulated in terms of bounded with $\xi \in \mathcal{H}_p$ (a suitable Hilbert space) operators. We are not very restrictive if it is assumed that there exists a double coprime factorization, i.e., $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and causal bounded $U, V, \tilde{U}$ and $\tilde{V}$ such that

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{2}$$

an assumption which is often made when setting the stabilization problem, see, e.g., Vidyasagar [1985]. Recall that $\begin{pmatrix} M \\ N \end{pmatrix}$ and $\begin{pmatrix} U \\ V \end{pmatrix}$ are determined only up to outer, i.e., stable with stable inverse, factors S and S'.

It is not hard to make the connection with the construction presented in the previous section: plants P, represented by $\underline{P} = \{\begin{pmatrix} M \\ N \end{pmatrix}\}$, are the finite points $\underline{\mathbb{P}}_f$ while the controllers $K = UV^{-1}$ in a feedback connection are described by the inverse relation (inverse graph), i.e., $\overline{K} = \{\begin{pmatrix} U \\ V \end{pmatrix}\}$. Well-posedness (feedback stability) of the pair (P, K) means that the matrix $\Xi_P = \begin{pmatrix} M & U \\ N & V \end{pmatrix}$ has a bounded causal inverse. Stable plants Q form a subset of the finite points, i.e., those for which $\begin{pmatrix} I \\ Q \end{pmatrix} \in \underline{\mathfrak{i}}^{-1}(\underline{Q})$. It is obvious that the zero plant is stabilized by the entire stable set, and only by that set. Since the image of $\underline{0}$ under the projectivity defined by Ξ_P is the plant P, it follows that the stabilizing controllers are described by by $\Pi_P\begin{pmatrix} Q \\ I \end{pmatrix} \sim \begin{pmatrix} K \\ I \end{pmatrix}$, i.e..

$$\mathcal{K}_{stab} = \{K \mid K = (U + MQ)(V + NQ)^{-1}, Q \text{ stable}\}$$

which is the well-known Youla parametrization

To emphasize this point we reformulate the standard stability result as follows:

Proposition 4.1. The plant P has a double coprime factorization if and only if there is a projectivity defined by an outer Ξ with Ξ_{11} invertible such that $\mathfrak{M}_\Xi(P) = 0$. Then all stabilizing controllers are given by $K = \mathfrak{M}_{L^*\Xi^{-1}L}(Q)$, Q stable, where $L = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

The proof is straightforward and it is omitted for brevity.

4.1 Linear relations and LFTs

If P is partitioned as $P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix}$ then, provided that the corresponding inverses exists, a lower and an upper LFT is defined as $\mathfrak{F}_l(P, K) = P_{zw} + P_{zu}K(I - P_{yu}K)^{-1}P_{yw}$ and $\mathfrak{F}_u(P, \Delta) = P_{yu} + P_{yw}\Delta(I - P_{zw}\Delta)^{-1}P_{zu}$. P is called the coefficient matrix of the LFT.

Möbius transformations $Z' = \mathfrak{M}_\Sigma(Z)$ relate two graph subspaces, $\mathcal{G}_Z$ and $\mathcal{G}_{Z'}$, through the projectivity Xi, i.e., $\mathcal{G}_{Z'} = \Xi\mathcal{G}_Z$ and inherit the group structure of the linear operators, i.e., $\mathfrak{M}_{\Xi_2} \circ \mathfrak{M}_{\Xi_1} = \mathfrak{M}_{\Xi_2\Xi_1}$.

It turns out that LFTs can be obtained in the same way as the Möbius transformations, by performing some interchange in the signal spaces and by considering linear relations instead of the linear operators, Shmulyan [1976, 1980].

If X and Y are two sets, a relation $T \subset X \times Y$ is defined as a set of pairs $(x, y) \in T$, where $x \in X, y \in Y$. If X and Y are linear spaces ($X \oplus Y = X \times Y$) a linear relation T is a linear subspace of $X \oplus Y$. If $x \in dom(T)$ then $T(x) = \{y \in Y : (x, y) \in T\}$ and correspondingly if $y \in ran(T)$, then $T^{-1}(y) = \{x \in X : (x, y) \in T\}$.

Let $T \subset X \times Y$ and $R \subset Y \times Z$ be linear relations. Then the product $RT \subset X \times Z$ is the linear relation defined by

$$RT = \{\{x, z\} \in X \times Z : \{x, y\} \in T, \{y, z\} \in R\}.$$

An operator $P : X \mapsto Y$ is equivalent to a special relation, the graph subspace $\mathcal{G}_P$. For details see, e.g., Arens [1961].

With $\mathcal{X} = \mathcal{X}_w \oplus \mathcal{X}_u$ and $\mathcal{Y} = \mathcal{Y}_z \oplus \mathcal{Y}_y$ consider $\mathcal{L} = \mathcal{X} \oplus \mathcal{Y}$ and $\tilde{\mathcal{L}} = (\mathcal{Y}_y \oplus \mathcal{X}_u) \oplus (\mathcal{X}_w \oplus \mathcal{Y}_z)$. Observe that we have $\tilde{\mathcal{L}} = \Pi_l\mathcal{L}$ with a permutation matrix Π_l. Thus every linear operator $P : \mathcal{X} \mapsto \mathcal{Y}$ induces a relation $\mathcal{R}_P \subset \tilde{\mathcal{L}}$ through its graph subspace, i.e.,

$$\mathcal{R}_P = \Pi_l\mathcal{G}_P \sim \left(\begin{array}{cc} P_{yw} & P_{yu} \\ 0 & I_u \\ \hline I_w & 0 \\ P_{zw} & P_{zu} \end{array}\right), \tag{3}$$

called scattering transformation. It turns out that evaluating this relation on the graph subspaces $\mathcal{G}_K$, i.e., on the linear operators $K : \mathcal{Y}_y \mapsto \mathcal{X}_u$, we obtain a graph subspace $\mathcal{G}_F = \mathcal{R}_T\mathcal{G}_K$ that corresponds to the linear operator

$F : \mathcal{X}_y \mapsto \mathcal{Y}_z$, provided that $(I - P_{yu}K)$ is boundedly invertible. This map is exactly the lower LFT $F = \mathfrak{F}_l(P, K)$. Analogously, by considering another permutation Π_r, one can obtain the expression of the upper LFT.

This construction extends the linearization trick already encountered for the Möbius transforms to the LFTs: on the level of equivalence classes, the map is linear while on the level of the representants the map is rational (Möbius, LFT). Moreover, the group structure on the representants is also present, however, the familiar matrix product should be changed to the more complex Redheffer (star) product, see, e.g., Zhou and Doyle [1999], that reflects the composition of the relations.

Note, that in this construction Π_l is a projectivity that sends a finite point $\underline{P}$ to a possible not finite $\underline{R}$, corresponding to the subspace $\mathcal{R}_P$. If the image is also a finite point, then the representant can be obtained by the Möbius transform $\hat{P} = \mathfrak{M}_{\Pi_l}(P)$ which is the Potapov-Ginsbourg transformation. P is in the domain of this transformation if $\begin{pmatrix} P_{yw} & P_{yu} \\ 0 & I_u \end{pmatrix}$ is invertible.

The relation $\mathfrak{F}_l(P, K) = \mathfrak{M}_{\hat{P}}(K)$, if it exists, between an LFT and a Möbius transformations has the advantage to use a more accessible operation (matrix product) instead of the star product for the factorizations that possibly simplify a given problem. This fact was widely exploited in the solution of the robust control problems, see, e.g., the factorization approach of Ball et al. [1991] or in the so called chain scattering-approach of Kimura [1997].

We conclude this section with an example that shows how suitable projectivities can reveal the basic structure of a control problem. We are to apply Proposition 4.1 for the stabilization problem related to LFTs, i.e., to find all internally stabilizing controllers K that makes $\mathfrak{F}_l(P, K)$ bounded. Even this is a known result, its geometric interpretation remains hidden. The novelty here is to show the role of the projectivity in the solution of the problem.

Proposition 4.2. The stabilization problem is solvable if and only if there is an outer Ξ with Ξ_{11} invertible such that $\mathfrak{M}_\Xi(P) = S$ is stable and $\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} = \mathfrak{M}_{L^*\Xi^{-1}L}(\begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix})$, Q stable. Then S can be chosen to be a model matching type, i.e., $S = \begin{pmatrix} S_{zw} & S_{zu} \\ S_{yw} & 0 \end{pmatrix}$.

The proof is by construction: under the stabilizability assumption, i.e., there is a stabilizing controller $\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$ for P, the double coprime factorization, with $K = \tilde{k}_{uu}^{-1}\tilde{k}_{uy} = k_{uy}k_{yy}^{-1}$, has the special structure

$$\left[\begin{array}{cc|cc} I_w & 0 & 0 & 0 \\ 0 & \tilde{k}_{uu} & 0 & -\tilde{k}_{uy} \\ \hline -n_{zw} & -\tilde{n}_{zu} & I_z & \tilde{n}_{zu}K \\ -\tilde{n}_{yw} & -\tilde{n}_{yu} & 0 & \tilde{m}_{yy} \end{array}\right] \left[\begin{array}{cc|cc} I_w & 0 & 0 & 0 \\ Kn_{yw} & m_{uu} & 0 & k_{uy} \\ \hline n_{zw} & n_{zu} & I_z & 0 \\ n_{yw} & n_{yu} & 0 & k_{yy} \end{array}\right] = I.$$

Compare this general fact with Green [1992]. Thus the projectivity $\Xi_s = \left[\begin{array}{cc|cc} I_w & 0 & 0 & 0 \\ 0 & \tilde{k}_{uu} & 0 & -\tilde{k}_{uy} \\ \hline 0 & 0 & I_z & 0 \\ 0 & -\tilde{n}_{yu} & 0 & \tilde{m}_{yy} \end{array}\right]$ has the property required by the proposition, with $\mathfrak{M}_{\Xi_s}(P) = \begin{pmatrix} n_{zw} & n_{zu} \\ \tilde{n}_{yw} & 0 \end{pmatrix}$. Actually one has $\mathfrak{F}_l(P, K) = n_{zw} + n_{zu}q\tilde{n}_{yw}$, where q is the Youla parameter of K. The details are simple algebraic computations and are left out. Note that P is either stabilized by any of the stabilizing controllers of the inner loop P_{yu} or it is not stabilizable at all.

4.2 The $\mathcal{H}_\infty$ problem

In this section we consider the suboptimal normalized $\mathcal{H}_\infty$ problem, i.e., we seek all controllers K that internally stabilize the loop and achieve $\|\mathfrak{F}_l(P, K)\| < 1$. While the conclusions of the section remain valid for much larger classes, including infinite dimensional LTI or LTV systems, we consider here only the finite dimensional LTI case, i.e., systems having a state space description.

We have already seen that it is enough to consider stable generalized plants of type $P = \begin{pmatrix} n_{zw} & n_{zu} \\ \tilde{n}_{yw} & 0 \end{pmatrix}$. There are basically two type of strategies, relevant for this paper, to solve the problem. Both methods assume either left or right invertibility of P. The first uses the scattering approach by augmenting the plant, if necessary, to obtain a well defined Potapov-Ginsburg transform $\hat{P}$, Ball et al. [1991], Kimura [1997]. Then a J-inner outer factorization $\hat{P} = \hat{\Theta}_a\hat{R}$, with a block tridiagonal structure of the outer factor that corresponds to the structure of the augmentation, solves the problem. The controllers are given by $\mathfrak{M}_{\hat{R}^{-1}}(H_a)$ with

$$H_a = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}, \quad \|H\| < 1,$$

while the closed loop is given by $\mathfrak{M}_{\hat{\Theta}_a}(H_a)$. Recall that Θ_a is an inner function, thus

$$\mathfrak{F}_l(P, K) = \mathfrak{M}_{\hat{\Theta}_a}(H_a) = \mathfrak{F}_l(\Theta_a, H_a) < 1. \tag{4}$$

For the details on J-inner and J-lossless functions see Dym [1989] and Kimura [1997].

The second approach, see Green et al. [1990], Green [1992], instead of augmentation uses two J-spectral factorizations. We briefly sketch the geometric content of the method: let us start from the scattering form, i.e.,

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \tilde{n}_{yw} & 0 \\ 0 & I_u \end{pmatrix} \xi = \hat{P}_{yu}\xi, \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} I_w & 0 \\ n_{zw} & n_{zu} \end{pmatrix} \xi = \hat{P}_{wz}\xi,$$

under the constraints $z = \mathfrak{F}_l(P, q)w = F_q w$ and $u = qy$.

We generically use the notation J_{pq} for $\begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$. Then, if we have the J-spectral factorizations

$$\hat{P}_{wz}^* J_{wz}\hat{P}_{wz} = V_r^* J_{wu}V_r, \quad \hat{P}_{yu}V_r^{-1}J_{wu}(\star)^* = W_r J_{yu}W_r^*,$$

with V_r, W_r outer, i.e., $\Phi_r = \hat{P}_{wz}V_r^{-1}$ is J-lossless and $\Psi_r = W^{-1}\hat{P}_{yu}V_r^{-1}$ is co-J-lossless, respectively. The first observation is that from $(\star)J_{wz}\Phi_r\tilde{\xi} = (\star)J_{wu}\tilde{\xi} < 0$ with $\tilde{\xi} = V\xi$ and using $(-q \ I_u)\,\hat{P}_{yu}\xi = (-q\tilde{n}_{yw} \ I_u)\,\xi = 0$ one has $\tilde{\xi}^\perp J_{wu}(\star) > 0$, i.e., $(-q \ I_u)\,\hat{P}_{yu}V_r^{-1}J_{wu}(\star) > 0$. Then we have that $(-q \ I_u)\,W_r J_{yu}(\star) > 0$ which is equivalent to $(\star)J_{yu}W_r^{-1}\begin{pmatrix} I_y \\ q \end{pmatrix} < 0$, which provides the desired controllers. The notation $(\star)$ stands for the adjoint of the corresponding left (right) hand side of the J-product. It turns out that for the considered class existence of the desired factorization is actually equivalent to the solvability of the problem.

In the original presentation an important part of the result is missing, namely that the closed loop is provided by an inner function Θ, whose scattered representation is $\begin{pmatrix} \Psi_r \\ \Phi_r \end{pmatrix}$. The existence of this function follows from the fact that $(\star)J_{wz}\Phi_r\tilde{\xi} < 0 \Leftrightarrow (\star)J_{yu}\Psi_r\tilde{\xi} < 0$ implies equality of the two indefinite quadratic form, see Bognár [1974], Shmulyan [1978]. It follows that $\|w\|^2 - \|z\|^2 = \|y'\|^2 - \|u'\|^2$, i.e., $\|w\|^2 + \|u'\|^2 = \|z\|^2 + \|y'\|^2$.

Thus the projectivity $\Xi_h = \Pi_l \mathrm{diag}(W_r^{-1}, I)$ maps the original plant P to Θ, moreover $\mathfrak{F}_l(P, q) = \mathfrak{F}_l(\Theta, h)$, with $q = \mathfrak{M}_{W_r}(h)$ on $\|h\| < 1$. Thus, we can continue the series of prototype configurations related to significant control problems that can be achieved with suitable projectivities:

Proposition 4.3. The suboptimal $\mathcal{H}_\infty$ problem is solvable if and only if there is a projectivity Ξ_h such that $\mathfrak{M}_{\Xi_h}(P) = \Theta$ is an inner function.

To conclude this section we return to formula (4). Since the closed loop system is parametrized naturally by contractions H_a through a hyperbolic motion it is natural to consider a hyperbolic distance, introduced as in subsection 2.1, that relates directly controllers with the closed loop plant.

5. CONCLUSIONS

The paper emphasizes Klein's approach to geometry and demonstrates that a natural framework to formulate different control problems is the world that contains as points equivalence classes determined by stabilizable plants and whose natural motions are the Möbius transforms. The fact that any geometric property of a configuration, which is invariant under a hyperbolic motion, may be reliably investigated after the data has been moved into a convenient position in the model, facilitates considerably the solution of the problems. This method provides a common background of robust control design techniques and suggests a unified strategy for problem solutions.

Besides the educative value a merit of the presentation for control engineers might be a unified view on the robust control problems that reveals the main structure of the problem at hand and give a skeleton for the algorithmic development.

REFERENCES

Allen, J. C. and Healy, D. M. (2003). Hyperbolic geometry, Nehari's theorem, electric circuits, and analog signal processing. *Modern Signal Processing*, 46:1–62.

Anderson, J. (2005). *Hyperbolic geometry.* Springer-Verlag, London.

Arens, R. (1961). Operational calculus of linear relations. *Pacific Journal of Mathematics*, 11(1):9–23.

Ball, J. A., Helton, J. W., and Verma, M. (1991). A factorization principle for stabilization of linear control systems. *Journal of Nonlinear and Robust Control*, 1:229–294.

Bognár, J. (1974). *Indefinite inner product spaces.* Springer–Verlag, Berlin.

Dym, H. (1989). *J–contractive matrix functions, reproducing kernel Hilbert spaces and interpolation.* AMS, Providence, R.I.

Gans, D. (1973). *An Introduction to Non-Euclidean Geometry.* Academic Press.

Green, G. M., Glover, K., Limebeer, D. J. N., and Doyle, J. C. (1990) A J-spectral factorization approach to $\mathcal{H}_\infty$ control. *SIAM J. Control and Opt.*, 28(3):1350-1371.

Green, M. (1992). $\mathcal{H}_\infty$ controller synthesis by J-lossless coprime factorization. *SIAM J. Control and Opt.*, 30(3):522-547.

Harris, L. A. (1973). *Bounded symmetric homogeneous domains in infinite dimensional spaces*, in Lecture Notes in Mathematics No. 364. Springer- Verlag, New York.

Hassibi, B., Sayed, A., and Kailath, T. (1999). *Indefinite Quadratic Estimation and Control: A Unified Approach to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Theories.* SIAM Studies in Applied Mathematics.

Helton, J. W. (1978). Orbit structure of the Möbius transformation semigroup action on $\mathcal{H}_\infty$ (broadband matching). *Adv. in Math. Suppl. Stud.*, 3:129–197.

Helton, J. W. (1980). The distance from a function to $\mathcal{H}_\infty$ in the Poincaré metric; electrical power transfer. *J. Functional Analysis*, 38(2):273–314.

Helton, J. W. (1982). Non–Euclidean functional analysis and electronics. *Bull. AMS*, 7(1):1–64.

Helton, J. W. (1987). *Operator theory, analytic functions, matrices and electrical engineering*, volume 68 of *CBMS Lecture Notes.* Amer. Math. Soc., Providence, RI.

Kelly, P. (1981). *The Non-Euclidean Hyperbolic Plane.* Springer-Verlag, New York.

Khargonekar, P. and Tannenbaum, A. (1985). Non–Euclidian metrics and the robust stabilization of systems with parameter uncertainty. *IEEE Trans. on Automatic Control*, 30(10):1005–1013.

Khatskevich, V. A. (1984). Generalized Poincaré metric on the operator ball. *Mathematical notes of the Academy of Sciences of the USSR*, 17(4):321–323.

Kimura, H. (1997). *Chain-scattering approach to $\mathcal{H}_\infty$ control.* Birkhauser.

Ostrovskii, M. I., Shulman, V. S., and Turowska, L. (2009). Unitarizable representations and fixed points of groups of biholomorphic transformations of operator balls. *Journal of Functional Analysis*, 257(8):2476–2496.

Schwarz, B. and Zaks, A. (1981). Matrix Möbius transformations. *Communications in Algebra*, 9(19):1913-1968.

Schwarz, B. and Zaks, A. (1985). Geometries of the projective matrix space. *Journal of Algebra*, 95(1):263-307.

Shmulyan, Y. L. (1976). The theory of linear relations and spaces with an indefinite metric. *Funkts. Anal. Prilozh*, 10(1):67–72.

Shmulyan, Y. L. (1978). General linear-fractional transformations of operator balls. *Siberian Mathematical Journal*, 19(2):293-298.

Shmulyan, Y. L. (1980). Transformers of linear relations in J-spaces. *Funkts. Anal. Prilozh.*, 14(2):39–43.

Soumelidis, A., Bokor, J., and Schipp, F. (2011). On hyperbolic wavelets. In *18th World Congress of the International Federation of Automatic Control, Milano*, pages 2309–2314.

Soumelidis, A., Schipp, F., and Bokor, J. (2011). Pole structure estimation from Laguerre representations using hyperbolic metrics on the unit disc. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, CDC-ECC 2011, Orlando, FL, USA*, pages 2136–2141.

Vidyasagar, M. (1985). *Control System Synthesis: A Factorization Approach.* MIT Press, Cambridge.

Zhou, K. and Doyle, J. C. (1999). *Essentials of Robust Control.* Prentice Hall.