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Interpolation of polytopic control Lyapunov functions for discrete–time linear systems

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Abstract: This paper proposes a method for interpolating two (or more) polytopic control Lyapunov functions (CLFs) for discrete–time linear systems subject to polytopic constraints, thereby combining different control objectives. The corresponding interpolated CLF is used for synthesis of a stabilizing controller by on–line optimization. Recursive feasibility and computational efficiency of the resulting optimization problem are established under suitable assumptions. The proposed method offers a smooth transient between different polytopic CLFs, while guaranteeing stability and constraints satisfaction. Moreover, it can also be designed such that it automatically yields a desired, optimal control law, within a region around the equilibrium. An illustrative example is presented to demonstrate the effectiveness of the developed synthesis method.

1. INTRODUCTION

Given a discrete–time linear system with state and input constraints, there exist controller design methods to achieve specific goals. However, most of the time it is difficult to achieve multiple goals with one controller. For example, a controller which is optimal in terms of frequency domain response or time domain response is normally quite aggressive. Therefore, in the presence of constraints, such an optimal controller can only be applied for a small set of states around the desired equilibrium, i.e., for the corresponding maximal admissible invariant set. On the other hand, a controller that has a large admissible set usually results in a slower time domain response and does not necessarily meet the specifications at the equilibrium. Therefore, there is a need for a solution that can combine different controllers or, even better, different controller synthesis objectives.

A possible solution is to use different design objectives to obtain suitable controllers for different goals, and then combine or merge the pre–designed controllers into a single controller. A simple way to combine the pre–designed controllers is to switch between different controllers, resulting in a piecewise affine (PWA) controllers defined over polytopic state–space partitions. A PWA controller can be synthesized using explicit model predictive control (MPC) [Bemporad et al., 2002], [Tøndel et al., 2003], overlapping contractive sets [Blanchini et al., 2008], [Spinu et al., 2011] or hybrid polytopic partitions [Spinu and Lazar, 2012]. However, switching between different controllers can lead to discontinuous control laws, which might result in noise and distortion. Therefore, it is of interest to reduce complexity and to guarantee a smooth transient of the control law from one region to another.

Interpolation techniques can be used to combine pre–designed, different controllers in a smooth way. In [Gutman and Cwikel, 1986], a controller valid within a polyhedral contractive set is obtained by interpolating between the pre–designed control inputs at the vertices and at the origin. More recently, this approach was generalized to interpolation between controllers defined for the boundary of two different polytopic invariant sets. In this approach [Nguyen et al., 2011b,a, 2013], the interpolation coefficients are optimized such that the resulting interpolation–based controller is as close as possible to the (optimal) controller that corresponds to the inner polyhedral invariant set. Interpolation techniques for pre–designed controllers can also be found in [Pluymers et al., 2005], where the controller is obtained as a convex combination of multiple controllers that are valid within different admissible sets. In common, these solutions interpolate between some existing, pre–designed controllers. Therefore, performance and robustness of the corresponding interpolated controllers is inherited from the pre–designed controllers used for interpolation, which may be a limiting factor.

The limitation of interpolating pre–designed controllers can be circumvented by combining different controller synthesis objectives at the design level. This leads to the idea of combining different control Lyapunov functions (CLFs) which are associated with different control objectives. [Andrieu and Prieur, 2010] proposed a method for uniting two CLFs for continuous–time dynamical systems. Later, [Grammatico et al., 2013, 2014] proposed a merging procedure between two CLFs using R–composition, which was introduced in [Balestrino et al., 2012]. A controller can then be synthesized using the resulting, *merged* CLF. This method is able to offer a smooth transient between different CLFs using gradient–type merging, if a *control–sharing* property holds for the different CLFs.

In this paper we also take a CLF approach for dealing with different controller synthesis objectives, while focusing on the setting of discrete–time dynamical systems. More specifically, we consider discrete–time linear systems subject to polytopic state and input constraints. Further more, we assume that two polyhedral CLFs, associated with different design objectives, are available. Then, we provide an analytic characterization of a corresponding interpolated function and we show how to use it as a parameterized CLF. Stabilizing controller synthesis method based on the parameterized CLF is formulated as an optimization problem. Recursive feasibility of this optimization problem is proven and, under an additional property called contractivity–sharing, it is shown that the problem can be formulated as a single linear program (LP) or quadratic program (QP). An illustrative example is used to demonstrate the effectiveness of the proposed synthesis method.

2. PRELIMINARIES

2.1 Notation

Let $\mathbb{R}$ and $\mathbb{Z}$ denote the set of real numbers and the set of integer numbers, respectively. The notation $\mathbb{Z}_{(c_1,c_2]}$ denotes the set $\mathbb{S} = \{k \in \mathbb{Z} | c_1 < k \leq c_2\}$. Let $\mathbb{R}^{n\times m}$ denote the set of real $n \times m$ matrices. For a matrix $Z \in \mathbb{R}^{n\times m}$, $[Z]_{ij} \in \mathbb{R}$ denotes the element on the i-th row and the j-th column of Z, $[Z]_{i\bullet} \in \mathbb{R}^{1\times m}$ denotes the i-th row of Z and $[Z]_{\bullet j} \in \mathbb{R}^{n\times 1}$ denotes the j-th column of Z. Let 1_n denote a vector in $\mathbb{R}^{n\times 1}$ with $[1_n]_i = 1$ for all $i \in \mathbb{Z}_{[1,n]}$. For a vector $x \in \mathbb{R}^n$, let $\|x\|_p$ denote the p-norm of x, and let $\|x\|$ denote an arbitrary norm of x. For a set $\mathbb{S}$, let $\partial\mathbb{S}$ denote its boundary and $int\mathbb{S}$ denote its interior. For a $\lambda \in \mathbb{R}$ and a set $\mathbb{S} \in \mathbb{R}^n$, let $\lambda\mathbb{S} := \{z | \exists x \in \mathbb{S} : z = \lambda x\}$.

2.2 Control Lyapunov functions and gauge functions

Consider a linear discrete-time system:

$$x(k+1) = Ax(k) + Bu(k),\ k \in \mathbb{Z}_+. \tag{1}$$

The domain of x is $\mathbb{X} = \{x \in \mathbb{R}^n | H_{\mathbb{X}} x \leq 1_{h_{\mathbb{X}}}\}$ and the domain of u is $\mathbb{U} = \{u \in \mathbb{R}^m | H_{\mathbb{U}} u \leq 1_{h_{\mathbb{U}}}\}$, where $h_{\mathbb{X}}$ and $h_{\mathbb{U}}$ are the numbers of hyperplanes of the polytopes $\mathbb{X}$ and $\mathbb{U}$, respectively.

Definition 2.1. A positive definite function $V(x)$ is a control Lyapunov function (CLF) for system (1) in a set $\mathbb{S}$ if for all $x \in \mathbb{S}$, there exists a control law $u \in \mathbb{U}$ such that $Ax + Bu \in \mathbb{S}$ and $V(Ax + Bu) \leq \rho V(x)$, for some $0 \leq \rho < 1$.

Definition 2.2. [Blanchini and Miani, 2008] A proper C-set is a convex and compact subset of $\mathbb{R}^n$ including the origin as an interior point.

Definition 2.3. [Blanchini and Miani, 2008] Given a proper C-set $\mathbb{S}$, its gauge function $\Psi_{\mathbb{S}}(x) : \mathbb{R}^n \to \mathbb{R}_+$ is defined as $\Psi_{\mathbb{S}}(x) = inf\{\lambda \geq 0 : x \in \lambda\mathbb{S}\}$.

Lemma 2.4. *Given two proper C-sets $\mathbb{S}_1$ and $\mathbb{S}_2$, the following statements are equivalent:*

(i) $\mathbb{S}_1 \subseteq \mathbb{S}_2$.
(ii) $\Psi_{\mathbb{S}_1}(x) \geq \Psi_{\mathbb{S}_2}(x)$ *for all* $x \in \mathbb{R}^n$.

For a proof of Lemma 2.4, the interested reader is referred to [Lazar et al., 2013] and the references therein.

2.3 Constrained control λ-contractive sets

Definition 2.5. A proper C-set $\mathbb{S} \subseteq \mathbb{X}$ is constrained control λ-contractive for system (1) if and only if for all $x \in \mathbb{S}$, there exists a $u \in \mathbb{U}$ such that $Ax + Bu \in \lambda\mathbb{S}$, where $0 \leq \lambda \leq 1$. If $\lambda = 1$ then the set is constrained control invariant for system (1).

Proposition 2.6. *If a proper C-set $\mathbb{S}$ is constrained control λ-contractive for system* (1) *then $\mu\mathbb{S}$ is also constrained control λ-contractive for system* (1) *for all* $0 \leq \mu \leq 1$.

Proposition 2.7. *Let $\mathbb{S}$ be a constrained control λ-contractive proper C-set for system* (1) *and let $\Psi_{\mathbb{S}}(x)$ denote the gauge function of $\mathbb{S}$. Then $\Psi_{\mathbb{S}}(x)$ is a set–induced CLF for system* (1).

Definition 2.8. Define Ω_λ as the set of all constrained control λ-contractive sets for system (1). A set $\tilde{\mathbb{P}}$ is said to be the maximal constrained control λ-contractive set for system (1) if $\tilde{\mathbb{P}} \in \Omega_\lambda$ and $\mathbb{S} \subseteq \tilde{\mathbb{P}}$ for all $\mathbb{S} \in \Omega_\lambda$.

2.4 Admissible constrained control λ-contractive sets

The only difference with respect to the previous definitions is that now a pre–designed controller is applied to the system. Therefore, admissibility of the pre–designed controller is now introduced.

Definition 2.9. A proper C-set $\mathbb{S} \subseteq \mathbb{X}$ is admissible constrained control λ-contractive for system (1) in closed–loop with a controller $u = u_{\mathbb{S}}(x)$ if and only if for all $x \in \mathbb{S}$ it holds that $u_{\mathbb{S}}(x) \in \mathbb{U}$ and $Ax + Bu_{\mathbb{S}}(x) \in \lambda\mathbb{S}$. If $\lambda = 1$ then the set is admissible constrained control invariant for system (1) in closed loop with controller $u = u_{\mathbb{S}}(x)$.

Definition 2.10. Define Γ_λ as the set of all admissible constrained control λ-contractive sets for system (1) with a given stabilizing controller $u = u_{\mathbb{Q}}(x)$. A set $\tilde{\mathbb{Q}}$ is said to be the maximal admissible constrained control λ-contractive set for system (1) with a feedback controller $u = u_{\mathbb{Q}}(x)$ if $\tilde{\mathbb{Q}} \in \Gamma_\lambda$ and $\mathbb{S} \subseteq \tilde{\mathbb{Q}}$ for all $\mathbb{S} \in \Gamma_\lambda$.

3. PROBLEM STATEMENT

Assume that we have two different control objectives for system (1). One is to minimize a predefined cost function $J(x,u)$ and the other is to maximize the region of attraction for the resulting closed–loop system. Two polytopic sets $\mathbb{Q}$ and $\mathbb{P}$, which correspond to two set–induced CLFs are designed to achieve the objectives separately. It should be noted that these two control objectives are just for the purpose of demonstration. The proposed method works for any two polytopic CLFs in general.

For the first objective, let $u_{\mathbb{Q}}(x) : \mathbb{R}^n \to \mathbb{R}^m$ be a stabilizing controller that minimizes a predefined cost function $J(x,u)$. Let

$$\mathbb{Q} = \{x \in \mathbb{X} | H_{\mathbb{Q}} x \leq 1_{h_{\mathbb{Q}}}\} \tag{2}$$

be a polytopic approximation of the maximal admissible constrained control $\lambda_{\mathbb{Q}}$-contractive set for system (1) in closed loop with controller $u_{\mathbb{Q}}(x)$, where $0 \leq \lambda_{\mathbb{Q}} \leq 1$. The set $\mathbb{Q}$ is normally small and cannot cover all the possible operational states. There is a need for another controller that is admissible outside the set $\mathbb{Q}$.

For the second objective, in order to maximize the region of attraction of the resulting closed–loop system, the set $\mathbb{P}$ is chosen as a polytopic approximation of the maximal constrained control $\lambda_{\mathbb{P}}$-contractive set for system (1), where $0 \leq \lambda_{\mathbb{P}} \leq \lambda_{\mathbb{Q}}$. It follows that $\mathbb{Q} \subseteq \mathbb{P}$. The set $\mathbb{P}$ can be represented as

$$\mathbb{P} = \{x \in \mathbb{X} | H_{\mathbb{P}} x \leq 1_{h_{\mathbb{P}}}\}. \tag{3}$$

Due to the constrained control contractive property of $\mathbb{P}$, there exists a stabilizing control law $u_{\mathbb{P}}(x) : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $x \in \mathbb{P}$, it holds that $u_{\mathbb{P}}(x) \in \mathbb{U}$ and $Ax + Bu_{\mathbb{P}}(x) \in \lambda\Psi_{\mathbb{P}}(x)\mathbb{P}$.

The two different CLFs can be used separately for controller synthesis, which yields the different controllers $u_{\mathbb{P}}(x)$ and $u_{\mathbb{Q}}(x)$. These controllers can then be combined by a switching rule, or by interpolation between the controllers of the two sets [Nguyen et al., 2011b, 2013]. As explained in the introduction, the performance of the resulting controllers of both methods is limited by the performance of the existing controllers.

Different from the above–mentioned approaches, in this paper we aim to first merge the two designed CLFs $\Psi_{\mathbb{Q}}(x)$ and $\Psi_{\mathbb{P}}(x)$ in a smooth way, *a priori* to controller synthesis. The controller is then synthesized from the resulting interpolated CLF.

4. INTERPOLATION OF CLFS

In this section, we propose a method to synthesize a controller from a changing–shape CLF, which is the interpolation of the original CLFs $\Psi_{\mathbb{P}}(x)$ and $\Psi_{\mathbb{Q}}(x)$. The method is called interpolation of CLFs (iCLFs).

4.1 Interpolation–based sets

As each candidate control Lyapunov function has its corresponding family of sublevel sets, interpolation of CLFs can be characterized *via* the interpolation of the corresponding sublevel sets, as follows. First, define the interpolation–based set

$$\Theta(\alpha) := \{x \in \mathbb{P} | \alpha\Psi_{\mathbb{P}}(x) + (1-\alpha)\Psi_{\mathbb{Q}}(x) \leq 1\}, \quad (4)$$

where $\alpha \in \mathbb{R}_{[0,1]}$.

To construct an interpolation–based set, we first divide the set $\mathbb{P}$ into a conic partition. Define a conic region:

$$\mathbb{C}_{(p,q)} := \{x \in \mathbb{P} | [H_{\mathbb{P}}]_{p\bullet}x - [H_{\mathbb{P}}]_{r\bullet}x \geq 0 \, \forall r \in \mathbb{Z}_{[1,h_{\mathbb{P}}]}, \; [H_{\mathbb{Q}}]_{q\bullet}x - [H_{\mathbb{Q}}]_{s\bullet}x \geq 0 \, \forall s \in \mathbb{Z}_{[1,h_{\mathbb{Q}}]}\}. \quad (5)$$

Define the following set of pairs of indexes:

$$I := \{(p,q) \in \mathbb{Z}_{[1,h_{\mathbb{P}}]} \times \mathbb{Z}_{[1,h_{\mathbb{Q}}]} | \mathbb{C}_{(p,q)} \neq \{0_n\}\}. \quad (6)$$

Define $n_l := \text{card} I$, which is the cardinal number of I, or in other words, n_l is the number of elements of I. Let $I(l)$ denote the l-th element of I. Define the following partition of the set $\mathbb{P}$:

$$\mathbb{P} = \bigcup_l \mathbb{P}_l, \text{ where } \mathbb{P}_l := \mathbb{C}_{I(l)}, \; l \in \mathbb{Z}_{[1,n_l]}. \quad (7)$$

Let the facets of $\mathbb{P}$ in a conic region $\mathbb{P}_l$, $l \in \mathbb{Z}_{[1,n_l]}$, be represented as $[H^c_{\mathbb{P}}]_l x \leq 1$. Similarly, let the facets of $\mathbb{Q}$ in a conic partition $\mathbb{P}_l$, $l \in \mathbb{Z}_{[1,n_l]}$, be represented as $[H^c_{\mathbb{Q}}]_l x \leq 1$.

In a conic region $\mathbb{P}_l$, it then holds that:

$$\alpha\Psi_{\mathbb{P}}(x) + (1-\alpha)\Psi_{\mathbb{Q}}(x) = \alpha[H^c_{\mathbb{P}}]_l x + (1-\alpha)[H^c_{\mathbb{Q}}]_l x. \quad (8)$$

Therefore, the facet of $\Theta(\alpha)$ in $\mathbb{P}_l$ can be represented as $[H_{\Theta(\alpha)}]_l x \leq 1$, where:

$$[H_{\Theta(\alpha)}]_l = \alpha[H^c_{\mathbb{P}}]_l + (1-\alpha)[H^c_{\mathbb{Q}}]_l. \quad (9)$$

As a result, the gauge function of $\Theta(\alpha)$ in $\mathbb{P}_l$ is:

$$[H_{\Theta(\alpha)}]_l x = \alpha[H^c_{\mathbb{P}}]_l x + (1-\alpha)[H^c_{\mathbb{Q}}]_l x. \quad (10)$$

Therefore, combining all the conic regions, the gauge function of $\Theta(\alpha)$ is:

$$\Psi_{\Theta(\alpha)}(x) = \alpha\Psi_{\mathbb{P}}(x) + (1-\alpha)\Psi_{\mathbb{Q}}(x). \quad (11)$$

Given two polytopic sets $\mathbb{P}$ and $\mathbb{Q}$, the illustration of the conic partition and $\Theta(\alpha)$ is given in Figure 1 for some values of $\alpha \in \mathbb{Z}_{[0,1]}$. It is important to note that the number of facets of $\Theta(\alpha)$ is always n_l, for all $\alpha \in \mathbb{R}_{(0,1)}$.

The following Lemmas establish certain useful properties of the interpolation–based set (4).

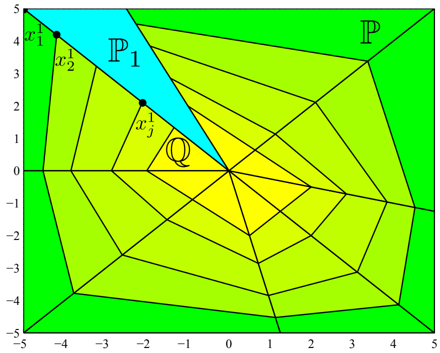


Fig. 1. Illustration of the sets $\mathbb{P}$ (green), $\mathbb{Q}$ (yellow) and $\Theta(\alpha)$ (gradient colors) for some values of $\alpha \in \mathbb{Z}_{[0,1]}$.

Lemma 4.1. The following statements are equivalent:

(i) $0 \leq \alpha_2 < \alpha_1 \leq 1$.
(ii) $\Theta(\alpha_2) \subset \Theta(\alpha_1)$.

Proof. *(i)⇒(ii)*. For all $x \in \mathbb{R}^n$, we have:

$$\begin{aligned}\Psi_{\Theta(\alpha_2)}(x) - \Psi_{\Theta(\alpha_1)}(x) &= (\alpha_2\Psi_{\mathbb{P}}(x) + (1-\alpha_2)\Psi_{\mathbb{Q}}(x)) \\ &\quad - (\alpha_1\Psi_{\mathbb{P}}(x) + (1-\alpha_1)\Psi_{\mathbb{Q}}(x)) \\ &= (\alpha_1 - \alpha_2)(\Psi_{\mathbb{Q}}(x) - \Psi_{\mathbb{P}}(x)).\end{aligned} \quad (12)$$

From the result of Lemma 2.4, we have $\Psi_{\mathbb{Q}}(x) - \Psi_{\mathbb{P}}(x) \geq 0$ for all $x \in \mathbb{P}$ since $\mathbb{Q} \subset \mathbb{P}$. Therefore, if $\alpha_2 < \alpha_1$ then $\Psi_{\Theta(\alpha_2)}(x) \geq \Psi_{\Theta(\alpha_1)}(x)$, which means $\Theta(\alpha_2) \subseteq \Theta(\alpha_1)$ due to the result of Lemma 2.4.

For all $x \in \Theta(\alpha_1)\backslash\mathbb{Q}$, it holds that $\Psi_{\mathbb{Q}}(x) - \Psi_{\mathbb{P}}(x) > 0$, resulting in $\Psi_{\Theta(\alpha_2)}(x) > \Psi_{\Theta(\alpha_1)}(x)$. We conclude that $\Theta(\alpha_2) \subset \Theta(\alpha_1)$.

(ii)⇒(i). Since $\Theta(\alpha_2) \subset \Theta(\alpha_1)$, there exists an $x \in \Theta(\alpha_1)\backslash\mathbb{Q}$ such that $\Psi_{\Theta(\alpha_2)}(x) > \Psi_{\Theta(\alpha_1)}(x)$. From (12) we conclude that $\alpha_2 < \alpha_1$. ■

Lemma 4.2. For any $x \in \mathbb{P}\backslash\mathbb{Q}$, if $\Psi_{\Theta(\alpha_1)}(x) \leq \sigma < 1$ then there exists an α_2 such that $0 < \alpha_2 < \alpha_1$ and $\Psi_{\Theta(\alpha_2)}(x) = 1$.

Proof. From

$$\Psi_{\Theta(\alpha_1)}(x) = \alpha_1\Psi_{\mathbb{P}}(x) + (1-\alpha_1)\Psi_{\mathbb{Q}}(x) \leq \sigma, \quad (13)$$

we have:

$$\alpha_1 \geq \frac{\Psi_{\mathbb{Q}}(x) - \sigma}{\Psi_{\mathbb{Q}}(x) - \Psi_{\mathbb{P}}(x)}. \quad (14)$$

Let

$$\alpha_2 = \frac{\Psi_{\mathbb{Q}}(x) - 1}{\Psi_{\mathbb{Q}}(x) - \Psi_{\mathbb{P}}(x)}, \quad (15)$$

which is equivalent to $\Psi_{\Theta(\alpha_2)}(x) = 1$. We have $\alpha_2 > 0$ since $\Psi_{\mathbb{Q}}(x) > 1 \geq \Psi_{\mathbb{P}}(x)$ for all $x \in \mathbb{P}\backslash\mathbb{Q}$.

Taking the difference between (15) and (14) we have:

$$\Delta\alpha = \alpha_1 - \alpha_2 \geq \frac{1-\sigma}{\Psi_{\mathbb{Q}}(x) - \Psi_{\mathbb{P}}(x)}. \quad (16)$$

Since $\sigma < 1$ and $\Psi_{\mathbb{Q}}(x) > \Psi_{\mathbb{P}}(x)$, it holds that $\alpha_2 < \alpha_1$. ■

From Lemma 4.2, we know that for any $x \in \mathbb{P}\backslash\mathbb{Q}$, there exists a minimal α corresponding to $\Psi_{\Theta(\alpha)}(x) = 1$, which means that

x is on the boundary of $\Theta(\alpha)$. Denote this unique α as $\alpha^*(x)$, which is a function of x:

$$\alpha^*(x) = \frac{\Psi_{\mathbb{Q}}(x) - 1}{\Psi_{\mathbb{Q}}(x) - \Psi_{\mathbb{P}}(x)}. \quad (17)$$

4.2 Interpolation of CLFs

For any $x \in \mathbb{P} \backslash \mathbb{Q}$, we would like to compute a control law u such that α is minimized, since smaller α means that x is closer to $\mathbb{Q}$. To guarantee stability, we introduce a parameterized control Lyapunov function [Lazar and Gielen, 2013]:

$$V(x, \alpha) = \Psi_{\Theta(\alpha)}(x). \quad (18)$$

To compute a control law u that minimizes the next step set $\Theta(\alpha^+)$ and guarantees stability, the following optimization problem is proposed:

Problem 4.3. Given x and α

$$\min_{\alpha^+, u} \alpha^+$$

subject to:

$$0 \leq \alpha^+ \leq 1,$$
$$H_{\mathbb{U}} u \leq 1_{h_{\mathbb{U}}},$$
$$V(Ax + Bu, \alpha^+) \leq \rho V(x, \alpha),$$

where $\lambda_{\mathbb{P}} \leq \rho < 1$.

Let $x(0)$ be the initial state and $\alpha(0) = 1$. At discrete time $k > 0$, Problem 4.3 is solved for $x := x(k)$ and $\alpha := \alpha(k)$ to obtain α^+ and u. Then let $\alpha(k+1) := \alpha^+$ and $u(k) := u$.

Theorem 4.4. Problem 4.3 is recursively feasible and the resulting control law guarantees asymptotic stability for the closed–loop system for all initial states $x(0) \in \mathbb{P}$.

Proof. For all initial states $x(0) \in \mathbb{P}$, Problem 4.3 is feasible with a trivial solution $\alpha(1) = 1$ due to the constrained control contractive property of $\mathbb{P}$. Now we will prove that if Problem 4.3 is feasible for $x(k-1)$ then it is feasible for $x(k)$, for all $k \geq 1$.

Since $\mathbb{Q} \subset \mathbb{P}$, from the results of Lemma 2.4 it holds for all $x \in \mathbb{P}$ that:

$$\Psi_{\mathbb{P}}(x) \leq \Psi_{\mathbb{Q}}(x). \quad (19)$$

Let $\alpha(k) := \alpha^+$ be the solution of Problem 4.3 for given $x(k-1)$ and $\alpha(k-1)$. As $V(x(k), \alpha(k))$ is a convex combination of $\Psi_{\mathbb{P}}(x(k))$ and $\Psi_{\mathbb{Q}}(x(k))$, it follows that

$$\Psi_{\mathbb{P}}(x(k)) \leq V(x(k), \alpha(k)) \leq \Psi_{\mathbb{Q}}(x(k)). \quad (20)$$

Since $x(k) \in \mathbb{P}$ and $\mathbb{P}$ is constrained control $\lambda_{\mathbb{P}}$-contractive, there exists a control law $u'(k) \in \mathbb{U}$ such that

$$\Psi_{\mathbb{P}}(Ax(k) + Bu'(k)) \leq \lambda_{\mathbb{P}} \Psi_{\mathbb{P}}(x(k)). \quad (21)$$

It follows from (20) and (21) that

$$\begin{aligned}\Psi_{\mathbb{P}}(Ax(k) + Bu'(k)) &\leq \lambda_{\mathbb{P}} V(x(k), \alpha(k)) \\ &\leq \rho V(x(k), \alpha(k)). \quad (22)\end{aligned}$$

If $u(k) = u'(k)$ and $\alpha(k+1) = 1$, then

$$\begin{aligned}V(x(k+1), \alpha(k+1)) &= V(Ax(k) + Bu'(k), 1) \\ &= \Psi_{\mathbb{P}}(Ax(k) + Bu'(k)) \\ &\leq \rho V(x(k), \alpha(k)). \quad (23)\end{aligned}$$

Therefore, $u = u'(k)$ and $\alpha^+ = 1$ is a feasible solution for Problem 4.3 for $x := x(k)$ and $\alpha^+ := \alpha(k)$. The recursive feasibility of the Problem is proved.

Due to the recursive feasibility of the Problem, for all $x(k) \in \mathbb{P}$ there exists a control law $u(k) \in \mathbb{U}$ and $\alpha(k+1) \in \mathbb{R}_{[0,1]}$ such that

$$\Psi_{\mathbb{P}}(x(k)) \leq V(x(k), \alpha(k)) \leq \Psi_{\mathbb{Q}}(x(k)), \quad (24)$$

and

$$V(x(k+1), \alpha(k+1)) \leq \rho V(x(k), \alpha(k)). \quad (25)$$

Therefore, $V(x(k), \alpha(k))$ is a parameterized Lyapunov function in $\mathbb{P}$ and hence guarantees asymptotic stability for system (1) in closed–loop with the controller obtained from Problem 4.3, for all initial states $x(0) \in \mathbb{P}$, in accordance with [Lazar and Gielen, 2013, Theorem 3.1]. ■

Problem 4.3 is a bilinear program. The objective function is a continuous and convex function of a scalar variable α^+, which takes value in the interval $[0, 1]$. Problem 4.3 is always feasible for $\alpha^+ = 1$, due to the constrained control contractive property of $\mathbb{P}$. Therefore, it can be solved by the bisection method, which allows reduction of a bilinear program to a sequence of linear programs or quadratic programs, depending on the cost function $J(x, u)$, i.e., if the cost is linear, then each problem is an LP and if the cost is quadratic then each problem is a QP. It should be noted that the first LP/QP, which corresponds to $\alpha^+ = 1$, is always feasible. Therefore, the bisection procedure can be stopped after any number of iterations, depending on the available computational time.

Remark 4.5. If ρ is chosen such that $\rho \geq \lambda_{\mathbb{Q}}$ then $\alpha^+ = 0$ and $u = u_{\mathbb{Q}}(x)$ is a feasible solution for Problem 4.3 when $x \in \mathbb{Q}$. Therefore, the optimal controller $u_{\mathbb{Q}}(x)$ is yielded when $x \in \mathbb{Q}$.

5. CONTRACTIVITY–SHARING

Solving a bilinear program using bisection can be computationally expensive if the optimum is sought for. However, under some certain properties of the two sets $\mathbb{P}$ and $\mathbb{Q}$, the computational load can be considerably reduced. The contractivity–sharing property of the two sets is introduced next for this purpose.

5.1 Contractivity–sharing

Definition 5.1. Given two sets $\mathbb{S}_1$ and $\mathbb{S}_2$ such that $\mathbb{S}_2 \subset \mathbb{S}_1 \subseteq \mathbb{X}$, the set $\mathbb{S}_1$ is 1-step constrained control contractive towards the set $\mathbb{S}_2$ if for any $x \in \mathbb{S}_1$, there exists a control law $u \in \mathbb{U}$ such that $Ax + Bu \in \mathbb{S}_2$.

Definition 5.2. (Contractivity–sharing) The sets $\mathbb{P}$ and $\mathbb{Q}$ have the contractivity–sharing property if there exists a sequence of interpolation–based sets $\Theta(\alpha_j)$, $j \in \mathbb{Z}_{[1,n_j]}$ such that:

(1) $\Theta(\alpha_1) = \mathbb{P}$, or equivalently $\alpha_1 = 1$.
(2) $\Theta(\alpha_{n_j}) = \mathbb{Q}$, or equivalently $\alpha_{n_j} = 0$.
(3) $\Theta(\alpha_{j+1}) \subset \Theta(\alpha_j)$, or equivalently $\alpha_{j+1} < \alpha_j$, for all $j \in \mathbb{Z}_{[1,n_j-1]}$.
(4) $\Theta(\alpha_j)$ is 1-step constrained control contractive towards $\Theta(\alpha^{j+1})$, for all $j \in \mathbb{Z}_{[1,n_j-1]}$.

It should be noted that the contractivity–sharing property does not always hold. Therefore, the following algorithm is proposed to verify the contractivity–sharing property of two polytopic constrained control λ-contractive sets:

Algorithm 5.3. Given two proper C-polytopic sets $\mathbb{P}$ and $\mathbb{Q}$, recall the conic partition as defined in (7).

(1) Denote all the vertices of all conic regions as x_1^i, where i is the index of the ray (see Figure 1). Let $\alpha_1 := 1$. Initialize $j := 1$.

(2) Solve the following optimization problem for $x := x_j^i$:

Problem 5.4. Given x:

$$\min_{\alpha^+, u} \alpha^+$$

subject to:

$$0 \leq \alpha^+ < \alpha^*(x),$$
$$H_{\mathbb{U}} u \leq 1_{h_{\mathbb{U}}},$$
$$\Psi_{\Theta(\alpha^+)}(Ax + Bu) \leq 1.$$

The control law u and α^+ are obtained. Let $u_j^i := u$, $\alpha_{j+1}^i := \alpha^+$ and $\alpha_{j+1} = \max_i \alpha_{j+1}^i$.

(3) There are three conditions for termination:
- *If Problem 5.4 is infeasible for any x_j^i, then $\mathbb{P}$ and $\mathbb{Q}$ do not have the contractivity–sharing property.*
- *If $\alpha_{j+1} = 0$ then $\mathbb{P}$ and $\mathbb{Q}$ have the contractivity–sharing property.*
- *If the predefined maximum number of iterations is reached then the test fails and no conclusion can be drawn.*

If none of the termination conditions is satisfied, let $j := j + 1$ and go back to step 2.

Algorithm 5.3 is just a sufficient check and it requires solving a number of optimization problems. For future work, we would like to derive an algebraic condition to verify the contractivity–sharing property.

Remark 5.5. The relation of the contractivity–sharing property with the control–sharing property introduced in [Grammatico et al., 2013] is discussed in this remark. To begin with, the control–sharing property concerns continuous–time dynamical systems, while the contractivity–sharing property concerns discrete–time dynamical systems. Moreover, the control–sharing property requires the existence of a control law such that both of the CLFs have a negative gradient. Therefore, a corresponding property of the control–sharing property for discrete–time dynamics should require the existence of a control law such that both CLFs decrease in one step. However, with the contractivity–sharing property, this is not necessarily the case, as one of the two CLFs can still increase in one step. Therefore, the contractivity–sharing property is less conservative than the obvious correspondent of the control–sharing property for discrete–time dynamics. This is to be expected, as usually, the discrete-time setting allows for relaxations of the standard assumptions employed by Lyapunov methods for continuous-time dynamical systems, see, e.g., [Lazar, 2006].

5.2 Interpolation of CLFs with contractivity–sharing

If $\mathbb{P}$ and $\mathbb{Q}$ have the contractivity–sharing property, Problem 4.3 can be replaced by Problem 5.4. At time $k > 0$, Problem 5.4 is solved for $x := x(k)$ to obtain the control law u and α^+. Let $u(k) := u$. Since α^+ is minimized, it follows that $\alpha^*(Ax(k) + Bu(k)) = \alpha^+$.

Theorem 5.6. If $\mathbb{P}$ and $\mathbb{Q}$ have the contractivity–sharing property then Problem 5.4 is recursively feasible and the resulting control law guarantees asymptotic stability for the closed–loop system for all initial states $x(0) \in \mathbb{P}$.

Proof. For any $x(k) \in \mathbb{P}\backslash\mathbb{Q}$, there exists a j such that $\alpha_{j+1} < \alpha^*(x(k)) \leq \alpha_j$. Since $\Theta(\alpha_j)$ is 1-step constrained control contractive towards $\Theta(\alpha_{j+1})$, there exists $u(k)$ such that $\alpha^*(Ax(k) + Bu(k)) \leq \alpha_{j+1} < \alpha^*(x(k))$. Therefore, Problem 5.4 is recursively feasible and the sequence $\alpha^*(x(k))$ reaches 0 in at most n_j steps.

When $\alpha^*(x(k)) = 0$, the state $x(k)$ is in $\mathbb{Q}$. By minimizing the cost function $J(x(k), u(k))$, the optimal controller $u_{\mathbb{Q}}(x(k))$ is yielded, which ensures stability for all $x(k) \in \mathbb{Q}$. Therefore the control law obtained from Problem 4.3 guarantees asymptotic stability for system (1) for all initial states $x(0) \in \mathbb{P}$. ■

If $\mathbb{P}$ and $\mathbb{Q}$ have the contractivity–sharing property then the property can be exploited to avoid bisection. Instead of minimizing $\alpha^*(Ax + Bu)$, we can search for a feasible suboptimal solution that satisfies $\alpha^*(Ax + Bu) < \alpha^*(x)$. Due to the result of Lemma 4.2, this means that there exists a $\sigma < 1$ such that:

$$\Psi_{\Theta(\alpha^*(x))}(Ax + Bu) = \sigma < 1. \tag{26}$$

We can fix a $\sigma < 1$ and minimize the cost function $J(x, u)$ as follows:

Problem 5.7. Given x:

$$\min_{u} J(x, u)$$

subject to:

$$H_{\mathbb{U}} u \leq 1_{h_{\mathbb{U}}},$$
$$\Psi_{\Theta(\alpha^*(x))}(Ax + Bu) \leq \sigma.$$

Problem 5.7 is just one single LP/QP, if the cost is linear/quadratic, which requires less computational demand than the bisection procedure. It returns a suboptimal solution α^*, but $J(x, u)$ is minimized, which means that the resulting controller is kept as close as possible to the optimal controller $u_{\mathbb{Q}}(x)$.

6. ILLUSTRATIVE EXAMPLE

The method is illustrated with a discrete-time double integrator:

$$x(k+1) = \begin{pmatrix} 1 & T_s \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} \frac{T_s^2}{2} \\ T_s \end{pmatrix} u(k), \tag{27}$$

where $T_s = 1$s is the sampling time.

The constraints are $\begin{pmatrix} -8 \\ -8 \end{pmatrix} \leq x \leq \begin{pmatrix} 8 \\ 8 \end{pmatrix}$ and $-1 \leq u \leq 1$.

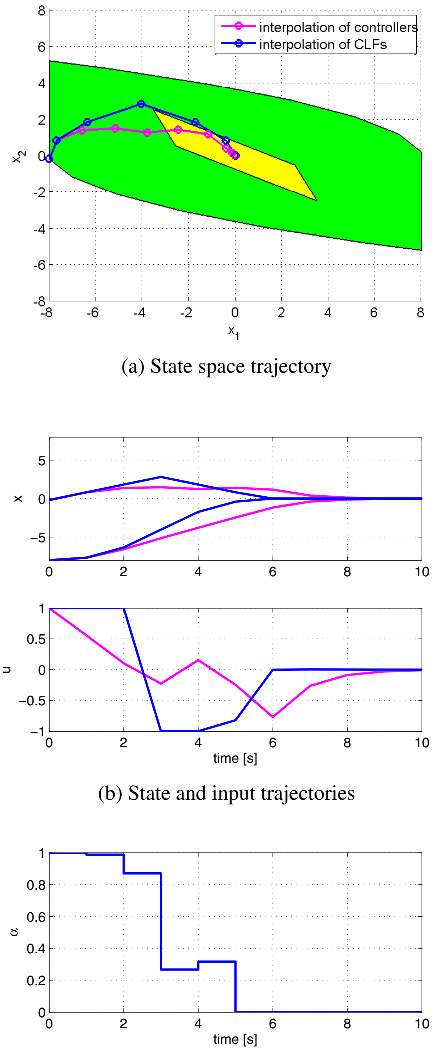
The optimal controller $u_{\mathbb{Q}}(x)$ is an LQR controller that minimizes the cost function

$$J(x, u) = x^T Q x + u^T R u + (Ax + Bu)^T P_N (Ax + Bu), \tag{28}$$

where $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R = 0.01$, and P_N is the solution of the discrete-time algebraic Riccati equation.

The set $\mathbb{P}$ is computed as the maximal constrained control contractive set for the system with the contraction rate $\lambda_{\mathbb{P}} = 0.98$. The set $\mathbb{Q}$ is computed as the maximal admissible control invariant set for the system in closed loop with controller $u_{\mathbb{Q}}(x)$. The computations are made using the procedure in [Blanchini and Miani, 2008, Section 5.3].

The simulation results of the iCLFs method for the discrete-time double integrator are shown in Figure 2. The simulation results of the interpolation of pre–designed controllers method [Nguyen et al., 2013] are also plotted for comparison. It can be seen that the iCLFs method exploits better the available range of admissible control inputs. As a result, the response of the iCLFs method is faster than that of the interpolation of pre–designed controllers method, for the considered initial condition.



(a) State space trajectory

(b) State and input trajectories

(c) Time trajectory of α

Fig. 2. Simulation results for discrete-time double integrator.

7. CONCLUSIONS

This paper proposed a method for interpolating different polytopic CLFs for linear discrete–time systems with polytopic state and input constraints. It was shown how the resulting interpolated function can be used as a parameterized CLF. A stabilizing controller was synthesized from the parameterized CLF by an optimization problem, which was proven to be recursively feasible. It was shown that the resulting controllers guarantees stability and constraints satisfaction. A desired optimal control law can be automatically yielded within a region around the equilibrium. Computational demand was discussed, and it was shown that under an additional property named contractivity–sharing, the optimization problem can be reformulated as a single LP/QP. The effectiveness of the proposed method was demonstrated by an illustrative example.

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