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Predictive and robust fault-tolerant control for Takagi-Sugeno systems

Marcin Witczak * **Christophe Aubrun** ** **Józef Korbicz** *

* *Institute of Control and Computation Engineering, University of Zielona Góra, ul. Podgórna 50, 65–246 Zielona Góra (e-mail: {M.Witczak, J.Korbicz}@issi.uz.zgora.pl)*
** *Centre de Recherche en Automatique de Nancy CRAN-UMR 7039, Nancy-Universite, CNRS, F-54506 Vandoeuvre-les-Nancy Cedex, France*

Abstract: The paper deals with the problem of robust predictive fault-tolerant control for non-linear discrete-time systems described by the Takagi-Sugeno models. The proposed approach consists of three steps, i.e. it starts from fault estimation, the fault is compensated with a robust controller, and finally, when the compensation is not successful then a suitable predictive action is performed. This appealing phenomenon makes it possible to enlarge the domain of attraction, which makes the proposed approach an efficient solution for the fault-tolerant control. The final part of the paper shows an illustrative example regarding the application of the proposed approach to the two-tank system.

Keywords: Fault diagnosis, fault identification, robust control, robust invariant set, predictive control, fault-tolerant control.

1. INTRODUCTION

A number of books was published in the last decade on the emerging problem of the Fault-Tolerant Control (FTC) (Blanke et al. (2006); De Oca et al. (2012)), which are based on the Fault Detection and Isolation (FDI) (Witczak (2007); Chen et al. (2011)). In particular, the book (Isermann (2011)), which is mainly devoted to fault diagnosis and its applications provides some general rules for the hardware-redundancy-based FTC. On the contrary, the work (Mahmoud et al. (2003)) introduces the concepts of the active and passive FTC. It also investigates the problem of performance and stability of the FTC under imperfect fault diagnosis. In particular, the authors consider (under a chain of some, not necessarily easy to satisfy assumptions) the effect of a delayed fault detection and an imperfect fault identification but the fault diagnosis scheme is treated separately during the design and no real integration of the fault diagnosis and the FTC is proposed. The FTC is also treated in a very interesting work (Noura and Chamseddine (2003)) where the number of practical case studies of FTC is presented, i.e., a winding machine, a three-tank system, and an active suspension system. Unfortunately, in spite of the incontestable appeal of the proposed approaches neither the FTC integrated with the fault diagnosis nor a systematic approach to non-linear systems are studied.

The proposed approach overcomes the above-mentioned difficulties and provides an elegant way of incorporating fault diagnosis (particularly fault identification) into the fault-tolerant control framework. Moreover, the non-linear system is described by the Takagi-Sugeno models (Takagi and Sugeno (1985)), which are frequently used in the literature. The proposed approach consists of three steps, i.e. it starts from fault estimation, the fault is compensated with a robust controller, and finally, when the compensation is not successful then a suitable predictive action is performed. The robust controller is designed without taking into account the input constraints related with the actuator saturation. Thus, to check the compensation feasibility, the robust invariant set is developed, which takes into account the input constraints. If the current state does not belong to the robust invariant set, then a suitable predictive control actions are performed in order to enhance the invariant set. This appealing phenomenon makes it possible to enlarge the domain of attraction, which makes the proposed approach an efficient solution for the fault-tolerant control. Indeed, the presented solution can be perceived as an extension of the recent developments in this area (Witczak et al. (2013)), which shows a fault estimation and compensation strategy for non-linear systems. The novelty of the scheme boils down to:

- introduction of robustness to exogenous disturbances, through the $\mathcal{H}_\infty$ approach,
- introduction of the triple stage procedure: fault estimation, fault compensation with robust controller, and predictive control enhancing the applicability of the approach,
- extension of the work of (Kouvaritakis et al. (2000)) to the case with exogenous disturbances,
- development of robust invariant set extending the usual framework proposed by (Kouvaritakis et al. (2000)).

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The paper is organised as follows. Section 2 presents preliminaries regarding the problem being undertaken. Robust fault estimation and control approach is proposed in Section 3. Subsequently, Section 4 presents the development of a robust invariant set while Section 5 presents an efficient robust predictive fault-tolerant control strategy, which enhances the performance of the overall scheme. The final part of the paper contains a numerical example, which shows the performance of the proposed approach.

2. A GENERAL DESCRIPTION OF THE FAULT-TOLERANT SCHEME

A non-linear dynamic system can be described in a relatively simple way by a Takagi-Sugeno fuzzy model, which uses series of locally linearised models from the nonlinear system, parameter identification of an a priori given structure or transformation of a nonlinear model using the nonlinear sector approach (see, e.g. (Takagi and Sugeno (1985); **?**); Korbicz et al. (2004)). According to this model, a non-linear dynamic systems can be linearised around a number of operating points. Each of these linear models represents the local system behaviour around the operating point. Thus, a fuzzy fusion of all linear model outputs describes the global system behaviour. A Takagi-Sugeno model is described by fuzzy IF-THEN rules. The presented structure may represent a non-linear system with control-affine state equation. It has a rule base of M rules, each having p antecedents, where ith rule is expressed as

$$\begin{aligned} R^i :\ \text{IF } s_k^1 \text{ is } F_1^i \text{ and } \ldots \text{ and } s_k^p \text{ is } F_p^i,\\ \text{THEN } \boldsymbol{x}_{f,k+1} = \boldsymbol{A}^i\boldsymbol{x}_{f,k} + \boldsymbol{B}\boldsymbol{u}_{f,k} + \boldsymbol{B}\boldsymbol{f}_k + \boldsymbol{W}\boldsymbol{w}_k, \end{aligned} \tag{1}$$

in which $\boldsymbol{x}_{f,k} \in \mathbb{R}^n$ stands for the state, is the output, and $\boldsymbol{u}_{f,k} \in \mathbb{R}^r$ denotes the nominal control input, $\boldsymbol{f}_k \in \mathbb{R}^r$ is the actuator fault, $i = 1, \ldots, M$, F_j^i $(j = 1, \ldots, p)$ are fuzzy sets and $\boldsymbol{s}_k = [s_k^1, s_k^2, \ldots, s_k^p]$ is a known vector of premise variables (Takagi and Sugeno (1985); Korbicz et al. (2004)). Additionally, $\boldsymbol{w}_k \in l_2$ is a an exogenous disturbance vector, while:

$$l_2 = \left\{ \mathbf{w} \in \mathbb{R}^n | \ \|\mathbf{w}\|_{l_2} < +\infty \right\}, \tag{2}$$

$$\|\mathbf{w}\|_{l_2} = \left(\sum_{k=0}^{\infty} \|\boldsymbol{w}_k\|^2 \right)^{\frac{1}{2}}. \tag{3}$$

Moreover, it is assumed that the control limits are given as follows:

$$-\bar{\boldsymbol{u}}_i \leq \boldsymbol{u}_{i,k} \leq \bar{\boldsymbol{u}}_i, \quad i = 1, \ldots, r. \tag{4}$$

where $\bar{\boldsymbol{u}}_i > 0$, $i = 1, \ldots, r$ are given control limits. Due to the simplicity of presentation, these limits are symmetrical around zero but with an appropriate scaling it is relatively easy to introduce non-symmetrical ones.

Given a pair of $(\boldsymbol{s}_k, \boldsymbol{u}_k)$ and a product inference engine, the final output of the normalized T-S fuzzy model can be inferred as:

$$\boldsymbol{x}_{f,k+1} = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)[\boldsymbol{A}^i\boldsymbol{x}_{f,k} + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{B}\boldsymbol{f}_k + \boldsymbol{W}\boldsymbol{w}_k], \tag{5}$$

where $h_i(\boldsymbol{s}_k)$ are normalised rule firing strengths defined as

$$h_i(\boldsymbol{s}_k) = \frac{\mathcal{T}_{j=1}^{p} \mu_{F_j^i}(s_k^j)}{\sum_{i=1}^{M} (\mathcal{T}_{j=1}^{p} \mu_{F_j^i}(s_k^j))} \tag{6}$$

and $\mathcal{T}$ denotes a t-norm (e.g., product). The term $\mu_{F_j^i}(s_k^j)$ is the grade of membership of the premise variable s_k^j. Moreover, the rule firing strengths $h_i(\boldsymbol{s}_k)$ $(i = 1, \ldots, M)$ satisfy the following constraints

$$\begin{cases} \sum_{i=1}^{M} h_i(\boldsymbol{s}_k) = 1, \\ 0 \leqslant h_i(\boldsymbol{s}_k) \leqslant 1, \quad \forall i = 1, \ldots, M. \end{cases} \tag{7}$$

The main objective of the subsequent part of the paper is to design the control strategy in such a way that the system (5) will converge to the origin irrespective of the presence of the fault $\boldsymbol{f}_k$. The proposed control scheme is as follows:

$$\boldsymbol{u}_{f,j} = \begin{cases} -\boldsymbol{K}\boldsymbol{x}_j - \hat{\boldsymbol{f}}_{k-1} + \boldsymbol{c}_j, \ j = k, \ldots, k + n_c - 1, \\ -\boldsymbol{K}\boldsymbol{x}_j - \hat{\boldsymbol{f}}_{k-1}, \qquad\qquad j \geq k + n_c. \end{cases} \tag{8}$$

where:

- n_c is the prediction horizon,
- $\boldsymbol{K}$ is the $\mathcal{H}_\infty$ controller designed to achieve robustness with respect to exogenous disturbances $\boldsymbol{w}_k$,
- $\hat{\boldsymbol{f}}_{k-1}$ is the fault estimate, which compensates the effect of a fault,
- $\boldsymbol{c}_j$ is a vector introducing additional design freedom, which should be exploited when the fault compensation does not provide the expected results due to the actuator saturation.

Note that beyond the control horizon n_c, $\boldsymbol{c}_j$ is set to zero, which denotes the feasibility of the $\boldsymbol{H}_\infty$ control. Thus, the design of the proposed control strategy boils down to solving a set of problems:

- to design a robust controller $\boldsymbol{K}$ in such way that a prescribed disturbance attenuation level is achieved with respect to $\boldsymbol{x}_{f,k}$ while guaranteeing its convergence to the origin,
- to estimate the fault $\boldsymbol{f}_k$,
- to determine a set of states for which the robust controller along with the fault compensation (under the control constraints) is feasible,
- to determine $\boldsymbol{c}_j$ in such a way as to enhance a set of states and, hence make the control problem feasible.

Since the general scheme is given, the remaining part of the paper is devoted to solving the above-mentioned design problems.

3. FAULT ESTIMATION AND ROBUST CONTROL

In this section, the fault estimation technique will be proposed, which along with the robust controller $\boldsymbol{K}$ will be used to compensate the effect of a fault and feed the system in such a way that the state $\boldsymbol{x}_{f,k}$ goes to the origin. Note that the designs of the fault estimator and the robust controller are realised for the unconstrained case. Moreover, the free control parameter $\boldsymbol{c}_j$ (cf. (8)) is set to zero.

Let us also assume that the system is controllable and the matrix $\boldsymbol{B}$ is a full rank one. Thus, following (Gillijns and De Moor (2007)), it is possible to compute $\boldsymbol{H} = \boldsymbol{B}^+$. Subsequently, multiplying (5) by $\boldsymbol{H}$ and then extracting $\boldsymbol{f}_k$ gives:

$$\boldsymbol{f}_k = \boldsymbol{H}\boldsymbol{A}(h_k)\boldsymbol{x}_{f,k} - \boldsymbol{u}_{f,k} - \boldsymbol{H}\boldsymbol{W}\boldsymbol{w}_k, \tag{9}$$

while its estimate can be given as:

$$\hat{\boldsymbol{f}}_k = \boldsymbol{H}\boldsymbol{A}(h_k)\boldsymbol{x}_{f,k} - \boldsymbol{u}_{f,k}, \tag{10}$$

with the associated estimation error

$$\boldsymbol{\varepsilon}_{f,k} = \boldsymbol{f}_k - \hat{\boldsymbol{f}}_k = -\boldsymbol{H}\boldsymbol{W}\boldsymbol{w}_k. \tag{11}$$

Note that in order to obtain $\hat{\boldsymbol{f}}_k$ it is necessary to have $\boldsymbol{x}_{f,k+1}$. Thus, the only choice to compensate $\boldsymbol{f}_k$ in (5) is to use $\hat{\boldsymbol{f}}_{k-1}$. This determines the above-proposed control strategy

$$\boldsymbol{u}_{f,k} = -\hat{\boldsymbol{f}}_{k-1} - \boldsymbol{K}\boldsymbol{x}_{f,k}. \tag{12}$$

Bearing in mind that in any physical system $\boldsymbol{f}_k$ is bounded, without a loss of generality, it is possible to write

$$\hat{\boldsymbol{f}}_k = \hat{\boldsymbol{f}}_{k-1} + \boldsymbol{v}_k, \quad \boldsymbol{v}_k \in l_2, \tag{13}$$

Thus, (12) can be written in an equivalent form, which will be used for further deliberations

$$\boldsymbol{u}_{f,k} = -\hat{\boldsymbol{f}}_k + \boldsymbol{v}_k - \boldsymbol{K}\boldsymbol{x}_{f,k}. \tag{14}$$

Substituting (14) into (5) gives

$$\boldsymbol{x}_{f,k+1} = \boldsymbol{A}_1\boldsymbol{x}_{f,k} + [\boldsymbol{I} - \boldsymbol{B}\boldsymbol{H}]\,\boldsymbol{W}\boldsymbol{w}_k + \boldsymbol{B}\boldsymbol{v}_k \tag{15}$$

with $\boldsymbol{A}_1(h) = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$. The equation (15) can be equivalently written as:

$$\boldsymbol{x}_{f,k+1} = \boldsymbol{A}_1(h)\boldsymbol{x}_{f,k} + \bar{\boldsymbol{W}}\bar{\boldsymbol{w}}_k \tag{16}$$

with $\bar{\boldsymbol{W}} = [[\boldsymbol{I} - \boldsymbol{B}\boldsymbol{H}]\,\boldsymbol{W} \quad \boldsymbol{B}]$, $\quad \bar{\boldsymbol{w}}_k = \left[\boldsymbol{w}_k^T, \quad \boldsymbol{v}_k^T\right]^T \in l_2$.

The following theorem constitutes the main result of this section.

Theorem 1. For a prescribed disturbance attenuation level $\mu > 0$ for the $\boldsymbol{x}_{f,k}$, the $\mathcal{H}_\infty$ controller design problem for the system (5) is solvable if there exist $\boldsymbol{U}$, $\boldsymbol{N}$ and $\boldsymbol{P} \succ \boldsymbol{0}$ such that the following LMIs are satisfied:

$$\begin{bmatrix} \boldsymbol{I} - \boldsymbol{P}^i & \boldsymbol{0} & \boldsymbol{A}^i\boldsymbol{U} - \boldsymbol{B}\boldsymbol{N} \\ \boldsymbol{0} & -\mu^2\boldsymbol{I} & \bar{\boldsymbol{W}}^T\boldsymbol{U}^T \\ \boldsymbol{U}^T(\boldsymbol{A}^i)^T - \boldsymbol{N}^T\boldsymbol{B}^T & \boldsymbol{U}\bar{\boldsymbol{W}} & \boldsymbol{P}^i - \boldsymbol{U} - \boldsymbol{U}^T \end{bmatrix} \prec \boldsymbol{0}, \tag{17}$$

$i = 1, \ldots, M$.

Proof. The problem of $\mathcal{H}_\infty$ controller design (cf. Li and Fu (1997); Zemouche et al. (2008)) is to determine the gain matrix $\boldsymbol{K}$ such that

$$\lim_{k\to\infty} \boldsymbol{x}_{f,k} = \boldsymbol{0} \quad \text{for } \bar{\boldsymbol{w}}_k = \boldsymbol{0} \tag{18}$$

$$\|\boldsymbol{x}_f\|_{l_2} \le \mu\|\bar{\boldsymbol{w}}_k\|_{l_2} \quad \text{for } \bar{\boldsymbol{w}}_k \ne \boldsymbol{0},\, \boldsymbol{e}_0 = \boldsymbol{0}. \tag{19}$$

In order to settle the above problem it is sufficient to find a Lyapunov function V_k such that:

$$\Delta V_k + \boldsymbol{x}_{f,k}^T\boldsymbol{x}_{f,k} - \mu^2\bar{\boldsymbol{w}}_k^T\bar{\boldsymbol{w}}_k < 0,\; k = 0, \ldots \infty, \tag{20}$$

where $\Delta V_k = V_{k+1} - V_k$. Indeed, if $\bar{\boldsymbol{w}}_k = \boldsymbol{0}$ then (20) boils down to

$$\Delta V_k + \boldsymbol{x}_{f,k}^T\boldsymbol{x}_{f,k} < 0,\; k = 0, \ldots \infty, \tag{21}$$

and hence $\Delta V_k < 0$, which leads to (18). If $\bar{\boldsymbol{w}}_k \ne \boldsymbol{0}$ then (20) yields

$$J = \sum_{k=0}^{\infty}\left(\Delta V_k + \boldsymbol{x}_{f,k}^T\boldsymbol{x}_{f,k} - \mu^2\bar{\boldsymbol{w}}_k^T\bar{\boldsymbol{w}}_k\right) < 0, \tag{22}$$

which can be written as

$$J = -V_0 + \sum_{k=0}^{\infty}\boldsymbol{x}_{f,k}^T\boldsymbol{x}_{f,k} - \sum_{k=0}^{\infty}\mu^2\bar{\boldsymbol{w}}_k^T\bar{\boldsymbol{w}}_k < 0, \tag{23}$$

Knowing that $V_0 = 0$ for $\boldsymbol{x}_{f,0} = \boldsymbol{0}$, (23) leads to (19).

Selecting the Lyapunov function as

$$V_k = \boldsymbol{x}_{f,k}^T\boldsymbol{P}(h_k)\boldsymbol{x}_{f,k}, \tag{24}$$

where

$$\boldsymbol{P}(h_k) = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)\boldsymbol{P}^i, \tag{25}$$

the inequality (20) becomes

$$\begin{aligned} &\Delta V + \boldsymbol{x}_{f,k}^T\boldsymbol{x}_{f,k} - \mu^2\bar{\boldsymbol{w}}_k^T\bar{\boldsymbol{w}}_k = \\ &\boldsymbol{x}_{f,k}^T\left[\boldsymbol{A}_1(h_k)^T\boldsymbol{P}(h_k)\boldsymbol{A}_1(h_k) + \boldsymbol{I} - \boldsymbol{P}(h_k)\right]\boldsymbol{x}_{f,k} + \\ &\boldsymbol{x}_{f,k}^T\left[\boldsymbol{A}_1(h_k)^T\boldsymbol{P}(h_k)\bar{\boldsymbol{W}}\right]\bar{\boldsymbol{w}}_k + \\ &\bar{\boldsymbol{w}}_k^T\left[\bar{\boldsymbol{W}}^T\boldsymbol{P}(h_k)\boldsymbol{A}_1(h_k)\right]\boldsymbol{x}_{f,k} + \\ &\bar{\boldsymbol{w}}_k^T\left[\bar{\boldsymbol{W}}^T\boldsymbol{P}(h_k)\bar{\boldsymbol{W}} - \mu^2\boldsymbol{I}\right]\bar{\boldsymbol{w}}_k < 0. \end{aligned} \tag{26}$$

It can be easily shown that (26) is equivalent to

$$\begin{bmatrix} \boldsymbol{A}_1(h)^T\boldsymbol{P}(h)\boldsymbol{A}_1(h) + \boldsymbol{I} - \boldsymbol{P}(h) & \boldsymbol{A}_1^T\boldsymbol{P}(h)\bar{\boldsymbol{W}} \\ \bar{\boldsymbol{W}}^T\boldsymbol{P}(h)\boldsymbol{A}_1(h) & \bar{\boldsymbol{W}}^T\boldsymbol{P}(h)\bar{\boldsymbol{W}} - \mu^2\boldsymbol{I} \end{bmatrix} \prec 0. \tag{27}$$

Applying Theorem 3 of (Oliveira et al. (1999)) to (27) and then substituting $\boldsymbol{A}_1(h_k)\boldsymbol{U} = (\boldsymbol{A}(h_k) - \boldsymbol{B}\boldsymbol{K})\boldsymbol{U} = \boldsymbol{A}(h_k)\boldsymbol{U} - \boldsymbol{B}\boldsymbol{N}$, gives (17), which completes the proof.

Finally, the design procedure boils down to solving (17) with respect to $\boldsymbol{U}$, $\boldsymbol{N}$ and $\boldsymbol{P}$, and then calculating

$$\boldsymbol{K} = \boldsymbol{N}\boldsymbol{U}^{-1}. \tag{28}$$

The objective of this section was to provide a fault estimation and compensation scheme without taking into the account the control limit. It is obvious fact that in the presence of faults, disturbances and control limits the set of the states that can be reached from $\boldsymbol{x}_{f,k}$ is significantly smaller that the one obtained without these unappealing phenomena. Thus, the objective of the subsequent section is to provide a useful description of such a set, while the Section 5 presents an on-line optimisation strategy that can be used for enlarging this set.

4. DERIVATION OF A ROBUST INVARIANT SET

As it was mentioned in the previous section, in order to maintain a desired system behaviour, the idea of an invariant set of state variables is to be employed (Blanchini (1999)).

In this section the ellipsoidal bounding will be used for describing the robust invariant set for

$$\boldsymbol{x}_{f,k+1} = \boldsymbol{A}_1(h_k)\boldsymbol{x}_{f,k} + \bar{\boldsymbol{W}}\bar{\boldsymbol{w}}_k \tag{29}$$

with an additional assumption that:

$$\bar{\boldsymbol{w}}_k^T\boldsymbol{Q}^{-1}\bar{\boldsymbol{w}}_k \le 1, \quad \boldsymbol{Q} \succ \boldsymbol{0}. \tag{30}$$

The proposed ellipsoidal bounding strategy can be perceived as an inner approximation of the exact invariant set (Gilbert and Tan (1991)). An obvious drawback to the proposed approach is that the obtained set is smaller than the exact one. However, the simplicity of the ellipsoidal description will make it possible to use it for online optimisation, which will be described in Section 5. In particular, $\boldsymbol{E}_{\boldsymbol{x}_f}$ is a robust invariant set for (29) if

$$\boldsymbol{x}_k \in \boldsymbol{E}_{\boldsymbol{x}_f} \implies \boldsymbol{x}_{k+1} \in \boldsymbol{E}_{\boldsymbol{x}_f}. \tag{31}$$

Thus, the ellipsoidal robust invariant set is given by

$$\boldsymbol{E}_{\boldsymbol{x}_f} = \{\boldsymbol{x}_f | \boldsymbol{x}_f^T \boldsymbol{P}(h_k)\boldsymbol{x}_f \ \le 1\}, \quad \boldsymbol{P}(h_k) \succ \boldsymbol{0}. \tag{32}$$

The above definition implies the following constraints:

$$\boldsymbol{x}_{f,k}^T \boldsymbol{P}(h_k)\boldsymbol{x}_{f,k} \le 1 \tag{33}$$

$$\boldsymbol{x}_{f,k+1}^T \boldsymbol{P}(h_k)\boldsymbol{x}_{f,k+1} \le 1 \tag{34}$$

which by applying the S-procedure along with (30) yield the following coupled constraint

$$\begin{aligned}&(\boldsymbol{x}_{f,k}^T\boldsymbol{A}_1(h_k)^T + \bar{\boldsymbol{w}}_k^T\bar{\boldsymbol{W}}^T)\boldsymbol{P}(h_k)(\boldsymbol{A}_1(h_k)\boldsymbol{x}_{f,k} + \bar{\boldsymbol{W}}\bar{\boldsymbol{w}}_k) - 1 + \\ &\gamma(1-\boldsymbol{x}_{f,k}^T\boldsymbol{P}(h_k)\boldsymbol{x}_{f,k}) + \beta(1-\bar{\boldsymbol{w}}_k^T\boldsymbol{Q}^{-1}\bar{\boldsymbol{w}}_k) = \\ &\boldsymbol{x}_{f,k}^T\boldsymbol{A}_1(h_k)^T\boldsymbol{P}^{-1}\boldsymbol{A}_1(h_k)\boldsymbol{x}_{f,k} + \boldsymbol{x}_{f,k}^T\boldsymbol{A}_1(h_k)^T\boldsymbol{P}(h_k)\bar{\boldsymbol{W}}\bar{\boldsymbol{w}}_k + \\ &\bar{\boldsymbol{w}}_k^T\bar{\boldsymbol{W}}^T\boldsymbol{P}(h_k)\boldsymbol{A}_1\boldsymbol{x}_{f,k} + \bar{\boldsymbol{w}}_k^T\bar{\boldsymbol{W}}^T\boldsymbol{P}(h_k)\bar{\boldsymbol{W}}\bar{\boldsymbol{w}}_k - 1 + \gamma \\ &-\gamma\boldsymbol{x}_{f,k}^T\boldsymbol{P}(h_k)\boldsymbol{x}_{f,k} + \beta - \beta\bar{\boldsymbol{w}}_k^T\boldsymbol{Q}^{-1}\bar{\boldsymbol{w}}_k \le 0\end{aligned}$$

with $\gamma > 0$ and $\beta > 0$. The above inequality can be described in a matrix form:

$$\begin{bmatrix}\boldsymbol{A}_1(h)^T\boldsymbol{P}(h)\boldsymbol{A}_1(h) - \gamma\boldsymbol{P}(h) & \boldsymbol{A}_1(h)^T\boldsymbol{P}(h)\bar{\boldsymbol{W}} & 0 \\ \bar{\boldsymbol{W}}^T\boldsymbol{P}(h)\boldsymbol{A}_1(h) & \bar{\boldsymbol{W}}^T\boldsymbol{P}(h)\bar{\boldsymbol{W}} - \beta\boldsymbol{Q}^{-1} & 0 \\ 0 & 0 & \gamma+\beta-1\end{bmatrix} \preceq \boldsymbol{0} \tag{35}$$

From (35), it is obvious that

$$\gamma + \beta \le 1 \Rightarrow \beta = 1-\gamma \Rightarrow 0 \le \gamma \le 1. \tag{36}$$

This leads directly to:

$$\begin{bmatrix}\boldsymbol{A}_1(h_k)^T \\ \bar{\boldsymbol{W}}^T\end{bmatrix}\boldsymbol{P}(h_k)\begin{bmatrix}\boldsymbol{A}_1(h_k) & \bar{\boldsymbol{W}}\end{bmatrix} + \begin{bmatrix}-\gamma\boldsymbol{P}(h_k) & 0 \\ 0 & -(1-\gamma)\boldsymbol{Q}^{-1}\end{bmatrix} \preceq \boldsymbol{0}, \quad 0 \le \gamma \le 1, \tag{37}$$

which using by Theorem 3 of (Oliveira et al. (1999)) can be written as

$$\begin{bmatrix}-\gamma\boldsymbol{P}^i & \boldsymbol{0} & (\boldsymbol{A}_1^i)^T\boldsymbol{U}^T \\ \boldsymbol{0} & -(1-\gamma)\boldsymbol{Q}^{-1} & \bar{\boldsymbol{W}}^T\boldsymbol{U}^T \\ \boldsymbol{U}\boldsymbol{A}_1^i & \boldsymbol{U}\bar{\boldsymbol{W}} & \boldsymbol{P}^i - \boldsymbol{U} - \boldsymbol{U}^T\end{bmatrix} \preceq \boldsymbol{0}, \tag{38}$$

$i = 1, \ldots, M, \quad 0 \le \gamma \le 1.$

Note that for a fixed $0 \le \gamma \le 1$, the inequality (38) becomes the usual LMI. Another strategy is to formulate (38) as a generalised eigenvalue optimisation problem. Both of them can be efficiently solved with the numerical packages like Matlab.

5. EFFICIENT PREDICTIVE FTC

The robust fault-tolerant control presented in Sec. 4 is based on the idea of estimating the fault, and then compensating it with a suitable increase or decrease of the control feeding the faulty actuator. In spite of the incontestable appeal of the proposed approach, its main drawback is that it does not take into account the fact that all actuators obey some saturation rules. Thus, the idea behind the approach presented in this section is as follows: when a saturation of a faulty actuator appears then perturb (or modify) the control strategy of the remaining actuators in such a way as to increase the robust invariant set and to make the overall control problem feasible. The subsequent part of this section is devoted to the implementation of such a strategy.

Thus, the objective of the subsequent part of this section is to develop a suitable control strategy that takes into account the actuator saturation. For this purpose, the efficient predictive control scheme introduced by Kouvaritakis et al. (2000) is utilised. In particular, the proposed scheme is suitably extended to cope with the external disturbances, and hence, achieving robustness.

It can be easily shown that the input constraints (4) for the ith input are (cf. (8))

$$-\bar{\boldsymbol{u}}_i \le \begin{bmatrix}-\boldsymbol{K}_i^T & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{f,k} \\ \boldsymbol{c}_{i,k} - \hat{\boldsymbol{f}}_{i,k-1}\end{bmatrix} \le \bar{\boldsymbol{u}}_i, i = 1, \ldots, r \tag{39}$$

where $\boldsymbol{K}_i$ stands for the ith row of $\boldsymbol{K}$. Thus, predictions at time k are generated as follows Kouvaritakis et al. (2000):

$$\boldsymbol{z}_{k+1} = \boldsymbol{Z}(h_k)\boldsymbol{z}_k + \tilde{\boldsymbol{W}}\bar{\boldsymbol{w}}_k. \tag{40}$$

where

$$\tilde{\boldsymbol{W}} = \begin{bmatrix}\bar{\boldsymbol{W}} \\ \boldsymbol{0}\end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix}\boldsymbol{A}(h_k) - \boldsymbol{B}\boldsymbol{K} & \boldsymbol{B}\boldsymbol{T} \\ \boldsymbol{0} & \boldsymbol{M}\end{bmatrix}, \tag{41}$$

$$\boldsymbol{M} = \begin{bmatrix}\boldsymbol{0}(n_c-1)r \times r & \boldsymbol{I} \\ \boldsymbol{0}r \times r & \boldsymbol{0}r \times (n_c-1)r\end{bmatrix}, \tag{42}$$

$$\boldsymbol{z}_k = \begin{bmatrix}\boldsymbol{x}_{f,k} \\ \boldsymbol{\omega}_k\end{bmatrix}, \quad \boldsymbol{\omega}_k = \begin{bmatrix}\boldsymbol{c}_k \\ \boldsymbol{c}_{k+1} \\ \cdots \\ \boldsymbol{c}_{k+n_c-1}\end{bmatrix}, \quad \boldsymbol{T} = [\boldsymbol{I}_{r\times r} \ \boldsymbol{0} \ \ldots \ \boldsymbol{0}]. \tag{43}$$

Note that the stability (in the $\boldsymbol{H}_\infty$ sense) of the autonomous system (40) is guaranteed by the stability of $\boldsymbol{A}(h_k) - \boldsymbol{B}\boldsymbol{K}$. Following Kouvaritakis et al. (2000), it can be pointed out that if there exists robust invariant set $\boldsymbol{E}_{\boldsymbol{x}_f}$ (cf. (32)) for (29), then there must exist at least one robust invariant set $\boldsymbol{E}_{\boldsymbol{z}}$ for (40). Thus, (38) can be easily adapted for (40), which gives the robust invariant set for the proposed fault-tolerant predictive scheme:

$$\begin{bmatrix}-\gamma\boldsymbol{P}^i & \boldsymbol{0} & (\boldsymbol{Z}^i)^T\boldsymbol{U}^T \\ \boldsymbol{0} & -(1-\gamma)\boldsymbol{Q}^{-1} & \tilde{\boldsymbol{W}}^T\boldsymbol{U}^T \\ \boldsymbol{U}\boldsymbol{Z}^i & \boldsymbol{U}\tilde{\boldsymbol{W}} & \boldsymbol{P}^i - \boldsymbol{U} - \boldsymbol{U}^T\end{bmatrix} \preceq \boldsymbol{0}, \quad 0 \le \gamma \le 1, \tag{44}$$

Since the robust invariant set for (40) is given then it is possible to introduce the input constraints (39). The easiest way to do this is to suitably scale $\boldsymbol{\omega}_k$ in (40) as follows, i.e. $\boldsymbol{\omega}_k$ is replaced by:

$$\bar{\boldsymbol{\omega}}_k = \begin{bmatrix}\boldsymbol{c}_k - \hat{\boldsymbol{f}}_{k-1} \\ \boldsymbol{c}_{k+1} \\ \cdots \\ \boldsymbol{c}_{k+n_c-1}\end{bmatrix}, \tag{45}$$

Thus, constraints (39) can be described as follows

$$-\bar{\boldsymbol{u}}_i \le \begin{bmatrix}-\boldsymbol{K}_i^T & \boldsymbol{e}_i^T\end{bmatrix}\boldsymbol{z}_k \le \bar{\boldsymbol{u}}_i, i = 1, \ldots, r \tag{46}$$

where $\boldsymbol{e}_i$ is the ith column of the identity matrix, or equivalently

$$|\begin{bmatrix}-\boldsymbol{K}_i^T & \boldsymbol{e}_i^T\end{bmatrix}\boldsymbol{z}_k| \le \bar{\boldsymbol{u}}_i, i = 1, \ldots, r \tag{47}$$

For $\boldsymbol{z}_k \in \boldsymbol{E}_{\boldsymbol{z}}$, the above inequality implies

$$\begin{aligned}&\left|\begin{bmatrix}-\boldsymbol{K}_i^T & \boldsymbol{e}_i^T\end{bmatrix}\boldsymbol{z}_k\right| = \left|\begin{bmatrix}-\boldsymbol{K}_i^T & \boldsymbol{e}_i^T\end{bmatrix}\boldsymbol{P}(h_k)^{\frac{1}{2}}\boldsymbol{P}(h_k)^{\frac{1}{2}}\boldsymbol{z}_k\right| \le \\ &\left\|\begin{bmatrix}-\boldsymbol{K}_i^T & \boldsymbol{e}_i^T\end{bmatrix}\boldsymbol{P}^{\frac{1}{2}}\right\|\left\|\boldsymbol{P}(h_k)^{-\frac{1}{2}}\boldsymbol{z}_k\right\| \le \\ &\left\|\begin{bmatrix}-\boldsymbol{K}_i^T & \boldsymbol{e}_i^T\end{bmatrix}\boldsymbol{P}(h_k)^{\frac{1}{2}}\right\|\end{aligned} \tag{48}$$

which is equivalent to

$$\left[-\boldsymbol{K}_i^T\ \boldsymbol{e}_i^T\right]\boldsymbol{P}(h_k)\left[-\boldsymbol{K}_i^T\ \boldsymbol{e}_i^T\right]^T-\bar{\boldsymbol{u}}_i^2\le\boldsymbol{0},\quad i=1,\ldots,r, \tag{49}$$

where $\boldsymbol{e}_i$ stands for the ith column of the identity matrix. Finally, using Theorem 3 of (Oliveira et al. (1999)), inequalities (49) can be written in an LMI form

$$\begin{bmatrix}-\bar{\boldsymbol{u}}_i^2 & [-\boldsymbol{K}_i^T\ \boldsymbol{e}_i^T](\boldsymbol{U}_u^i)^T\\ \boldsymbol{U}_u^i[-\boldsymbol{K}_i^T\ \boldsymbol{e}_i^T]^T & \boldsymbol{P}^j-\boldsymbol{U}_u^i-(\boldsymbol{U}_u^i)^T\end{bmatrix}\preceq\boldsymbol{0}, \tag{50}$$

$i=1,\ldots,r,\quad j=1,\ldots,M.$

If the robust invariant set along with input constraints is described in a form of LMIs, then it is possible to solve them and simultaneously maximize the invariant set. For that purpose, various criteria can be selected, e.g.:

- maxmisation of the determinant of $\boldsymbol{P}(h_k)$, which corresponds to the volume of the invariant set,
- maxmisation of the trace of $\boldsymbol{P}(h_k)$, which corresponds to the sum of the axes of the ellipsoid describing invariant set.

Taking into account the structure of $\boldsymbol{P}(h_k)$, which is a weighted sum of matrices, to maximise the size of the $\boldsymbol{E}_{\boldsymbol{x}_f}$ the following sum of traces should be maximised:

$$\max\operatorname{trace}\left(\sum_{i=1}^{M}\left(\boldsymbol{T}_z\boldsymbol{P}^i\boldsymbol{T}_z^T\right)\right)= \tag{51}$$

$$\max\operatorname{trace}\left(\operatorname{diag}\left(\boldsymbol{T}_z\boldsymbol{P}^1\boldsymbol{T}_z^T,\ldots,\boldsymbol{T}_z\boldsymbol{P}^M\boldsymbol{T}_z^T\right)\right) \tag{52}$$

with

$$\boldsymbol{x}_{f,k}=\boldsymbol{T}_z\boldsymbol{z}_k, \tag{53}$$

under the constraints (44) and (50). The algorithm for computing $\boldsymbol{c}_k$ in (40) is also inspired by Kouvaritakis et al. (2000) and boils down to perform, at each sampling time, the following minimisation

$$\boldsymbol{\omega}_k^*=\min_{\boldsymbol{\omega}_k}\boldsymbol{\omega}_k^T\boldsymbol{\omega}_k,\qquad s.t.\ \boldsymbol{z}_k^T\boldsymbol{P}(h_k)\boldsymbol{z}_k\le1, \tag{54}$$

which can be equivalently written as:

$$\begin{aligned}\boldsymbol{\omega}_k^*=\min_{\boldsymbol{\omega}_k}\boldsymbol{\omega}_k^T\boldsymbol{\omega}_k,\qquad s.t.\ &\boldsymbol{x}_{f,k}^T\boldsymbol{P}_{1,1}(h_k)\boldsymbol{x}_{f,k}+\\ &2\boldsymbol{x}_{f,k}^T\boldsymbol{P}_{1,2}(h_k)\bar{\boldsymbol{\omega}}_k+\\ &\bar{\boldsymbol{\omega}}_k^T\boldsymbol{P}_{2,2}(h_k)\bar{\boldsymbol{\omega}}_k\le1,\end{aligned} \tag{55}$$

where $\boldsymbol{P}_{1,1}(h_k)$, $\boldsymbol{P}_{1,2}(h_k)$ and $\boldsymbol{P}_{2,2}(h_k)$ are block partitions of $\boldsymbol{P}(h_k)$ conformal to the partition of $\boldsymbol{z}_k=[\boldsymbol{x}_{f,k}^T\ \bar{\boldsymbol{\omega}}_k^T]^T$. Thus, if the $\mathcal{H}_\infty$ control is feasible then $\boldsymbol{\omega}=\boldsymbol{0}$, otherwise the solution lies on the boundary of $\boldsymbol{E}_{\boldsymbol{z}}$ described by (55). This means that when $\boldsymbol{\omega}=\boldsymbol{0}$ is contained in $\boldsymbol{E}_{\boldsymbol{z}}$ described by (55), then there is no need for optimisation and the opitimal solution is $\boldsymbol{\omega}=\boldsymbol{0}$. Otherwise, as indicated in (Kouvaritakis et al. (2000)), the above optimisation problem has a unique solution and can be very efficiently solved with, e.g., the Newton-Raphson algorithm (Kouvaritakis et al. (2000)). Thus, the structure of whole robust predictive fault-tolerant control can be summarized as follows:

Off-line computation:

(1) for a predefined disturbance attenuation level $\mu>0$, design a robust controller $\boldsymbol{K}$ by solving (17),
(2) determine the robust invariant set by solving (52) under the constraints (44) and (50).

On-line computation: for each k,

(1) compute the fault estimate $\hat{\boldsymbol{f}}_{k-1}$ with (10),
(2) solve the optimisation problem (55),
(3) implement the first element of $\boldsymbol{\omega}_k$, i.e. $\boldsymbol{c}_k$.

6. ILLUSTRATIVE EXAMPLE

Let us consider a two-tank system presented in Fig. 1, which is composed of two-tanks that are fed with two pumps that provide the liquid into the first and second tank, respectively. Let us consider a mathematical model

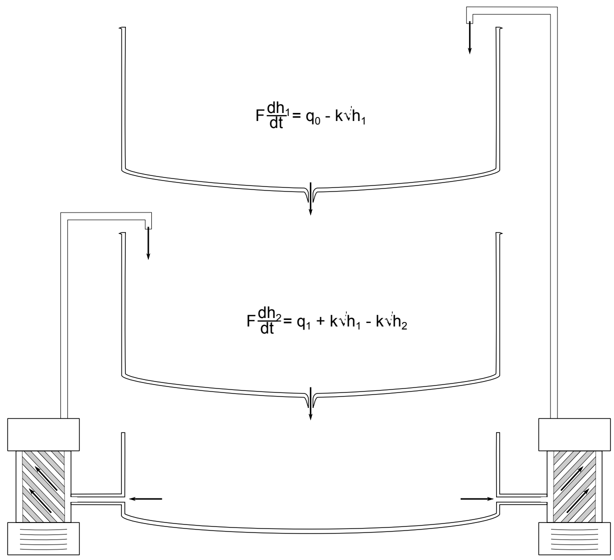

Fig. 1. Two-tank system

of a two-tank system

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u} \tag{56}$$

with

$$\begin{aligned}\boldsymbol{A}&=\begin{bmatrix}-\dfrac{k}{2F\sqrt{h_1^s}} & 0\\ \dfrac{k}{2F\sqrt{h_1^s}} & \dfrac{k}{2F\sqrt{h_2^s}}\end{bmatrix}\\ \boldsymbol{B}&=\begin{bmatrix}\dfrac{1}{F} & 0\\ 0 & \dfrac{1}{F}\end{bmatrix}\end{aligned} \tag{57}$$

where $\boldsymbol{x}=[(h_1-h_1^s),(h_2-h_2^s)]^T$, h_1 and h_2 are liquid levels in the tanks, $F=12.566cm^2$ is tanks cross-section area, $k=3.667cm^{5/2}s^{-1}$ flow constant. The model was linearised around two different points and discretised using Euler method. Moreover, the control limit $\bar{\boldsymbol{u}}_i=0.2$, $i=1,2$ was established. The robust controller was designed with (17) for $\mu=0.9$ and then the robust invariant set was computed by solving (52) (for $\boldsymbol{Q}=0.01\boldsymbol{I}$) under the constraints (44) and (50). The computations were performed for $n_c=0$, $n_c=4$ and $n_c=8$. Exemplary sets are shown in Fig. 2. From this result it is evident that the size of the invariant set can be suitably enlarged with n_c. If the controller and the matrix $\boldsymbol{P}$ underlying the robust invariant set are given, then it is possible to proceed to the on-line implementation of the proposed approach. In the sequel $n_c=4$ was used. The fault scenarios being considered are related to the 10% decrease

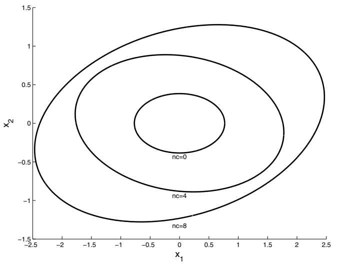

Fig. 2. Exemplary invariant sets for $n_c = 0$, $n_c = 4$ and $n_c = 8$

of the performance of either first or second actuator. The results related to the first fault are shown in Fig. 3, while Fig. 4 presents the effects of the second fault. From these results, it is clear that the proposed FTC outperforms the usual robust control scheme without the FTC mechanism.

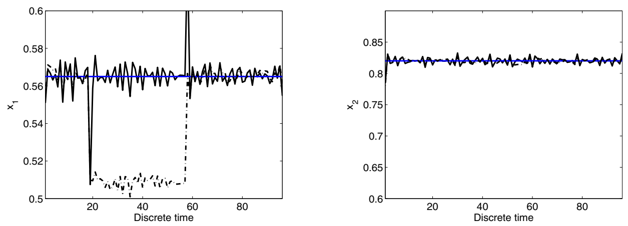

Fig. 3. Set point, state for the FTC (solid line) and state without FTC (dash-dot line) for the first fault

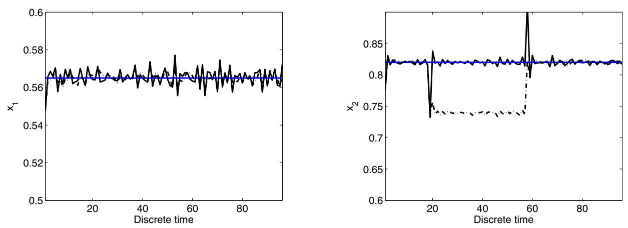

Fig. 4. Set point, state for the FTC (solid line) and state without FTC (dash-dot line) for the second fault

7. CONCLUSIONS

The contribution of the paper can be divided into a few important points: extension of the efficient predictive control to the robust case with exogenous external disturbances acting on the system, development of robust fault estimation and compensation scheme, and an integration of the developed schemes within a unified robust predictive fault-tolerant control framework. It is worth to note that the framework was also suitably extended to non-linear systems that can be described with the Takagi-Sugeno models. All the proposed approaches can be efficiently implemented, i.e. the off-line computations boils down to solving a number of linear matrix inequalities while the on-line computation reduces to the application of the Newton-Raphson method. The proposed approach was applied to the benchmark example of the two-tank system. The achieved results show the high performance of the approach.

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