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On Dubins Paths to Intercept a Moving Target at a Given Time ⋆

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Abstract: In this work we examine the problem of intercepting a moving target with a pursuer that only moves forwards at constant speed and whose radius of turn is bounded from below. We assume that the target is moving on a straight line at constant speed and that the target's velocity is known to the pursuer with no measurement error. We analyze both the problem of minimum-time interception, as well as the problem of interception at a predefined time. We establish lower and upper bounds on the minimum time to interception that are easy to compute. We examine the relation between shortest paths and minimum time interception paths, give conditions for the two types of paths to coincide and show cases where they differ. Finally, we propose two algorithms for the elongation of an admissible path and provide conditions that guarantee continuous elongation. The above analysis is also conducted in scenarios where the target is located near or inside the circles of minimum turning radius that correspond to the pursuer's initial configuration.

Keywords: Dubins, moving target, minimum time interception, interception at a given time.

1. INTRODUCTION

We consider the problem of interception of a non-maneuvering target by a pursuer that only moves forwards in constant speed and whose radius of turn is bounded from below. The target is assumed to move at constant speed on a trajectory that is known to the pursuer. In this work we wish to gain a better understanding of the minimum-time interception problem and the ability to continuously elongate minimum-time paths in order to have control on the interception time.

A car-like robot that only moves forwards with constant speed on a path of bounded curvature, is often called a Dubins vehicle. It is named after L.E. Dubins (1957) who studied planar continuously differentiable shortest paths between fixed initial and final positions and orientations. Dubins proved that such curves exist and are necessarily a sub-path of a path of type CSC or of type CCC, where S is a straight line segment and C is an arc of a circle whose radius is the vehicle's minimum turning radius r. If C describes a clockwise (resp. counter-clockwise) turn it will be replaced by R (resp. L). Thus, the shortest path for a Dubins vehicle from any initial to any final configuration belongs to the set of 6 admissible paths $\mathcal{D} = \{LSL, RSR, RSL, LSR, RLR, LRL\}$. The Dubins vehicle model could be used as a simplified representation of an uninhabited aerial vehicle(UAV), robot or missile whose motion is planar.

We call the problem of finding a shortest path for a Dubins vehicle without a terminal angle constraint the "relaxed Dubins" problem. Boissonnat and Bui (1994) formulated the optimal control problem with a free terminal angle. The transversality necessary condition for optimality, together with the fact that all line segments and inflection points must lie on the same straight line, imply that the set of six candidates for optimal paths for the constrained terminal angle problem, reduces to a set of four possible paths for the relaxed version of the problem: $\mathcal{RD} = \{RS, LS, RL, LR\}$.

Dubins and relaxed Dubins optimal paths have been considered also for the interception of a moving target. Under the assumptions of a pursuer modeled as a Dubins vehicle and a constant velocity target, Looker (2008) suggests a search algorithm for finding the shortest CS path to interception. The suggested algorithm is based on a numeric solution for a single implicit equation for the minimum time to interception, developed from a rigorous analysis of the model constraints.

Bhatia and Frazzoli (2008) examine the rendezvous problem for a team of Dubins vehicles. For a pre-assigned destination point far enough (four times the length of the vehicle's minimum turning radius) from all team members, they propose a decentralized approximation algorithm for minimum-time rendezvous with equal separation angles between successive team members at the destination.

In this paper, we analyze the problems of minimum-time interception and the problem of interception at a predefined time. Because many interesting phenomena occur when the target is located near or inside the pursuer's circle of minimum turning radius, we analyse such sce-

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narios thoroughly. We give lower and upper bounds on the minimum time to interception that are easy to compute, examine the relation between shortest paths and minimum-time interception paths, and we show that shortest paths may not be the optimal strategy for achieving a minimum-time interception. We propose two algorithms for path elongation and show that paths cannot always be elongated continuously.

2. PROBLEM FORMULATION

We consider the problem of a pursuer modeled as Dubins vehicle trying to intercept a moving, yet non-maneuvering target, at a predefined time. We assume that:

(1) The velocities of the pursuer and target are coplanar.
(2) The target travels on a straight line at constant speed.
(3) The pursuer is modeled as a Dubins vehicle.
(4) There are no obstacles.
(5) The pursuer has full information about the target's future trajectory and speed.

The kinematic equations of a Dubins vehicle are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} V cos\alpha \\ V sin\alpha \\ u\dfrac{V}{r} \end{pmatrix}, \tag{1}$$

with (x, y) being the inertial position coordinates, α is the orientation, measured counter-clockwise from the x-axis, V is the speed, r is the minimum turning radius and u is the control satisfying $|u| \leq 1$.

The subscripts T and P, used throughout this paper, refer to the target and pursuer, respectively. We use the notation $\Omega \in \mathbb{R}^3$ for a configuration, which is a position and orientation triplet expressed in some inertial Cartesian frame. We denote an inertial position with no orientation constraint by $\omega \in \mathbb{R}^2$.

Now, we formulate the problem of interception at a predefined time as an optimal control problem. Without loss of generality we assume the scenario starts at time $t = 0$ with the configurations of the pursuer and the target being $\Omega_P^0 = (x_P^0, y_P^0, \alpha_P^0)$ and $\Omega_T^0 = (x_T^0, y_T^0, \alpha_T^0)$, respectively. The scenario ends at the predefined time $t_{req} \geq 0$. The solution to the problem will be a control that minimizes the miss distance between the adversaries at t_{req}. We are, thus, lead to following optimal control problem.

Minimize the cost

$$J = (x_P(t_{req}) - x_T(t_{req}))^2 + (y_P(t_{req}) - y_T(t_{req}))^2, \tag{2}$$

subject to the constraints

$$\begin{pmatrix} \dot{x}_P \\ \dot{y}_P \\ \dot{\alpha}_P \\ \dot{x}_T \\ \dot{y}_T \\ \dot{\alpha}_T \end{pmatrix} = \begin{pmatrix} V_P cos\alpha_P \\ V_P sin\alpha_P \\ u\dfrac{V_P}{r} \\ V_T cos\alpha_T \\ V_T sin\alpha_T \\ 0 \end{pmatrix}, \tag{3}$$

$$|u| \leq 1, \tag{4}$$

$$\begin{aligned} \Omega_P(0) &= \Omega_P^0, \\ \Omega_T(0) &= \Omega_T^0. \end{aligned} \tag{5}$$

One may obtain a numerical solution for this non-linear optimal control problem, but in this work we aim at a better understanding of the problem, using an analytic approach. We divide the solution into two stages. First we seek the minimum time required for capture (zero-miss interception) denoted as t_{min}. Sufficient conditions for capture were suggested by Cockayne (1967); Section 3 covers those conditions and provides helpful guidelines for finding t_{min}. Once we know that capture is possible and we have t_{min}, we address in Section 4 the problem of interception at a predefined time $t_{req} > t_{min}$, using path elongation algorithms. Conclusions are presented in Section 5.

3. MINIMUM-TIME INTERCEPTION

We start by defining some helpful notation. We often use the splitting of the 2D space to the left-hand-side (LHS) and right-hand-side (RHS). LHS (resp. RHS) represents the half plane located to the left (resp. right) of the axis defined by Ω_P^0. We denote the left and right circles of minimum turning radius, tangent to the pursuer's initial configuration Ω^0, as D_L and D_R. The length of the shortest path of a Dubins vehicle from Ω^0 to Ω^f will be denoted by $Dub(\Omega^0, \Omega^f)$. The shortest path from Ω^0 to ω^f will be denoted by $RDub(\Omega^0, \omega^f)$. The trajectory of the target will be denoted by $\gamma(t) : [0, \infty) \to \mathbb{R}^2$. Given a time t, the image γ indicates the target's inertial position. The function $\tilde{f}_P^s(\omega) : \mathbb{R}^2 \to [0, \infty)$, denoted as a time-to-reach (TTR) function, represents the time it takes the pursuer to reach from its initial configuration to any point $\omega \in \mathbb{R}^2$, using some feasible path-planning strategy s. $\tilde{f}_P^*(\omega) = \dfrac{RDub(\Omega_P^0, \omega)}{V_P}$ is a TTR function of a pursuer that uses the relaxed Dubins shortest path strategy. The composition $\tilde{f}_P^s(\gamma(t))$ indicates the time it takes a pursuer that uses some feasible path-planning strategy s to reach the point $\gamma(t) \in \mathbb{R}^2$ on the target's trajectory. Since the trajectory of the target is fixed (Assumption 2), we simplify the notation by setting $f_P^s(t) \triangleq \tilde{f}_P^s(\gamma(t))$. The TTR function of the target $\tilde{f}_T(\cdot) : \gamma(t) \to [0, \infty)$ is actually the left inverse of $\gamma(t)$ and thus we get $f_T(t) \triangleq \tilde{f}_T(\gamma(t)) = t : [0, \infty) \to [0, \infty)$. We notice that all TTR functions are non-negative by definition and that $f_T(t)$ is a continuous monotonically strictly increasing function.

For a pursuer and a target modeled as Dubins vehicles, Cockayne (1967) shows that the pursuer will be able to capture the target from *any initial state* if and only if $V_P > V_T$ and $\dfrac{V_P^2}{r_P} \geq \dfrac{V_T^2}{r_T}$, where r_P and r_T are the minimum turning radii of the pursuer and target, respectively. In our scenario of interest, the target travels at constant velocity while the pursuer is free to maneuver (under its acceleration limitations), thus a speed advantage is a sufficient condition for capture. It is not a necessary condition, in general, however. Consider, for example, the case of adversaries that are initially aligned on a collision triangle. In that case the pursuer will capture the target (without maneuvering) even if it is inferior in the sense of speed. In terms of the TTR functions, when interception is possible there exist a strategy s and a non-negative real number $\hat{t}$ such that: $f_P^s(\hat{t}) = f_T(\hat{t})$.

Theorem 1. If the target's trajectory does not enter neither one of the two circles of minimum turning radius, tangent the pursuer's initial velocity, f_P^* is a continuous function.

Proof. Under the above constraint on the target's trajectory, both functions that compose $f_P^*(t) = \tilde{f}_P^*(\gamma(t))$ are continuous. The continuity of γ is an assumption based on the fact that it describes a trajectory of a real object. It can be shown that the function $\tilde{f}_P^*(\omega) : \mathbb{R}^2 \setminus D_L \cup D_R \to [0, \infty)$ can be expressed in terms of continuous functions. Continuity is maintained even when the target crosses from the LHS to the RHS of the plane or vice versa.

Lemma 2. A relaxed Dubins shortest path reaches every point on the target's trajectory in minimum time.

Proof. Implied directly from the optimality of the relaxed Dubins shortest paths shown by Boissonnat and Bui (1994).

Lemma 3. If f_P^* is a continuous function and interception is possible, there must exist a $\hat{t}$ such that: $f_P^*(\hat{t}) = f_T(\hat{t})$.

Proof. Interception is achieved at t_1 that satisfies

$$f_P^s(t_1) = f_T(t_1). \tag{6}$$

From Lemma 2 we get that

$$f_P^*(t_1) \leq f_P^s(t_1) = f_T(t_1). \tag{7}$$

Assuming the pursuer and target start at different initial positions, we get

$$f_P^*(0) > f_T(0) = 0. \tag{8}$$

From (7), (8), and the continuity of f_P^* and f_T, we deduce that $\exists \hat{t} : 0 < \hat{t} \leq t_1$ and $f_P^*(\hat{t}) = f_T(\hat{t})$.

Theorem 4. If f_P^* is a continuous function and interception is possible, then the pursuer can intercept the target in minimum time by following a relaxed Dubins shortest path.

Proof. Under the specified assumptions, Lemma 3 implies that when interception is possible using some path planning strategy s, it must also be possible using the relaxed Dubins shortest path strategy, that is achieved for $\hat{t}$ that satisfies

$$f_P^*(\hat{t}) = f_T(\hat{t}). \tag{9}$$

We define t_1 as the minimal value that satisfies equation (9). Suppose there exists $t_2 < t_1$ that satisfies $f_P^s(t_2) = f_T(t_2)$ for $s \neq *$. Lemma 2 and the definition of t_1 imply that

$$f_P^*(t_2) < f_P^s(t_2) = f_T(t_2). \tag{10}$$

From (8), (10), and from the continuity of f_P^* and f_T we deduce that $\exists t_3 : 0 < t_3 < t_2 < t_1$ and $f_P^*(t_3) = f_T(t_3)$. That contradicts the definition of t_1.

Theorem 5. A minimum-time interception of a moving target may require the pursuer to use a strategy that will not result in the shortest path to the interception point.

Proof. Consider the example described in Figure 1. We choose the values of $r = 1$ and $V_P = 1$. By choosing the target's speed to be $V_T = \frac{\sqrt{3}}{4\pi} V_P$ we get that the target reaches point B in the same time it takes the pursuer to complete a 2π turn. Next, we show that the pursuer cannot intercept the target inside its left turning circle. For that, we consider relaxed Dubins shortest path strategy, suggesting a path of type $RL_{>\pi}$ from point C to each and every point $\gamma(t)$ between A and B. The corresponding TTR function would be $f_P^{RL_{>\pi}}(t)$. We show that these paths are too long to enable interception inside the pursuer's left turning circle. The path marked with squares in Figure 1 is the shortest path from C to A. The pursuer will surely be late to meet the target at point A because the target leaves A at $t = 0$, when the scenario starts:

$$f_P^{RL_{>\pi}}(0) > f_T(0) = 0. \tag{11}$$

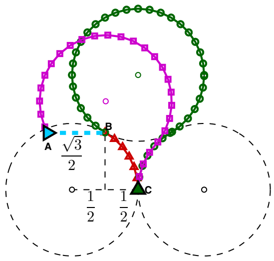
The values for the problem parameters we chose above impose that interception is achieved if the pursuer travels on the path marked with circles, a path of length 2π. We denote the time it takes the target to reach point B as t_B and thus we have

$$f_P^{RL_{>\pi}}(t_B) = f_T(t_B). \tag{12}$$

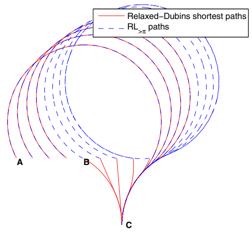
The function $f_P^{RL_{>\pi}}$ is a monotonically strictly increasing function for $0 \leq t \leq t_B$ because as the final point of the path gets closer to point B, both segments for the right and left turns get longer. We also recall that f_T is a linear function. We can consider the $RL_{>\pi}$ path planing strategy to points on the target's trajectory as if each point $\gamma(t)$ corresponds to a roll angle of the left turning circle over the fixed right turning circle (see Figure 1b). Ignoring its roll, we can decouple the translation of the left turning circle and say that it moves upwards and right. The right motion does not effect the length of the left turn because the target's path is parallel to that direction. The upwards motion does in fact elongate the length of the left turn segment. As the left turning circle rolls over the fixed right turning circle, the component of the upwards motion decreases and so does the rate of the left turn length elongation, suggesting that $f_P^{RL_{>\pi}}$ is a concave function. We can conclude by saying that $f_P^{RL_{>\pi}}$ and f_T do not intersect at any point $\gamma(\hat{t})$ satisfying $\hat{t} < t_B$.

To summarize, we showed that the pursuer cannot intercept the target inside its left turning circle. Minimum-time interception is achieved at point B on the path marked with circles, whereas the relaxed Dubins shortest path to point B is the path marked with triangles, reaching point B before the target does and thus is not an interception path.

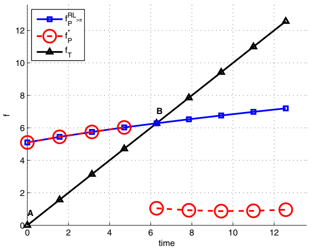
Figure 2 illustrates another interception scenario for which $r = 1$ and $V_P = 1$, and $V_T = \frac{2\sqrt{3}}{5} V_P$. We consider six different path planning strategies for the pursuer, specified in the legend of Figure 2b. Among the covered strategies, minimum-time interception is achieved using the $RL_{<\pi}$ strategy which does not belong to the $\mathcal{RD}$ set of shortest path candidates, suggesting that the set of minimum-time interception path candidates does not coincide with the set of shortest path candidates.



(a) The counter-example used in the proof of Theorem 5. The target is heading from point A towards point B on a straight line. The pursuer is initially located at point C, heading upwards.

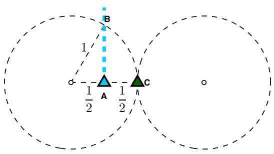


(b) Paths of the pursuer to points on the target's trajectory.

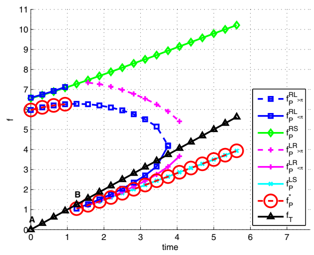


(c) TTR functions – the discontinuity of f_P^* occurs at t_B. $f_P^{RL_{>\pi}}$ is continuous in the examined domain and intersects f_T at t_B.

Fig. 1. $r = 1$, $V_P = 1$, $V_T = \frac{\sqrt{3}}{4\pi} V_P$ - Minimum time interception is achieved with a path longer than the shortest path to the interception point.



(a) Scenario Configuration - The target is heading from point A towards point B. The pursuer is initially located at point C, heading upwards.



(b) TTR functions - considering the specified strategies, $f_P^{RL_{<\pi}}$ is the first to intersect f_T.

Fig. 2. $r = 1$, $V_P = 1$, $V_T = \frac{2\sqrt{3}}{5} V_P$ - Minimum time interception is achieved using the $RL_{<\pi} \notin \mathcal{RD}$ strategy.

Proposition 6. For a pursuer having a speed and a maneuverability advantage over its target $t_{min} \leq \dfrac{Dub(\Omega_P^0, \Omega_T^0)}{V_P - V_T}$.

Proof. If the pursuer follows the target's trajectory, it will eventually capture the target due to its speed advantage. The pursuer can align its velocity with the target's trajectory by following a Dubins path from Ω_P^0 to Ω_T^0. It takes the pursuer $\dfrac{Dub(\Omega_P^0, \Omega_T^0)}{V_P}$ units of time to complete this path. During that time, the target travels $\dfrac{V_T}{V_P} Dub(\Omega_P^0, \Omega_T^0)$ distance units. This distance is then closed with the relative speed of the pursuer and target: $V_P - V_T$. To summarize, the total duration of the path described above is:

$$Dub(\Omega_P^0, \Omega_T^0)\left[\frac{1}{V_P} + \frac{V_T}{V_P(V_P - V_T)}\right] = \frac{Dub(\Omega_P^0, \Omega_T^0)}{V_P - V_T}.$$

The following lemma will be used to establish a lower bound on t_{min} while considering a target that follows a straight line trajectory.

Lemma 7. Assuming that interception is possible, a pursuer with no acceleration constraints and constant speed achieves minimum-time interception of a non-maneuvering target by heading straight to the interception point, namely moving on the smallest available [1] collision triangle defined by the adversaries' relative geometry and their speed ratio.

Proof. Let us assume that minimum time interception is achieved by a pursuer taking a non-straight path to the interception point, we indicate this strategy by $NStrt$. We mark the interception point by $\gamma(t_1)$:

$$f_P^{NStrt}(t_1) = f_T(t_1). \tag{13}$$

A pursuer heading straight (a strategy indicated by $Strt$) towards point $\gamma(t_1)$ would surely precede the $NStrt$ pursuer and the target to that point, because a straight line is the shortest path between two points

$$f_P^{Strt}(t_1) < f_P^{NStrt}(t_1) = f_T(t_1). \tag{14}$$

Assuming the pursuer and target start at different initial positions we get

$$f_P^{Strt}(0) > f_T(0) = 0. \tag{15}$$

We notice that f_P^{Strt} is a continuous function and we recall that f_T is also continuous. Combining the property of continuity with equations (14) and (15) we get that $\exists \hat{t} : 0 < \hat{t} < t_1$ and $f_P^{Strt}(\hat{t}) = f_T(\hat{t}) \equiv t_{SCT}$.[2] That contradicts the definition of $\gamma(t_1)$ as the point of minimum time interception. $\hat{t}$ can easily be derived by finding the smallest available collision triangle, which is defined by the adversaries' relative geometry and their speed ratio.

Corollary 8. For any pursuer modeled as a Dubins vehicle that sets to intercept a constant-speed, non-maneuvering target, $t_{min} \geq t_{SCT}$.

Proof. A lower bound can be obtained by relaxing some of a problem's constraints. In this case we relax the acceleration constraint that the Dubins vehicle is subjected to and the corollary follows from Lemma 7.

4. INTERCEPTION AT A GIVEN TIME

We now address the problem formulated in Section 2 of interception at a predefined time t_{req}. We focus on the interesting scenario where $t_{req} \geq t_{min}$.

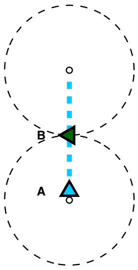
First, we notice that if a Dubins vehicle can reach a point at time $\hat{t}$, it does not mean that it can reach that point at any given time $t > \hat{t}$, as can be illustrated in the scenario described in Figure 3. Furthermore, this example shows

[1] For a target with a speed advantage over the pursuer, it was shown by Shima (2011) that, when interception is possible, the pursuer can choose between two admissible collision triangles.

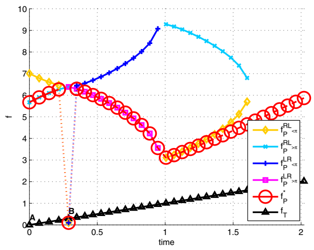
[2] SCT stands for Smallest Collision Triangle.

that the existence of a point on the target's trajectory, for which the pursuer can precede the target, is not a sufficient condition for capture, equivalently,

$$\exists t_1 : f_P^s(t_1) < f_T(t_1) \not\Rightarrow \exists t_2 : f_P^s(t_2) = f_T(t_2). \quad (16)$$



(a) Scenario Configuration - The target is heading from point A towards point B. The pursuer is initially located a small distance, ϵ, to the right of point B and heading towards it.



(b) TTR functions - f_P^*'s discontinuity occurs at t_B.

Fig. 3. $r = 1$, $V_P = 1$, $V_T = 3V_P$ - The pursuer gets to a singular region (that gets smaller with ϵ getting smaller) on the target's trajectory before the target does, but will never capture it.

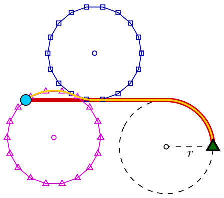
In view of the last scenario, we provide conditions under which the implication in (16) does hold.

Lemma 9. For given initial configuration and final position $\{\Omega^0,\omega^f\}$ and a CS path connecting them, if the length of the straight line segment satisfies $|S| \geq r\sqrt{8}$ it is possible to extend the path length by an arbitrary length $\Delta x > 0$.

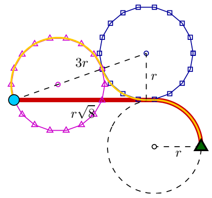
Proof. We follow the approach of Bhatia and Frazzoli (2008) who give a similar proof for the case of a fixed terminal angle. Extending the path length by $2\pi rn, n \in \mathbb{N}$, is trivial by making n loops around a point on the original path. Let us assume, then, that $\Delta x = mod(\Delta x, 2\pi r)$. We suggest the path modification shown in Figure 4 for an LS path: take a right turn on the circle marked with squares followed by a left turn on the circle marked with triangles (the same approach is applicable for the modification of an RS path). Consider the circle marked with triangles as if it were connected to a hinge located on the final position. We can rotate the circle marked with triangles counter-clockwise around the hinge like the sequence described in Figures 4a→ 4b→4c→ 4d. The circle marked with squares sits on top of the straight line and is tangent to the circle marked with triangles. As the circle marked with triangles moves from 4a to 4b the circle marked with squares rolls right on the straight line segment. The circle marked with squares rolls left when the circle marked with triangles moves from 4b through 4c to 4d. This method enables a continuous modification of the path length for $0 < \Delta x < 2\pi r$.

Figure 5a shows LHS $|S|$-loci of the final position. For every point outside the LHS $|S|$-locus a LS path would have a straight line segment longer than $|S|$. A RHS $|S|$-locus can be obtained by reflecting the LHS $|S|$- locus around the axis defined by Ω_P^0.

Lemma 10. The length of the Dubins path defined by the initial and final configurations $\{\Omega^0 = (x^0, y^0, \alpha^0), \Omega^f = (x^f, y^f, \alpha^0 + \pi)\}$ can be extended by an arbitrary length $\Delta x > 0$.



(a)



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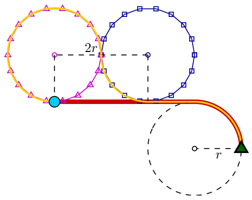

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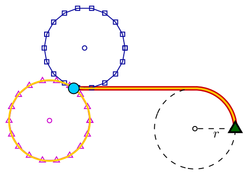
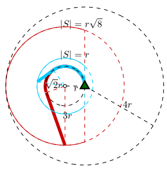
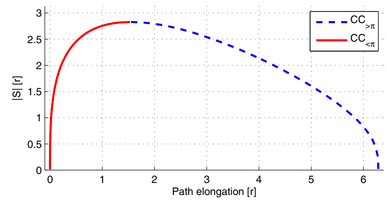


(d)

Fig. 4. Elongation of an LS path



(a) LHS $|S|$-loci of the final position



(b) The straight line segment length $|S|$ required for the elongation of a CS path according to the method described in the proof of Lemma 9.

Fig. 5. CS path - length of the straight line segment $|S|$

Proof. If the initial and final orientations are heading in opposite directions, to extend the path length by Δx one can add two straight line segments of length $\frac{\Delta x}{2}$ after the initial and before the final positions, aligned with α^0 and $\alpha^0 + \pi$, respectively. We can look at this method as if we are translating the initial and final configurations while maintaining the relative geometry.

Lemma 11. Given the initial configuration and final position $\{\Omega^0 = (\omega^0, \alpha^0), \omega^f\}$ satisfying $|\omega^f - \omega^0| \geq 4r$, the length of the relaxed Dubins shortest path connecting them can be extended by an arbitrary length $\Delta x > 0$.

Proof. Bhatia and Frazzoli (2008) showed that, given an initial configuration $\Omega^0 = (\omega^0, \alpha^0)$ and a final position ω^f satisfying $|\omega^f - \omega^0| \geq 4r$, the length of the optimal path for a Dubins vehicle is a continuous function of the final orientation α^f. The relaxed Dubins shortest path reaches the final position in an arbitrary orientation α_{RD}^f. We can refer to the final orientation as a constraint and describe a process of rotation of the final orientation from the value of α_{RD}^f towards $\alpha^0 + \pi$. For a given final orientation we can compute the Dubins shortest path. According to the theorem by Bhatia and Frazzoli, the process of rotation of the final orientation maintains a continuous modification of the length of the original path. If we met the desired path length during the rotation process, our goal is achieved. Otherwise, Lemma 10 implies that an arbitrary elongation of the path length can be obtained if the final orientation of the path is $\alpha^0 + \pi$.

We propose two algorithms for planning a path with a duration that equals t_{req}. We assume a Dubins vehicle with speed v and minimum turning radius r and $\omega^0 \neq \omega^f$. A successful finish of an algorithm is indicated by ***finish***, a failure is indicated by ***quit***.

Algorithm 1. Elongation 1

(1) Define an auxiliary variable x and $x \leftarrow 0$.
(2) Given an initial configuration $\Omega^0 = (\omega^0, \alpha^0)$ and a final position ω^f, compute a CS path according to the value of x:
  (a) $x = 0$: $x \leftarrow 1$. If ω^f is located on the LHS (resp. RHS) of the plane, outside D_L (resp. D_R), use a LS path (resp. RS) and continue to Step 3, otherwise return to Step 2.
  (b) $x = 1$: $x \leftarrow 2$. If ω^f is located on the LHS (resp. RHS) of the plane, use a RS path (resp. LS) and continue to Step 3.
  (c) $x = 2$: **quit**.
(3) Denote the length of the straight line segment and the duration of the path calculated in Step 2 as $|S|$ and t_{CS}, respectively. If $t_{req} \geq t_{CS}$, calculate the corresponding path elongation $\Delta x = (t_{req} - t_{CS})v$ and continue to Step 4, otherwise **quit**.
(4) Use Figure 5b to find a sufficient length of straight line segment $|S|_{suf}$ that corresponds to the required elongation $mod(\Delta x, 2\pi r)$.
(5) If $|S| \geq |S|_{suf}$ continue to Step 6, otherwise, return to Step 2.
(6) If $\Delta x \geq 2\pi r$ make a loop around the initial position, set Δx to be $\Delta x - 2\pi r$ and return to Step 6, otherwise continue to Step 7.
(7) If $\Delta x > 0$ continue to Step 8, otherwise **finish**.
(8) Elongate the path according to the method described in the proof of Lemma 9 and **finish**.

Algorithm 2. Elongation 2

(1) Given an initial configuration $\Omega^0 = (\omega^0, \alpha^0)$ and a final position ω^f, compute the relaxed Dubins shortest path and denote its duration as t_{RD}.
(2) If $t_{req} \geq t_{RD}$ calculate the corresponding path elongation $\Delta x = (t_{req} - t_{RD})v$ and continue to Step 3, otherwise t_{req} is unfeasible - **quit**.
(3) Compute the shortest Dubins path defined by the configuration's pair $\{\Omega^0, (\omega^f, \alpha^0 + \pi)\}$ and the difference $\widehat{\Delta x}$ between its length and the length of the path calculated in Step 1. If $\widehat{\Delta x} < \Delta x$ continue to Step 4, otherwise jump to Step 5.
(4) Add two straight line segments of length $\dfrac{\Delta x - \widehat{\Delta x}}{2}$ after the initial and before the final positions, aligned with α^0 and $\alpha^0 + \pi$, respectively, and **finish**.
(5) Create a set of N final orientations $\alpha^i, i \in 1..N$, equally spaced between 0 and 2π.
(6) $\forall i = 1..N$ Compute the shortest Dubins path defined by the pair of configurations $\{\Omega^0, (\omega^f, \alpha^i)\}$ and the difference $\widehat{\Delta x^i}$ between its length and the length of the path calculated in Step 1.
(7) If there exists a path i satisfying $|\widehat{\Delta x^i} - \Delta x| < \epsilon$, for a positive number ϵ as small as we wish, **finish**, otherwise continue to Step 8.
(8) If there exist two successive paths j and k ($k = j+1$ or $j = N$ and $k = 1$) such that $\widehat{\Delta x^j} < \Delta x < \widehat{\Delta x^k}$ create a set of N finial orientations $\alpha^i, i \in 1..N$, equally spaced between α^j and α^k and return to Step 6 [3], otherwise **quit**.

Notice that, given Ω^0, $\{\omega^f \mid \exists CS\ path : |S| \geq r\sqrt{8}\} \subset \{\omega^f \mid |\omega^f - \omega^0| \geq 4r\}$, as can be observed in Figure 5a. For a given pair of an initial configuration and a final position, *Elongation 1* and *Elongation 2* do not necessarily require the preconditions of $|S| \geq r\sqrt{8}$ and $|\omega^f - \omega^0| \geq 4r$, respectively. However, *Elongation 1* is not applicable if the length of the straight line segment is not sufficiently large (see: Figure 5b), whereas *Elongation 2* can always generate a path with an arbitrary duration starting from the duration of the relaxed Dubins shortest path defined by $\{\Omega^0 = (x^0, y^0, \alpha^0), \Omega^f = (x^f, y^f, \alpha^0 + \pi)\}$.

5. CONCLUSION

The problems of intercepting a moving target in minimum time and at a predefined time were investigated. We gave conditions for shortest paths to coincide with minimum-time interception paths and gave examples where this is not the case. We provided easy-to-compute, lower and upper bounds on the minimum time to interception. For the problem of interception at a predefined time, we proposed two path elongation algorithms, and discussed the contribution of each one of them and their ability to generate a continuous elongation.

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[3] The process described in steps 6-8 might not converge if $|\omega^f - \omega^0| < 4r$ and thus the number of its iterations should be bounded.