Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Delay-range-dependent robust BIBO stabilization of 2D discrete delayed systems via LMI approach *

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Abstract—**This paper focuses on delay-dependent bounded input bounded output (BIBO) stabilization of a class of two-dimensional (2D) discrete delayed systems described by Fornasini and Marchesini (FM) state-space models. The main contribution is to propose a linear matrix inequality (LMI) method to construct sufficient conditions on delay-range-dependent 2D BIBO stability, where the stability criterion guarantees that bounded input can lead to bounded output though the time delay has a direct effect on the system stability. To solve the problem, a novel relevant concept of 2D BIBO stability is first introduced. Then, the condition is formulated in terms of LMIS through the Lyapunov–Krasovskii function. Subsequently, the existence condition can be extended to the uncertain case. Meanwhile, the relation between BIBO stability and asymptotical stability is expatiated. The generality of the proposed design method is shown by results through constant delay case. Finally, an illustrative example of a causal 2D system is selected to demonstrate the effectiveness and merits of the proposed method.**

I. Introduction

There has been the widespread concern to bounded input bounded output (BIBO) stability of one-dimensional (1D) systems in recent years [1-3]. BIBO stability means that any bounded input yields a bounded output, which is the system property. As an important system performance index, BIBO stability of the system can be considered in many aspects such as the free system dynamics, the basic single or double loop modulators, the issues connected with bilinear input/output maps, and so on. The investigation on BIBO stability for 1D system has become a hotspot and has been extended to areas such as neutral delay systems [4]. However, no much research on BIBO stability for two-dimensional (2D) delay systems has been done yet.

*Research supported in part by the National Natural Science Foundation of China (61104058); Hong Kong, Macao and Taiwan Science & Technology Cooperation Program of China (2013DFH10120); Guangdong academician workstation project (2012B090500010); Guangzhou scientific and technological project (12190007); and Doctor Scientific Research Foundation of Liaoning Province (20121046).

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In the past few decades, the study on 2D system has achieved significant results [5-6] in both theory and application areas such as multi-dimensional digital filtering, linear image processing, signal processing, process control, and so on. Because time delays may lead to oscillation, instability, and poor performance, study on 2D systems with time delays using the Lyapunov functional method has received considerable attention in recent years [7-9].

In practice, n-dimensional (nD) problems often show the property that the underlying signal is unbounded with respect to only one coordinate direction, for example, the spatio-temporal 3D signal of a video camera. In real world, the scenarios lead to the definition of practical BIBO stability, a property that can be analyzed by comparably simple tests. In control theory, BIBO stability is a useful property for a plant since it guarantees that control actions stay inside predefined amplitude bounds. Due to signal processing and if signals are not decreasing in case of zero input after certain time, the output signal becomes almost independent on the input signal which is definitely harmful in most signal processing applications. Designing the appropriate controller to force the output signal to reduce to some predefined bounds in the case of the non-zero input is thus an urgent problem in applications. In some chemical processes such as the barrel temperature in the batch process, for a given bounded control input, the system's actual output temperature is difficult to achieve zero error tracking for the given output. Our goal is to preserve the error changing in a range, which means bounded. From the above analysis, study on BIBO stability of 2D system is valuable in both reality and theory.

To date, there exist the following methods focusing on the BIBO stability of 1D system, such as Lipschitz continuous combined with differential equation method [10], matrix Riccati equation and Bihari-type inequality technique [1, 11], Razumikhin technique [3], and son on. However, to the best of the authors' knowledge, the BIBO stability problem for 2D systems has not been fully investigated with only limited results based on the frequency domain method [12]. For the 2D system with time delay, nearly no research results have been available up to now.

In process control, we have known that if the system reference input signal is bounded when the state cannot be convergent to zero, we expect the output signal be convergent to some predefined bounds. Inspired by this and according to the bounded reference input signal, we are interested in analyzing the delay-range dependent BIBO stability issues for a class of uncertain systems described by the 2D-FM state-space model with state delays varying in a range. As we all know, the LMI approach is an efficient method to solve many control problems. Here we use this method to solve the considered problem. Different from the existing results on the analysis of BIBO control, the work of our paper has the following features: (1) A new relevant concept of 2D BIBO stability is first presented. (2) To avoid the complex calculations, a novel Lyapunov functional candidate is chosen and a differential inequality is added to the difference Lyapunov functional for 2D systems. Then based on linear matrix inequality (LMI) method, sufficient conditions for the existence of the delay-range dependent state feedback controller is derived that ensures the closed-loop system BIBO stable and can be easily solved by LMI Toolbox in Matlab. (3) In order to make the study more practical, the existence conditions are easily extended to the uncertain case according to some common methods of transformation. This paper also expatiates on the relation of BIBO stability with asymptotical stability. Finally, applications to a causal 2D system show that the proposed 2D controller achieves the design objectives despite the fact that the time delay item may lead to the system instability.

II. PROBLEM FORMULATION

Consider the following discrete-time model with interval time-varying delay:

$$\begin{cases} x(i+1,j+1)=A_1x(i,j+1)+A_2x(i+1,j) \\ \qquad +A_{1d}x(i-d_1(i),j+1)+A_{2d}x(i+1,j-d_2(j)) \\ \qquad +B_1u(i,j+1)+B_2u(i+1,j) \\ \quad y(i,j)=Cx(i,j) \end{cases} \tag{1}$$

where $0\le i,j\in\mathbb{Z}$ ($\mathbb{Z}$ is integer set) are horizontal and vertical coordinates, respectively; $x(i,j)\in R^n$, $u(i,j)\in R^m$ and $y(i,j)\in R^l$ represent, respectively, the state, control input, and output of the system; $\{A_1,A_2,A_{1d},A_{2d},B,C\}$ are constant matrices of appropriate dimensions; the time-varying delays $d_1(i)$ and $d_2(j)$ are along horizontal and vertical directions, respectively, and satisfy

$$d_{1L}\le d(i)\le d_{1H},\quad d_{2L}\le d(j)\le d_{2H} \tag{2}$$

where d_{1L}, d_{1H}, d_{2L} and d_{2H} are constant positive scalars representing the lower and upper delay bounds along horizontal and vertical directions, respectively.

The boundary conditions associated with the 2D system (1) are denoted by

$$\begin{cases} x(i,j)=s_{ij},\quad j=0,1,2,\cdots,\quad i=-d_{1H},-d_{1H}+1,\cdots,0, \\ x(i,j)=t_{ij},\quad i=0,1,2,\cdots,\quad j=-d_{2H},-d_{2H}+1,\cdots,0 \\ \quad s_{00}=t_{00} \end{cases} \tag{3}$$

s_{ij} and t_{ij} are given vectors.

The main objective of this paper is to design a feedback controller in the form

$$u(i,j)=Kx(i,j)+r(i,j) \tag{4}$$

such that the considered delayed system (1) is BIBO stabilized, where $r(i,j)$ is the reference input and K , to be determined, is the feedback gain matrix with appropriate dimensions. Substituting (4) into (1), the corresponding closed-loop system is described by

$$\begin{cases} x(i+1,j+1)=(A_1+B_1K)x(i,j+1)+(A_2+B_2K)x(i+1,j) \\ \qquad +A_{1d}x(i-d_1(i),j+1)+A_{2d}x(i+1,j-d_2(j)) \\ \qquad +B_1r(i,j+1)+B_2r(i+1,j) \\ \qquad =A_{1K}x(i,j+1)+A_{2K}x(i+1,j) \\ \qquad +A_{1d}x(i-d_1(i),j+1)+A_{2d}x(i+1,j-d_2(j)) \\ \qquad +B_1r(i,j+1)+B_2r(i+1,j) \\ \quad y(i,j)=Cx(i,j) \end{cases} \tag{5}$$

Now, according to [13], we extend the idea and achieve the new definitions, where we also introduce some lemmas for a precise formulation of our results.

Definition 1. Define

$$\|z(\cdot,\cdot)\|_{\infty,\infty}=\sup_{i,j\to\infty}\|z(i,j)\|$$

as a real-valued signal vector $z(i,j)\in L^n_{\infty,\infty}$, if $\|z(i,j)\|_{\infty,\infty}<\infty$.

The control system given by (1) is said to be 2D BIBO stabilized by the control law (4), if for every reference input $r(i,j)\in L^n_{\infty,\infty}$, there exists some positive constants θ_1 and θ_2 satisfying

$$\|y(i,j)\|_E\le\theta_1\|r\|_{\infty,\infty}+\theta_2$$

Lemma 1 [14]. For any vector $\delta(t)\in R^n$, two positive integers κ_0 , κ_1 , and a matrix $0<R\in R^{n\times n}$, the following inequality holds

$$-(\kappa_1-\kappa_0+1)\sum_{t=\kappa_0}^{\kappa_1}\delta^T(t)R\delta(t)\le-\sum_{t=\kappa_0}^{\kappa_1}\delta^T(t)R\sum_{t=\kappa_0}^{\kappa_1}\delta(t)$$

III. MAIN RESULTS

In this section, the main goal is to design an 2D controller to ensure the corresponding closed-loop system (5) BIBO is stable. The main results are as follows.

Theorem 1. For some given scalars $0\le d_{1L}\le d_{1H}$, $0\le d_{2L}\le d_{2H}$, if there exist symmetric positive matrices L , $\bar{Q}_1$, $\bar{Q}_2$, $\bar{W}_1$, $\bar{W}_2$, $\tilde{W}_1$, $\tilde{W}_2$, X_1, $X_2\in R^{(n+l)\times(n+l)}$, matrices $Y\in R^{m\times(n+l)}$ and positive scalars α and β such that the following LMI holds

$$\begin{bmatrix} \Pi_{11} & \mathbf{0} & \Pi_{13} & \Pi_{14} \\ * & \Pi_{22} & \Pi_{23} & \mathbf{0} \\ * & * & \Pi_{33} & \mathbf{0} \\ * & * & * & \Pi_{44} \end{bmatrix}<0 \tag{6}$$

where

$$\Pi_{11}=\begin{bmatrix}\phi_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & L & \mathbf{0}\\ \mathbf{0} & \phi_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} & L\\ \mathbf{0} & \mathbf{0} & -\bar{Q}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\bar{Q}_2 & \mathbf{0} & \mathbf{0}\\ L & \mathbf{0} & \mathbf{0} & \mathbf{0} & \phi_{13} & \mathbf{0}\\ \mathbf{0} & L & \mathbf{0} & \mathbf{0} & \mathbf{0} & \phi_{14}\end{bmatrix},$$

$$\Pi_{13}=$$

$$\begin{bmatrix} t(LA_1^T+Y^TB_1^T) & LA_1^T+Y^TB_1^T-L & LA_1^T+Y^TB_1^T\\ t(LA_2^T+Y^TB_2^T) & LA_2^T+Y^TB_2^T & LA_2^T+Y^TB_2^T-L\\ tLA_{1d}^T & LA_{1d}^T & LA_{1d}^T\\ tLA_{2d}^T & LA_{2d}^T & LA_{2d}^T\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},$$

$$\Pi_{14}=\begin{bmatrix} L & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & L & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}^T,\ \Pi_{23}=\begin{bmatrix} B_1^T & B_1^T & B_1^T\\ B_2^T & B_2^T & B_2^T\end{bmatrix},$$

$$\Pi_{22}=diag\begin{bmatrix}-\alpha I & -\alpha I\end{bmatrix},$$

$$\Pi_{33}=diag\begin{bmatrix}-tL & -d_{1H}^{-2}X_1 & -d_{2H}^{-2}X_2\end{bmatrix},$$

$$\Pi_{44}=diag\begin{bmatrix}-\beta I & -\beta I\end{bmatrix},$$

$$\phi_{11}=-t_1L+(d_{1H}-d_{1L}+1)\bar{Q}_1+\bar{W}_1-LX_1^{-1}L,$$

$$\phi_{12}=-(t-t_1)L+(d_{2H}-d_{2L}+1)\bar{Q}_2+\bar{W}_2-LX_2^{-1}L,$$

$$\phi_{13}=-\tilde{W}_1-X_1,\ \phi_{14}=-\tilde{W}_2-X_2,$$

and choose the gain of the control law in (4) as

$$K=YL^{-1} \tag{7}$$

then the closed-loop 2D system (5) is 2D BIBO controllable.

Proof. Choose the following Lyapunov function candidate

$$V(i+\theta,j+\tau)=V_h(i+\theta,j+\tau)+V_v(i+\theta,j+\tau) \tag{8}$$

where

$$V_h(i+\theta,j+\tau)=\sum_{l=1}^{5}V_l(i+\theta,j+\tau),$$

$$V_v(i+\theta,j+\tau)=\sum_{l=6}^{10}V_l(i+\theta,j+\tau)$$

$$V_1(i+\theta,j+\tau)=x^T(i+\theta,j+\tau)t_1Px(i+\theta,j+\tau),$$

$$V_2(i+\theta,j+\tau)=\sum_{r=i+\theta-d_1(i+\theta)}^{i+\theta-1}x^T(r,j+\tau)Q_1x(r,j+\tau),$$

$$V_3(i+\theta,j+\tau)=\sum_{r=i+\theta-d_{1H}}^{i+\theta-1}x^T(r,j+\tau)W_1x(r,j+\tau),$$

$$V_4(i+\theta,j+\tau)=\sum_{s=-d_{1H}}^{-d_{1L}}\sum_{r=i+\theta+s}^{i+\theta-1}x^T(r,j+\tau)Q_1x(r,j+\tau),$$

$$V_5(i+\theta,j+\tau)=d_{1H}\sum_{s=-d_{1H}}^{-1}\sum_{r=i+\theta+s}^{i+\theta-1}\eta^T(r,j+\tau)R_1\eta(r,j+\tau),$$

$$V_v(i+\theta,j+\tau)=\sum_{l=6}^{10}V_l(i+\theta,j+\tau),$$

$$V_6(i+\theta,j+\tau)=x^T(i+\theta,j+\tau)(t-t_1)Px(i+\theta,j+\tau),$$

$$V_7(i+\theta,j+\tau)=\sum_{k=j+\tau-d_2(j+\tau)}^{j+\tau-1}x^T(i+\theta,k)Q_2x(i+\theta,k),$$

$$V_8(i+\theta,j+\tau)=\sum_{k=j+\tau-d_{2H}}^{j+\tau-1}x^T(i+\theta,k)W_2x(i+\theta,k),$$

$$V_9(i+\theta,j+\tau)=\sum_{s=-d_{2H}}^{-d_{2L}}\sum_{k=j+\tau+s}^{j+\tau-1}x^T(i+\theta,k)Q_2x(i+\theta,k),$$

$$V_{10}(i+\theta,j+\tau)=d_{2H}\sum_{s=-d_{2H}}^{-1}\sum_{k=j+\tau+s}^{j+\tau-1}\eta^T(i+\theta,k)R_2\eta(i+\theta,k),$$

$$\eta(r,j)=x(r+1,j)-x(r,j),$$

$$\eta(i,k)=x(i,k+1)-x(i,k),$$

P, Q_1, Q_2, W_1, W_2, R_1 and R_2 are positive definite matrices to be determined, and t_1 and t are given positive scalars. Design

$$\begin{aligned}\Delta V(i+1,j+1)&=V_h(i+1,j+1)-V_h(i,j+1)\\&\quad+V_v(i+1,j+1)-V_v(i+1,j)\\&=\sum_{l=1}^{10}\Delta V_l(i+1,j+1)\end{aligned} \tag{9}$$

We can obtain combined from Lemma 1 and (9) that

$$\Delta V(i+1,j+1)\le\varphi^T(i,j)\Psi\varphi(i,j)+\alpha\varphi_4^T(i,j)\varphi_4(i,j) \tag{10}$$

$$\Psi=\Psi_1+\Lambda_1^TtP\Lambda_1+d_{1H}^2\Lambda_2^TR_1\Lambda_2+d_{2H}^2\Lambda_3^TR_2\Lambda_3 \tag{11}$$

where

$$\varphi^T(i,j)=\begin{bmatrix}\varphi_1^T(i,j) & \varphi_2^T(i,j) & \varphi_3^T(i,j) & \varphi_4^T(i,j)\end{bmatrix},$$

$$\varphi_1^T(i,j)=\begin{bmatrix}x^T(i,j+1) & x^T(i+1,j)\end{bmatrix},$$

$$\varphi_2^T(i,j)=\begin{bmatrix}x^T(i-d(i),j+1) & x^T(i+1,j-h(j))\end{bmatrix},$$

$$\varphi_3^T(i,j)=\begin{bmatrix}x^T(i-d_{1H},j+1) & x^T(i+1,j-d_{2H})\end{bmatrix},$$

$$\varphi_4^T(i,j)=\begin{bmatrix}r^T(i,j+1) & r^T(i+1,j)\end{bmatrix},$$

$$\Psi_1 = \begin{bmatrix} \Psi_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \Psi_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} \\ * & * & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & -Q_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & -W_1-R_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -W_2-R_2 & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -\alpha I & \mathbf{0} \\ * & * & * & * & * & * & * & -\alpha I \end{bmatrix}$$

$\Psi_{11} = -t_1 P + (d_{1H} - d_{1L} + 1)Q_1 + W_1 - R_1$,

$\Psi_{22} = -(t - t_1)P + (d_{2H} - d_{2L} + 1)Q_2 + W_2 - R_2$,

$\Lambda_1 = \begin{bmatrix} A_{1K} & A_{2K} & A_{1d} & A_{2d} & \mathbf{0} & \mathbf{0} & B_1 & B_2 \end{bmatrix}$,

$\Lambda_2 = \begin{bmatrix} (A_{1K} - \mathrm{I}) & A_{2K} & A_{1d} & A_{2d} & \mathbf{0} & \mathbf{0} & B_1 & B_2 \end{bmatrix}$,

$\Lambda_3 = \begin{bmatrix} A_{1K} & (A_{2K} - I) & A_{1d} & A_{2d} & \mathbf{0} & \mathbf{0} & B_1 & B_2 \end{bmatrix}$.

Define $L = P^{-1}$, $X_1 = R_1^{-1}$, $X_2 = R_2^{-1}$, $LW_1L = \bar{W}_1$, $LW_2L = \bar{W}_2$, $LQ_1L = \bar{Q}_1$, $LQ_2L = \bar{Q}_2$, $X_1W_1X_1 = \tilde{W}_1$, $X_2W_2X_2 = \tilde{W}_2$, pre- and post-multiply (6) using $diag[P,\ P,\ P,\ P,\ R_1,\ R_2,\ I,\ I,\ P,\ R_1,\ R_2, I,\ I]$, $Y = KL$, and use Schur complements, the inequality (6) is transformed into the following inequality

$$\Psi \le -\beta^{-1} \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & * & \mathbf{0} \end{bmatrix} \tag{12}$$

Hence, for any $\varphi(i,j)$ and $\beta^{-1} = \varepsilon$, the following inequality hold:

$$\Delta V(i+1, j+1) \le -\varepsilon\varphi_1^T(i,j)\varphi_1(i,j) + \alpha\varphi_4^T(i,j)\varphi_4(i,j)$$

Let

$$M_1 = m\|\varphi\|_{d(i),d(j)}^2 \text{ and } M_2 = m\left(\frac{b}{a}\right)^2 \tag{13}$$

where

$$m = \Big[t\lambda_{\max}(P) + d_{1H}(\lambda_{\max}(Q_1) + \lambda_{\max}(W_1)) + d_{2H}(\lambda_{\max}(Q_2) + \lambda_{\max}(W_2)) + \frac{(d_{1H}+d_{1L})(d_{1H}-d_{1L})}{2}\lambda_{\max}(Q_1) + \frac{(d_{2H}+d_{2L})(d_{2H}-d_{2L})}{2}\lambda_{\max}(Q_2) + 2d_{1H}^2(d_{1H}+1)\lambda_{\max}(R_1) + 2d_{2H}^2(d_{2H}+1)\lambda_{\max}(R_2)\Big],$$

$$\|\varphi\|_{d(i),d(j)}^2 = \sup_{\substack{-d_{1H}\le i\le 0,\ j=0,1,2,\cdots or \\ -d_{2H}\le j\le 0,\ i=0,1,2,\cdots}} \|x(i,j)\|^2 .$$

Now consider

$$V(i,j) \le M = M_1 + M_2 \tag{14}$$

And for all $i \ge i_0$ $(i_0 = -d_{1H},\ -d_{1H}+1, \cdots, 0)$, $j = 0,\ 1, \cdots$ or $j \ge j_0$ $(j_0 = -d_{2H},\ -d_{2H}+1, \cdots, 0)$, $i = 0,\ 1, \cdots$, assume that $V(i,j) \le V_h(i_0, j) + V_v(i, j_0)$ holds. Then

$$\begin{aligned} V(i,j) &\le V_h(i_0,j) + V_v(i,j_0) \\ &\le \Big[t\lambda_{\max}(P) + d_{1H}(\lambda_{\max}(Q_1) + \lambda_{\max}(W_1)) + d_{2H}(\lambda_{\max}(Q_2) \\ &\quad + \lambda_{\max}(W_2)) + \frac{(d_{1H}+d_{1L})(d_{1H}-d_{1L})}{2}\lambda_{\max}(Q_1) \\ &\quad + \frac{(d_{2H}+d_{2L})(d_{2H}-d_{2L})}{2}\lambda_{\max}(Q_2)\Big]\|\varphi\|_{d(i),d(j)}^2 \\ &\quad + \frac{d_{1H}^2(d_{1H}+1)}{2}\lambda_{\max}(R_1)\|\eta(r,j)\|^2 \\ &\quad + \frac{d_{2H}^2(d_{2H}+1)}{2}\lambda_{\max}(R_2)\|\eta(i,k)\|^2 \end{aligned} \tag{15}$$

Yet for $r \in \{-d_{1H},\ -d_{1H}+1, \cdots, -1\}$ and $k \in \{-d_{2H},\ -d_{2H}+1, \cdots, -1\}$, $\|\eta(r,j)\|^2 \le (\|x(r+1,j)\| + \|x(r,j)\|)^2 \le 4\|\varphi\|_{d(i),d(j)}^2$ and $\|\eta(i,k)\|^2 \le (\|x(i,k+1)\| + \|x(i,k)\|)^2 \le 4\|\varphi\|_{d(i),d(j)}^2$ hold, which shows that $V(i,j) \le M_1$ holds.

If $V(i,j) \le V_h(i_0,j) + V_v(i,j_0)$ doesn't hold, then there exists scalars s and t, when $i \ge i_0$ and $j \ge j_0$, and for $\forall s \in [i_0, i)$, and $\forall t \in [j_0, j)$, $V(i,j) \ge V_h(s,j) + V_v(i,t)$ holds. Thus

$$0 \le \Delta V(i+1, j+1) \le -\varepsilon\varphi_1^T(i,j)\varphi_1(i,j) + \alpha\varphi_4^T(i,j)\varphi_4(i,j) \tag{17}$$

So

$$\|\varphi_1\| \le \frac{\alpha}{\varepsilon}\|\varphi_4\| \tag{16}$$

That is

$$\|x(i,j)\| \le \frac{b}{a} = \frac{\alpha}{\varepsilon}\|r(i,j)\| \tag{17}$$

Similar to the above proof, one obtains that

$$V(i,j) \le m\left(\frac{b}{a}\right)^2 \tag{18}$$

In summary

$$V(i,j) \le M = M_1 + M_2 \tag{19}$$

Because

$$t\lambda_{\min}(P)\|x(i,j)\|^2 \le V(i,j) \le M \quad (20)$$

It is seen that

$$\|x(i,j)\| \le \sqrt{\frac{m\left(\|\varphi\|_{d(i),d(j)}^2 + \left(\frac{b}{a}\right)^2\right)}{t\lambda_{\min}(P)}} \le \sqrt{\frac{m}{t\lambda_{\min}(P)}}\left(\|\varphi\|_{d(i),d(j)} + \frac{b}{a}\right)$$

Thus

$$\|y(i,j)\| = \|C\|\|x(i,j)\| \le \|C\|\sqrt{\frac{m}{t\lambda_{\min}(P)}}\|\varphi\|_{d(i),d(j)} + \|C\|\sqrt{\frac{m}{t\lambda_{\min}(P)}}\frac{b}{a}$$

Let

$$\theta_1 = \frac{\|C\|\alpha}{\varepsilon}\sqrt{\frac{m}{t\lambda_{\min}(P)}}, \theta_2 = \|C\|\sqrt{\frac{m}{t\lambda_{\min}(P)}}\|\varphi\|_{d(i),d(j)}$$

For $\forall r(i,j) \in L_{\infty,\infty}^n$, we have

$$\|y(i,j)\| \le \theta_1\|r\|_{\infty,\infty} + \theta_2 \quad (21)$$

This completes the proof of Theorem 1.

Remark 1. The ultimate goal of the above theorem is to get sufficient conditions for the existence of the control gain of the 2D BIBO control law $u(i,j)$. With such conditions, we can access the control law (7) by combining with the convex optimal algorithm used in [15] if there exist nonlinear elements, such as $LX_i^{-1}L$ (i=1, 2), and so on. Thus, the 2D BIBO law with the form (4) is obtained. On the other hand, note that if $r(i,j)=0$, sufficient conditions on BIBO stability will be transformed into asymptotical stability. Moreover, if the interval time delay is reduced to a constant value, the above conditions will be transformed into sufficient delay-dependent stability conditions. Only for Lyapunov–Krasovskii function can Q_1 and Q_2 be chosen as zero matrices.

Remark 2. In practice, it is inevitable that the system is disturbed by other external factors, which shows that considering BIBO control of the uncertainty delay system is certainly of theoretical significance and practical value. If the uncertainties are considered here, we can easily obtain the results by using Theorem 1.

IV. AN NUMERICAL EXAMPLE

To demonstrate the effectiveness of the design method, a causal 2D system with time delays is chosen with the following form [16]:

$$\begin{cases} y(m+1,n+1) = a_1 y(m+1,n) + a_2 y(m,n+1) \\ \qquad -a_1 a_2 y(m,n) + a_3 y(m+1, n-d_2(n)) \\ \qquad + a_4 y(m-d_1(m), n+1) + u(m,n) \end{cases} \quad (22)$$

Denote $x^T(i,j) = \begin{bmatrix} y^T(m,n+1) - a_1 y^T(m,n) & y^T(m,n)\end{bmatrix}$, and assume that the measurement output is given by $y(i,j) = Cx(i,j)$. It is easy to see that Eq. (22) can be converted to the state-space system (1) with

$$A_1 = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 1 & a_1 \end{bmatrix}, A_{1d} = \begin{bmatrix} a_4 & a_1 a_4 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$A_{2d} = \begin{bmatrix} a_3 & a_1 a_3 \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 1 \end{bmatrix}.$$

Here we will consider two cases to analyze the impact of time delay on the stability of the system.

(1) A zero-mean stationary random field

As shown in [15], if $0 < a_1, a_2 < 1$ and $0 < a_3, a_4 < 1$, Eq. (22) is a system with a zero-mean stationary random field, where $u(m,n)$ is an uncorrelated sequence with zero mean. Let $a_1 = 0.2$, $a_2 = 0.3$, $a_3 = 0.15$, $a_4 = 0.03$. Solving Theorem 1, the delays with the ranges of $1 \le d(i) \le 6$ and $1 \le d(j) \le 4$ can be obtained. Then the system is asymptotically stable. However, when the elements that affect the stability of the system in matrix A_{1d} or A_{2d} is chosen sufficient large, for example, $a_4 = 0.5$ and others remain unchanged, the output of the system diverges. The output response is convergent again if we design a controller with the zero reference input by solving Theorem 1, which is $u(i,j) = \begin{bmatrix} -0.1254 & -0.2020 \end{bmatrix} x(i,j)$ as shown in Figure 1. This verifies that the time delay item may cause the system to be unstable. The system can be controlled by the proposed controller in this paper. If the reference input is chosen as a non-zero value but is bounded $r(i,j) \ne 0$, it is easily obtained that the output response will be convergent, which is shown in Figure 2.

Remark 3. Under the condition $r(i,j) = 0$, the results seem similar to those in [7-9]. It is obvious that they are different. In [7-9], the authors considered the stability of the batch process. For this kind of systems, the state must run in the finite time. On the other hand, for some general 2D systems, the difference is not obvious if $i \in [0, x_f]$, where x_f is constant value.

(2) A non-stationary random field

Let $a_2 = 1.3$ and other elements remain unchanged., the system is still unstable. Deign the controller with the non-zero reference input and solve Theorem 1. The controller is obtained as $u(i,j) = \begin{bmatrix} -1.0978 & -0.0083 \end{bmatrix} x(i,j)$ and $r(i,j)$ is chosen by $r(i,j) = 1$. We can see the system is BIBO stable, as shown in Figure 3. If $r(i,j)$ is chosen by $r(i,j) = 0$, the BIBO stability is transformed into asymptotically stability.

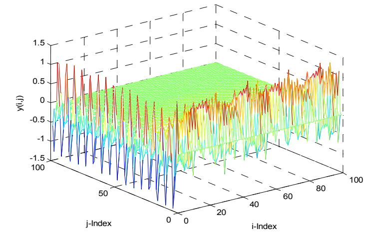

Figure 1. Output responses affected by the time-delay item and with the proposed controller $u(i,j)=[-0.1254 \quad -0.2020]x(i,j)$

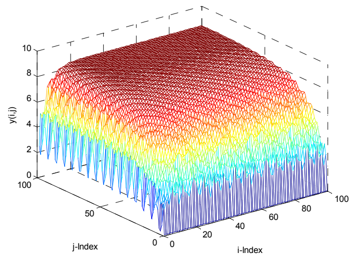

Figure 2. Output responses affected by the time-delay item and with the proposed controller $u(i,j)=[-0.1254 \quad -0.2020]x(i,j)+1$

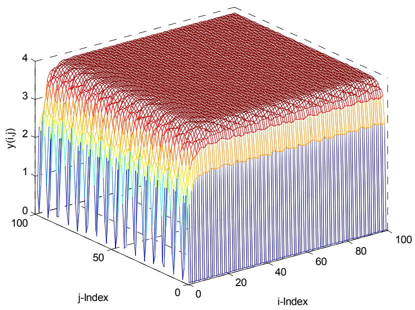

Figure 3. Output responses with the controller $u(i,j)=[-1.0978 \quad -0.0083]x(i,j)+1$

V. CONCLUSION

Based on LMI methods, a solution to the design problem of a 2D BIBO controller is proposed for a class of two-dimensional (2D) discrete delayed systems described by the FM second state-space model with interval time-varying delay. By choosing a 2D Lyapunov function and adding a differential inequality to the Lyapunov function, the proposed conditions are formulated in terms of LMIs, which can be extended to the uncertain case. Furthermore, the relation of BIBO stability with asymptotical stability is expatiated. The results may be converted to constant delay case without loss of generality. The illustrative example of a causal 2D system has demonstrated the effectiveness and merits of the proposed method.

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