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Stability analysis of discrete–time general homogeneous systems

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Abstract: In this paper, stability analysis of discrete–time homogeneous nonlinear dynamics is considered. In particular, necessary and sufficient conditions for semiglobal $\mathcal{KL}$–stability are established. These conditions lead to equivalence between semiglobal $\mathcal{KL}$–stability and semiglobal exponential stability of general homogeneous dynamics. Furthermore, systematic stability analysis methods based on solving optimization problems are developed. These methods are tractable even for homogeneous systems in state spaces of high dimensions.

Keywords: Discrete–time systems, Stability analysis, Lyapunov function, Contractive sets, Homogeneous functions, Nonlinear systems.

1. INTRODUCTION

A rich class of dynamical systems is described by homogeneous functions. In fact, homogeneous systems can be seen as a particular class of nonlinear systems, with a behavior as complex as that of the general nonlinear dynamics (Sepulchre and Aeyels, 1996), (Aeyels and De Leenheer, 2002). Furthermore, homogeneous dynamics include also relevant classes of hybrid systems, such as switched linear dynamics under arbitrary switching (Tuna, 2008), conewise linear dynamics (Lazar et al., 2013).

The stability analysis of continuous–time homogeneous systems was considered, for example, in (Hahn, 1967) and in (Rosier, 1992), where it was shown that asymptotic stability of any homogeneous approximation of general nonlinear control systems is sufficient for (local) asymptotic stability of the original system. Furthermore, for homogeneous asymptotically stable differential equations the existence of a homogeneous Lyapunov function was guaranteed (Rosier, 1992). In (Sepulchre and Aeyels, 1996) necessary conditions for homogeneous stabilization are introduced, i.e. for the existence of a stabilizing feedback leaving the closed–loop system homogeneous, since in this case the existence of a homogeneous Lyapunov function is guaranteed from (Rosier, 1992). A class of continuous–time positive homogeneous systems was considered in (Aeyels and De Leenheer, 2002), for which a generalization of the Perron–Frobenius theorem was provided for stability analysis. Also the notion of D–stability (i.e. stability of systems transformed into diagonal form) was studied for positive homogeneous systems, e.g. in (Bokharaie, 2012). Therein, D–stability was extended for nonlinear systems and some results are established for different classes of positive nonlinear systems, like homogeneous cooperative systems. Non–autonomous continuous–time homogeneous systems were considered in (Grüne, 2000) and (Grüne et al., 2000), and it was shown therein that any asymptotically controllable homogeneous control system admits a homogeneous control Lyapunov function and a stabilizing, possibly discontinuous, homogeneous state feedback law.

Although continuous–time homogeneous systems have been studied in the literature, there are few results which consider discrete–time homogeneous systems. A particular type of discrete-time homogeneous dynamics, i.e. homogeneous systems of order one, or as it will be defined here, homogeneous dynamics of order zero, w.r.t. the standard dilation map (matrix), has been considered in (Lazar et al., 2013) from the perspective of a generalization of the concept of λ–contractive sets, i.e. (k, λ)–contractive sets. This generalization yields a non–conservative Lyapunov–type tool, i.e. finite–time Lyapunov functions, which allows an easier verification of conditions for stability analysis of the considered class of systems. Based on this, scalable stability tests for switched linear systems were derived in (Lazar et al., 2013).

In this paper, general, discrete–time homogeneous dynamics, i.e. homogeneous dynamics of order greater than one, are considered. This is a richer class of systems, allowing the inclusion of, for example, polynomial nonlinearities in the dynamics. By considering more complex dynamics, also the stability analysis methods become much more complex, even for low order systems. As such, the goal of this paper is twofold. First, the problem of establishing non–conservative Lyapunov–type tools for general homogeneous nonlinear dynamics is considered. For this, we establish necessary and sufficient conditions for semiglobal $\mathcal{KL}$–stability which lead to equivalence between semiglobal $\mathcal{KL}$–stability and semiglobal exponential stability of general homogeneous dynamics. Second, the goal is to develop systematic stability analysis verification methods for the above mentioned type of systems. These methods materialize into solving optimization problems which follow two approaches. One approach is set–theoretical and is yielded by the concept of (k, λ)–contractive sets, whilst the other is a functional approach, generated by finite–time Lyapunov functions.

The paper is organized as follows. In Section 2 some preliminaries, general homogeneous systems and some instrumental results are introduced, while the problem is formally defined in Section 3. The main results of the paper and illustrative example are given in Section 4. Conclusions are summarized in Section 5.

2. PRELIMINARIES

The sets of non–negative integers and non–negative reals are denoted by $\mathbb{N}_+$, and $\mathbb{R}_+$, respectively. Given two sets $\mathcal{S}$ and $\mathcal{P}$, $\mathcal{S}_\mathcal{P}$ is defined as $\mathcal{S}_\mathcal{P} := \mathcal{S} \cap \mathcal{P}$. We write $\mathbb{N}_{\geq b}$, $\mathbb{R}_{\geq b}$ for $\mathbb{N}_{[b,\infty)}$ and $\mathbb{R}_{[b,\infty)}$. For a vector $x \in \mathbb{R}^n$, let $\|x\|$ denote an arbitrary Hölder norm. A map $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be sub–additive if the inequality $f(x+y) \leq f(x) + f(y)$ holds componentwise for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. A map $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be sublinear if $f(\cdot)$ is sub–additive and $f(\alpha x) = \alpha f(x)$, for all $\alpha \in \mathbb{R}_+$. A set $\mathcal{X} \subset \mathbb{R}^n$ is a C–set if it is compact, convex, and contains the origin. A set $\mathcal{X} \subset \mathbb{R}^n$ is a proper C–set if it is a C–set and contains the origin in its interior. The collection of non–empty compact sets in $\mathbb{R}^n$ is denoted by $\mathrm{Com}(\mathbb{R}^n)$. The polar set of a C–set $\mathcal{X} \in \mathbb{R}^n$ is defined as $\mathcal{X}^* := \{x : \forall y \in \mathcal{X},\ y^\top x \leq 1\}$. A polyhedron is the (convex) intersection of a finite number of open and/or closed half–spaces and a polytope is a closed and bounded polyhedron. Given a non–empty closed convex set $\mathcal{X} \subseteq \mathbb{R}^n$ the function $\mathrm{support}(\mathcal{X}, \cdot)$ defined as $\mathrm{support}(\mathcal{X}, \xi) := \sup_x\{\xi^\top x : x \in \mathcal{X}\}$ for $\xi \in \mathbb{R}^n$, is called the support function of the set $\mathcal{X}$ at the point ξ. Given a proper C–set $\mathcal{X} \subset \mathbb{R}^n$ the function $\mathrm{g}(\mathcal{X}, \cdot)$ defined by $\mathrm{g}(\mathcal{X}, \xi) := \inf_\mu\{\mu : \xi \in \mu\mathcal{X},\ \mu \geq 0\}$ for $\xi \in \mathbb{R}^n$, is called the Minkowski (gauge) function of the set $\mathcal{X}$ at the point ξ. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{s\to\infty} \varphi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

Recall the following relevant properties of the Minkowski (gauge) functions associated with proper C–sets, which were established in (Raković and Lazar, 2012); see also (Lazar et al., 2013).

Lemma 2.1. Let $\mathcal{X}$ and $\mathcal{Z}$ be any two proper C–sets in $\mathbb{R}^n$. Then $\mathcal{Z} \subseteq \mathcal{X}$ if and only if:

$$\forall x \in \mathbb{R}^n, \quad \mathrm{g}(\mathcal{X}, x) \leq \mathrm{g}(\mathcal{Z}, x). \tag{1}$$

Lemma 2.2. Let $\mathcal{X}$ be any proper C–set in $\mathbb{R}^n$. Then, for all $\alpha \in \mathbb{R}_+ \setminus \{0\}$:

$$\forall x \in \mathbb{R}^n, \quad \mathrm{g}(\alpha\mathcal{X}, x) = \frac{1}{\alpha}\,\mathrm{g}(\mathcal{X}, x). \tag{2}$$

2.1 Homogeneous dynamics

Consider dynamical systems of the form:

$$x^+ = \Phi(x), \tag{3}$$

where $x \in \mathbb{R}^n$ is the current state, $x^+ \in \mathbb{R}^n$ is the successor state and $\Phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary map with $\Phi(0) = 0$.

Homogeneous maps are defined more commonly, simply as in the definition below (Hahn, 1967).

Definition 2.3. A map $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be homogeneous of order r, where r is a real nonnegative scalar, if for all $x \in \mathbb{R}^n$ and all $\alpha > 0$, it holds that:

$$f(\alpha x) = \alpha^r f(x). \tag{4}$$

A more general definition of homogeneous functions, which recovers Definition 2.3 as a particular case, as given in, e.g. (Grüne, 2000) and (Aeyels and De Leenheer, 2002) is the following.

Definition 2.4. Consider a map $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$. If there exist real scalars $r_i > 0$, $i = 1, \ldots, n$, $\tau \geq 0$, and $\alpha > 0$ such that it holds that:

$$f(\Lambda_\alpha x) = \alpha^\tau \Lambda_\alpha f(x),\ \alpha \in \mathbb{R}_+,\ \Lambda_\alpha = \mathrm{diag}(\alpha^{r_1}, \ldots, \alpha^{r_n}), \tag{5}$$

then it is said that the map f is homogeneous of order τ with respect to the dilation matrix Λ_α.

For the remainder of the paper, dynamics defined by maps which satisfy the condition in Definition 2.4 w.r.t. the dilation matrix will be considered, and functions which satisfy (5) will be called *homogeneous functions (HFs) of order τ*.

Remark 2.5. If $r = (1, \ldots, 1)^\top$, the dilation matrix (5) is called the *standard dilation matrix*. Observe that Definition 2.3 recovers a particular case of Definition 2.4 when the standard dilation matrix is considered and $\tau = r - 1$.

Denote by Λ_α^s the standard dilation matrix.

Assumption 2.6. The map $\Phi(\cdot)$ is homogeneous of order τ.

Definition 2.7. A map $\Phi(\cdot)$ is called $\mathcal{K}$–bounded in $\mathbb{X} \subseteq \mathbb{R}^n$ if for all $x \in \mathbb{X}$, there exists a $\kappa \in \mathcal{K}$ such that $\|\Phi(x)\| \leq \kappa(\|x\|)$. If there exists a nonnegative real constant Γ, such that $\kappa(\|x\|) = \Gamma\|x\|$, then the map $\Phi(\cdot)$ is called Lipschitz–bounded in $\mathbb{X} \subseteq \mathbb{R}^n$.

Lemma 2.8. Suppose the map $\Phi(\cdot)$ satisfies Assumption 2.6. Then the map $\Phi(\cdot)$ is Lipschitz–bounded in $\mathbb{R}^n$.

Proof. Define the ball $\mathcal{B}_1 := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and observe that $\Phi(\cdot)$ is bounded on $\mathcal{B}_1$ by convention, i.e., $\Phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$. Let $\Gamma := \sup\{\|\Phi(x)\| : x \in \mathcal{B}_1\}$. Then Γ exists and it is finite since $\mathcal{B}_1$ is compact, $\Phi(\cdot)$ is bounded on $\mathcal{B}_1$ and the norm is bounded on bounded sets. As for all x on the boundary of $\mathcal{B}_1$, it holds that $\|x\| = 1$, this further yields that $\|\Phi(x)\| \leq \Gamma\|x\|$ for all $x \in \partial\mathcal{B}_1$. Similarly, for any $x \in \mathbb{R}^n$ there exists a pair $(\alpha, \xi) \in \mathbb{R}_+ \times \partial\mathcal{B}_1$ such that $x = \alpha\xi$. Observe that $x = \alpha\xi = \Lambda_\alpha^s\xi$ holds for any $x \in \mathbb{R}^n$. This and Assumption 2.6 imply that $\forall x \in \mathbb{R}^n$:

$$\|\Phi(x)\| = \|\Phi(\Lambda_\alpha^s\xi)\| = \|\alpha^\tau\Lambda_\alpha^s\Phi(x)\| \leq \alpha^\tau\|\Lambda_\alpha^s\|\Gamma\|\xi\|.$$

But,

$$\alpha^\tau\|\Lambda_\alpha^s\|\Gamma\|\xi\| = \alpha^{\tau-1}\|\Lambda_\alpha^s\|\Gamma\|\alpha\xi\| = \alpha^{\tau-1}\|\Lambda_\alpha^s\|\Gamma\|x\|.$$

Since $\|\Lambda_\alpha^s\| \leq \alpha$, it implies that $\|\Phi(x)\| \leq \alpha^\tau\Gamma\|x\|$, which further implies that the map $\Phi(\cdot)$ is Lipschitz–bounded in $\mathbb{R}^n$, with the Lipschitz constant $\alpha^\tau\Gamma$. ■

For HFs of order zero w.r.t the standard dilation matrix, the Lipschitz constant will be equal to Γ.

The following result is a consequence of Assumption 2.6 and Lemma 2.8.

Corollary 2.9. Suppose that the map $\Phi(\cdot)$ satisfies Assumption 2.6. Then, it holds that $\Phi(0) = 0$.

Next, Lemma II.5 from (Lazar et al., 2013) will be extended for homogeneous systems of order τ.

Lemma 2.10. Let $\mathcal{Y}$ and $\mathcal{Z}$ be any two proper C–sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ and let $\mu(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ be any HF of order τ. Then $\{\mu(x) : x \in \mathcal{Z}\} \subseteq \mathcal{Y}$ if and only if

$$\forall x \in \mathbb{R}^n, \quad \mathrm{g}(\mathcal{Y}, \mu(x)) \leq \mathrm{g}(\mathcal{Z}, x)^{\tau+1}. \tag{6}$$

Proof. Fix any two proper C–sets, in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, $\mathcal{Z}$ and $\mathcal{Y}$. Fix any HF of order τ $\mu(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ such that $\{\mu(x) : x \in \mathcal{Z}\} \subseteq \mathcal{Y}$. Take any $x \in \mathbb{R}^n$. Then $x \in \mathrm{g}(\mathcal{Z}, x)\mathcal{Z}$ and, consequently, $\mu(x) \in \{\mu(x) : x \in \mathrm{g}(\mathcal{Z}, x)\mathcal{Z}\}$. Since $\mathcal{Z}$ is a proper C–set, $\mathrm{g}(\mathcal{Z}, x) \geq 0$ and $\mu(\cdot)$ is homogeneous of order τ, it follows from Fact II.1 in (Lazar et al., 2013) that $\{\mu(x) : x \in \mathrm{g}(\mathcal{Z}, x)\mathcal{Z}\} = \Lambda_\alpha^s \mathrm{g}(\mathcal{Z}, x)^\tau \{\mu(x) : x \in \mathcal{Z}\} = \alpha^{\tau+1}\{\mu(x) : x \in \mathcal{Z}\}$, where $\alpha = \mathrm{g}(\mathcal{Z}, x)$. Since $\mathcal{Z}$ and $\mathcal{Y}$ are proper C–sets, $\{\mu(x) : x \in \mathcal{Z}\} \subseteq \mathcal{Y}$, and $\mathrm{g}(\mathcal{Z}, x) \geq 0$ it follows that $\{\mu(x) : x \in \mathrm{g}(\mathcal{Z}, x)\mathcal{Z}\} = \mathrm{g}(\mathcal{Z}, x)^{\tau+1}\{\mu(x) : x \in \mathcal{Z}\} \subseteq \mathrm{g}(\mathcal{Z}, x)^{\tau+1}\mathcal{Y}$. In turn, $\mu(x) \in \mathrm{g}(\mathcal{Z}, x)^{\tau+1}\mathcal{Y}$ which implies that $\mathrm{g}(\mathrm{g}(\mathcal{Z}, x)^{\tau+1}\mathcal{Y}, \mu(x)) \leq 1$ and, thus, it results by Lemma 2.2 that $\mathrm{g}(\mathcal{Y}, \mu(x)) \leq \mathrm{g}(\mathcal{Z}, x)^{\tau+1}$. Conversely, take any $\mu(x) \in \{\mu(x) : x \in \mathcal{Z}\}$. Then, $\mu(x) \in \mathrm{g}(\mathcal{Y}, \mu(x))\mathcal{Y}$ and because $\mathrm{g}(\mathcal{Y}, \mu(x)) \leq \mathrm{g}(\mathcal{Z}, x)^{\tau+1}$, it results from Fact II.1 in (Lazar et al., 2013) that $\mu(x) \in \mathrm{g}(\mathcal{Z}, x)^{\tau+1}\mathcal{Y}$. Since for any $x \in \mathcal{Z}$, $\mathrm{g}(\mathcal{Z}, x) \leq 1$ it further results that for any $\mu(x) \in \{\mu(x) : x \in \mathcal{Z}\}$, $\mu(x) \in \mathcal{Y}$. ■

3. PROBLEM FORMULATION

Recall the discrete–time dynamics defined in (3), for which we introduce the following notation.

For a given $k \in \mathbb{N}_+$, the iterated map $\Phi^k(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is given by:

$$\Phi^0(x) := x, \quad \Phi^k(x) := \Phi(\Phi^{k-1}(x)).$$

The following notation will be used to denote the set-valued map $\overline{\Phi}(\cdot) : \mathrm{Com}(\mathbb{R}^n) \to \mathrm{Com}(\mathbb{R}^n)$ defined by:

$$\overline{\Phi}(\mathcal{S}) := \{\Phi(x) : x \in \mathcal{S}\},$$

where $\mathcal{S} \in \mathrm{Com}(\mathbb{R}^n)$. Given $k \in \mathbb{N}_+$ and a proper C–set $\mathcal{S}$ in $\mathbb{R}^n$ the iterated set–valued map $\overline{\Phi}^k(\cdot) : \mathrm{Com}(\mathbb{R}^n) \to \mathrm{Com}(\mathbb{R}^n)$ is defined by:

$$\overline{\Phi}^0(\mathcal{S}) := \mathcal{S}, \quad \overline{\Phi}^k(\mathcal{S}) := \overline{\Phi}(\overline{\Phi}^{k-1}(\mathcal{S})).$$

Denote the solution of the system (3) at any discrete–time instant $k \in \mathbb{N}_+$ by:

$$x(k, \xi) := \Phi^k(\xi),$$

for any initial condition $\xi \in \mathbb{R}^n$. Next, regional, semiglobal and global $\mathcal{KL}$–stability will be defined.

Definition 3.1. The system (3) is called *$\mathcal{KL}$–stable in $\mathcal{S}$*, if for a given compact set $\mathcal{S} \subset \mathbb{R}^n$, which contains the origin in its interior, there exists a function $\beta_\mathcal{S} \in \mathcal{KL}$ such that $\|x(k, \xi)\| \leq \beta_\mathcal{S}(\|\xi\|, k)$ for all $(\xi, k) \in \mathcal{S} \times \mathbb{N}_+$. If in addition $\beta_\mathcal{S}(s, k) := c_\mathcal{S}\mu_\mathcal{S}^k s$ for some $(c_\mathcal{S}, \mu_\mathcal{S}) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$, then the system (3) is called *exponentially stable in $\mathcal{S}$* (ES in $\mathcal{S}$).

Definition 3.2. The system (3) is called *semiglobally $\mathcal{KL}$–stable*, if for *any* compact set $\mathcal{S} \subset \mathbb{R}^n$, which contains the origin in its interior, there exists a function $\beta_\mathcal{S} \in \mathcal{KL}$ such that $\|x(k, \xi)\| \leq \beta_\mathcal{S}(\|\xi\|, k)$ for all $(\xi, k) \in \mathcal{S} \times \mathbb{N}_+$. If in addition $\beta_\mathcal{S}(s, k) := c_\mathcal{S}\mu_\mathcal{S}^k s$ for some $(c_\mathcal{S}, \mu_\mathcal{S}) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$, then the system (3) is called *semiglobally exponentially stable* (semi–GES).

Definition 3.3. The system (3) is called *$\mathcal{KL}$–stable in $\mathbb{R}^n$* if there exists a function $\beta \in \mathcal{KL}$ such that $\|x(k, \xi)\| \leq \beta(\|\xi\|, k)$ for all $(\xi, k) \in \mathbb{R}^n \times \mathbb{N}_+$. If in addition $\beta(s, k) := c\mu^k s$ for some $(c, \mu) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$, then the system (3) is called *globally exponentially stable* (GES).

The main objective of this paper is to derive necessary and sufficient stability conditions which allow systematic verification methods for general homogeneous dynamics. In what follows, we will restrict our attention to analysis of semiglobal $\mathcal{KL}$–stability and semi–GES. To this aim, three problems will be addressed. First, necessary and sufficient conditions for semiglobal $\mathcal{KL}$–stability analysis of general homogeneous dynamics need to be developed. Second, the conditions under which semiglobal $\mathcal{KL}$–stability is equivalent with semi–GES will be investigated. Finally, systematic methods for checking the developed stability conditions will be provided.

4. MAIN RESULTS

In order to solve the above defined problem we will make use of the following concept introduced in (Lazar et al., 2013).

Definition 4.1. Given a real scalar $\lambda \in [0, 1)$ and a $k \in \mathbb{N}_+$, a proper C–set $\mathcal{S} \subset \mathbb{R}^n$ is called a *(k, λ)–contractive* set for the system $x^+ = \Phi(x)$ and constraint set $\mathbb{X}$ if and only if $\mathcal{S} \subseteq \mathbb{X}$ and, for all $i \in \mathbb{N}_{[1,k-1]}$, $\Phi^i(x) \in \mathbb{X}$ and $\Phi^k(x) \in \lambda\mathcal{S}$, for all $x \in \mathcal{S}$, i.e. for all $i \in \mathbb{N}_{[1,k-1]}$, $\overline{\Phi}^i(\mathcal{S}) \subseteq \mathbb{X}$ and $\overline{\Phi}^k(\mathcal{S}) \subseteq \lambda\mathcal{S}$.

Proposition 4.2. Suppose that Assumption 2.6 holds and let the proper C–set $\mathcal{S}$ be a (k, λ)–contractive set for the dynamics (3) and constraint set $\mathbb{X}$. Then for all $\alpha \in \mathbb{R}_+$, $\alpha \in (0, 1)$, the set $\alpha\mathcal{S}$ if a (k, λ)–contractive set for the dynamics (3).

The proof of Proposition 4.2 is omitted here due to space constraints. With this result, it is shown that every subunitary scaling of a (k, λ)–contractive set of a general homogeneous system is also a (k, λ)–contractive set. The consequence is that it doesn't allow for global conditions, as shown in the remainder of this section. However, this property will turn out to be useful for analyzing semiglobal stability properties.

The following two results are recalled from (Lazar et al., 2013) and apply to general nonlinear discrete–time dynamics.

Theorem 4.3. Suppose the dynamics (3) is semiglobally $\mathcal{KL}$–stable. Let $\mathcal{S} \subseteq \mathbb{X} \subseteq \mathbb{R}^n$ denote an arbitrary proper C–set such that $\overline{\Phi}^i(\mathcal{S}) \subseteq \mathbb{X}$ for all $i \in \mathbb{N}_+$. Then for any real scalar $\lambda \in (0, 1]$ there exists a $k = k(\lambda, \mathcal{S}) \in \mathbb{N}_+$ such that $\mathcal{S}$ is a (k, λ)–contractive set with respect to $\mathbb{X}$.

Theorem 4.4. Suppose the map $\Phi(\cdot)$ is $\mathcal{K}$–bounded in $\mathbb{X}$, $\mathbb{X} \subseteq \mathbb{R}^n$ contains the origin in its interior, and let $\mathcal{S}$ be a $(k, 1)$–contractive proper C–set with respect to $\mathbb{X}$. Furthermore, let $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$, a real scalar $\rho \in (0, 1)$, $k \in \mathbb{N}_+$, and let $V : \mathbb{X} \to \mathbb{R}_+$ be a function such that:

$$\forall x \in \mathbb{X}\,, \kappa_1(\|x\|) \leq V(x) \leq \kappa_2(\|x\|), \tag{7}$$
$$\forall x \in \mathcal{S}\,, V(\Phi^k(x)) \leq \rho V(x). \tag{8}$$

Then dynamics (3) *is $\mathcal{KL}$–stable in $\mathcal{S}$ with respect to $\mathbb{X}$. If the map $\Phi(\cdot)$ is Lipschitz-bounded and* (7) *holds with $\kappa_1(s) = a_1 s$ and $\kappa_2(s) = a_2 s$, for some $a_1, a_2 > 0$, then dynamics* (3) *is GES in $\mathcal{S}$.*

A function $V(\cdot)$ which satisfies the conditions (7)-(8) is called a finite–time Lyapunov function (FTLF) associated with a $(k, 1)$–contractive proper C–set $\mathcal{S}$, relative to $\mathcal{S}$ and with respect to $\mathbb{X}$.

If condition (8) in Theorem 4.4 holds for any proper C–set $\mathcal{S}$, i.e. for each set we have a FTLF $V_{\mathcal{S}}(\cdot)$, then semiglobal $\mathcal{KL}$–stability is implied. This is because any compact set $\mathcal{S}$ with the origin in its interior can be seen as a subset of a proper C–set, i.e. the convex hull of $\mathcal{S}$ for example.

Theorem 4.3 provides a necessary condition (semiglobal $\mathcal{KL}$–stability) for the existence of (k, λ)–contractive proper C–sets for general homogeneous dynamics. In turn, Theorem 4.4 provides a sufficient result in terms of FTLFs for regional $\mathcal{KL}$–stability of general homogeneous dynamics. In order to have necessary and sufficient conditions in terms of (k, λ)–contractive sets and FTLFs, the next equivalence result is instrumental.

4.1 Equivalence theorem

Theorem 4.5. Suppose that Assumption 2.6 holds and let $k \in \mathbb{N}_+$ and $(\lambda, \rho) \in (0, 1) \times (0, 1)$. The system described by (3) *admits a (k, λ)–contractive proper C–set with respect to $\mathbb{X} \subseteq \mathbb{R}^n$ if and only if it admits a sublinear FTLF associated with a $(k, 1)$–contractive proper C–set $\mathcal{P}$, relative to $\mathcal{P}$ and with respect to $\mathbb{X} \subseteq \mathbb{R}^n$.*

Proof. Let the proper C–set $\mathcal{S} \subset \mathbb{R}^n$ be a (k, λ)–contractive set for the dynamics (3) and constraint set $\mathbb{X}$. First it will be shown that (7) holds. Define $c_2 = \min_{x \in \partial\mathcal{S}} \|x\| > 0$ and $c_1 = \max_{x \in \partial\mathcal{S}} \|x\| > 0$. Observing that

$$\{x \in \mathbb{R}^n \,:\, \|x\| \leq c_2\} \subseteq \mathcal{S} \subseteq \{x \in \mathbb{R}^n \,:\, \|x\| \leq c_1\},$$

yields by Lemma 2.1 and Lemma 2.2 that

$$c_1^{-1}\|x\| \leq \mathrm{g}(S, x) \leq c_2^{-1}\|x\|.$$

Hence, letting $V(x) := \mathrm{g}(S, x)$ for all $x \in \mathbb{R}^n$, condition (7) holds with $\kappa_i(s) := a_i s$, $a_i = c_i^{-1}$, $i \in \{1, 2\}$. Next, for any $x \in \mathcal{S}$ it holds that $\Phi^k(x) \in \lambda\mathcal{S}$. From Lemma 2.10 it results that $\mathrm{g}(\lambda\mathcal{S}, \Phi^k(x)) \leq \mathrm{g}(\mathcal{S}, x)^{\tau+1}$. Since for any $x \in \mathcal{S}$, $\mathrm{g}(\mathcal{S}, x) \leq 1$, it follows that $\mathrm{g}(\mathcal{S}, x)^{\tau+1} \leq \mathrm{g}(\mathcal{S}, x)$. As such, it is obtained that $\mathrm{g}(\lambda\mathcal{S}, \Phi^k(x)) \leq \mathrm{g}(\mathcal{S}, x)$ and from Lemma 2.2 that $\mathrm{g}(\mathcal{S}, \Phi^k(x)) \leq \lambda\, \mathrm{g}(\mathcal{S}, x)$. Thus, (8) holds for $V(x) = \mathrm{g}(S, x)$ and $\rho = \lambda$. Observe that $\mathcal{S}$ is (k, λ)–contractive with $\lambda \in (0, 1)$ and hence, $\mathcal{S}$ is a $(k, 1)$–contractive set. Then, letting $\mathcal{P} := \mathcal{S}$ yields that $V(\cdot)$ is a sublinear FTLF associated with a $(k, 1)$–contractive proper C–set $\mathcal{P}$, relative to $\mathcal{P}$ and with respect to $\mathbb{X} \subseteq \mathbb{R}^n$.

Conversely, let $V(\cdot)$ be a sublinear FTLF associated with a $(k, 1)$–contractive proper C–set $\mathcal{P}$, with respect to $\mathbb{X}$. Then there exists a unique proper C–set $\mathcal{S}$ such that $\mathrm{support}(\mathcal{S}, x) = V(x)$ for any real vector x (Schneider, 1993, Thereom 1.7.1). Furthermore, from (Schneider, 1993, Thereom 1.7.6) it results that $V(x) = \mathrm{support}(\mathcal{S}, x) = \mathrm{g}(\mathcal{S}^*, x)$, for any $x \in \mathcal{S}$. Let $\beta^* := \max_\beta\{\beta > 0 : \beta\mathcal{S}^* \subseteq \mathcal{P}\}$. For any $x \in \mathbb{R}^n$, from Lemma 2.2 it holds that $\mathrm{g}(\beta^*\mathcal{S}^*, x) = \frac{1}{\beta^*}\mathrm{g}(\mathcal{S}^*, x) = \frac{1}{\beta^*}V(x)$. Since $\alpha V(x)$ is also a sublinear function and satisfies (7) and (8) for any $\alpha \in \mathbb{R}_+$, then $\frac{1}{\beta^*}V(x)$ also satisfies the conditions (7), (8) for all $x \in \mathcal{P}$. Because $\frac{1}{\beta^*}V(x) = \mathrm{g}(\beta^*\mathcal{S}^*, x)$, it results from (8) that $\mathrm{g}(\beta^*\mathcal{S}^*, \Phi^k(x)) \leq \rho\, \mathrm{g}(\beta^*\mathcal{S}^*, x)$. This implies that $\mathrm{g}(\rho\beta^*\mathcal{S}^*, \Phi^k(x)) \leq \mathrm{g}(\beta^*\mathcal{S}, x)$, for any $x \in \beta^*\mathcal{S}^*$. Since $\mathrm{g}(\beta^*\mathcal{S}, x) \geq 0$ and it is subunitary, it follows from the previous inequality that

$$\mathrm{g}(\rho\beta^*\mathcal{S}^*, \Phi^k(x))\, \mathrm{g}(\beta^*\mathcal{S}^*, x)^{r-1} \leq \mathrm{g}(\beta^*\mathcal{S}, x)^\tau$$

and, furthermore, that

$$\mathrm{g}(\rho\beta^*\mathcal{S}^*, \Phi^k(x)\, \mathrm{g}(\beta^*\mathcal{S}^*, x)^{\tau-1}) \leq \mathrm{g}(\beta^*\mathcal{S}, x)^\tau.$$

By Lemma 2.10 it is obtained that

$$\Phi^k(x)\, \mathrm{g}(\beta^*\mathcal{S}^*, x^{\tau-1}) \in \rho\beta^*\mathcal{S}^*,$$

for any $x \in \beta^*\mathcal{S}^*$. This further implies that the set $\beta^*\alpha\mathcal{S}$, with $\alpha = \frac{1}{\mathrm{g}(\beta^*\mathcal{S}^*, x^{\tau-1})}$ is a (k, λ)–contractive set with $\lambda = \rho$. ■

This result answers the first problem that we addressed. By Theorems 4.3 and 4.4 through Theorem 4.5 we have obtained necessary and sufficient conditions for semiglobal $\mathcal{KS}$–stability of system (3). The following result is a consequence of Theorem 4.5 and Theorem 4.3.

Theorem 4.6. Suppose that Assumption 2.6 holds and that the dynamics (3) *is semiglobally $\mathcal{KL}$–stable. Let $V : \mathbb{X} \to \mathbb{R}_+$ be an arbitrary sublinear function and let $\mathcal{S}$ denote the unique proper C–set such that* $\mathrm{support}(\mathcal{S}, x) = V(x)$*, for all $x \in \mathbb{X}$. Then there exist κ_1, $\kappa_2 \in \mathcal{K}_\infty$, a real scalar $\rho \in (0, 1)$ and a $k = k(\rho, \mathcal{S}^*) \in \mathbb{N}_+$ such that $V(\cdot)$ is a FTLF associated with $\mathcal{S}^*$, relative to $\mathcal{S}^*$ and with respect to $\mathbb{X}$.*

Proof. The proof follows easily from Theorem 4.3 and Theorem 4.5. ■

Note that in Theorem 4.6 the function $V(\cdot)$ is dependent on the choice of the set $\mathcal{S}$. Furthermore, it provides a necessary conditions for the existence of FTLFs, in terms of semiglobal $\mathcal{KL}$–stability. Thus, from Theorem 4.4 and Theorem 4.6 we obtained necessary and sufficient conditions by means of FTLFs, for semiglobal $\mathcal{KL}$–stability of discrete–time systems which are homogeneous of order τ.

From the hypothesis of Theorem 4.5 the function $V(\cdot)$ can be chosen arbitrarily from within the class of sublinear functions. For any of these functions there exist positive a_i, $i \in \{1, 2\}$, such that the inequality in (7) holds for $\mathcal{K}$–functions defined as $\kappa_i := a_i s$. As such, the next result follows from Lemma 2.8, Theorem 4.6 and Theorem 4.4.

Corollary 4.7. Suppose that Assumption 2.6 holds. Then the dynamics (3) *is semiglobally $\mathcal{KL}$–stable if and only if it is semi–GES.*

Corollary 4.7 answers the second problem considered in this paper. It provides an equivalence result between semiglobal $\mathcal{KL}$–stability and semi–GES for general homogeneous systems. Finally, a stability theorem by means of (k, λ)–contractive sets is provided. This is a consequence of Theorem 4.5 and Theorem 4.4.

Theorem 4.8. Suppose that the Assumption 2.6 holds. Suppose that there exists a proper C–set $\mathcal{S} \subseteq \mathbb{X} \subset \mathbb{R}^n$ and $\lambda \in (0, 1)$ such that $\mathcal{S}$ is (k, λ)–contractive with respect *to* $\mathbb{X}$ *for the dynamics* (3)*. Then the dynamics* (3) *is* $\mathcal{KL}$*–stable in* $\mathcal{S}$*.*

Theorem 4.8 renders a sufficient condition for regional $\mathcal{KL}$–stability. Together with Theorem 4.3 it yields necessary and sufficient conditions by means of (k, λ)–contractive sets for discrete–time homogeneous dynamics of order τ.

4.2 Verification

So far, we have provided non–conservative conditions for semiglobal $\mathcal{KL}$–stability of general homogeneous dynamics. In this section two approaches for checking these conditions will be provided. The first one is based on Theorem 4.4 and the second one exploits the equivalence theorem, Theorem 4.5. In what follows, the stability check tools starting from the hypothesis of Theorem 4.4 will be derived, for which two requirements need to be met.

(i) Find proper C–sets $\mathcal{S}$ and $\mathbb{X}$, $\mathcal{S} \subseteq \mathbb{X} \subseteq \mathbb{R}^n$, such that $\mathcal{S}$ is a $(k, 1)$–contractive set for the dynamics (3).
(ii) Find a FTLF associated with $\mathcal{S}$, relative to $\mathcal{S}$ and with respect to $\mathbb{X}$, i.e. find a function $V : \mathbb{X} \to \mathbb{R}_+$ which satisfies conditions (7) and (8) for the system (3).

Due to the (k, λ)–contractive set concept introduced in Definition 4.1, to solve the above problems we can pick *any* proper C–set $\mathcal{S}$ and *any* sublinear function $V(\cdot)$ which satisfy the conditions in Definition 4.1 and inequalities (7). As such, the stability analysis problem to be solved is not related to finding a particular set $\mathcal{S}$ and particular function V, but to finding a finite positive integer k, such that the conditions above are met for any arbitrary set and function.

$(k, 1)$*–contractive verification : solution to* (i):

Consider the proper C–sets $\mathcal{S}$ and $\mathbb{X}$, such that $\mathcal{S} \subseteq \mathbb{X} \subseteq \mathbb{R}^n$. Let the cost function $F_k : \mathcal{S} \to \mathbb{R}$, be defined as:

$$F_k(x) := -\,\mathrm{g}(\mathcal{S}, \Phi^k(x)), \tag{9}$$

for some $k \in \mathbb{N}_+$, and consider the following minimization problems:

$$\inf_{x \in \mathcal{S}} F_k(x)\,, \tag{10}$$

$$\inf_{x \in \mathbb{X}} F_i(x)\,, \quad i \in \mathbb{N}_{[1,k-1]}. \tag{11}$$

The cost function (9) has been defined by using the Minkowski function as the $(k, 1)$–contractive set condition $\Phi^k(x) \in \mathcal{S}$, for any $x \in \mathcal{S}$ can be written equivalently as $\mathrm{g}(\mathcal{S}, \Phi^k(x)) \le 1$.

Proposition 4.9. Let $x_{\mathcal{S}}$ *and* $x^i_{\mathbb{X}}$ *denote the global optima of* (10) *and* (11)*, respectively. If the value functions* $F_k(x_{\mathcal{S}})$ *and* $F_i(x^i_{\mathbb{X}})$ *satisfy* $F_k(x_{\mathcal{S}}) \ge 0$ *and* $F_i(x^i_{\mathbb{X}}) \ge 0$*, for all* $i \in \mathbb{N}_{[1,k-1]}$*, then the set* $\mathcal{S}$ *is* $(k, 1)$*–contractive for the system* (3)*, with respect to* $\mathbb{X}$*.*

The conditions in Proposition 4.9 guarantee that a chosen proper C–set is (k, λ)–contractive because they are equivalent with $\mathrm{g}(\mathcal{S}, \Phi^k(x)) \le 1$ and $\mathrm{g}(\mathcal{S}, \Phi^i(x)) \le 1$, $i = 1, \ldots, k-1$. Next, the conditions under which existence of global optima of optimization problems is guaranteed will be briefly recalled. For more detailed explanations, see for example (Borwein and Lewis, 2006).

Remark 4.10. If the constraint set is compact and the cost function continuous, then the optimum exists and it is attainable, but may not be unique. Additionally if the cost function to be minimized is convex and differentiable and the constraint set is convex, then any locally optimum of the considered optimization problem is globally optimal. If $\Phi(\cdot)$ is Lipschitz continuous then there are ways to guarantee convergence to the global optimum (Mladineo, 1986).

If the map $\Phi(\cdot)$ describing the dynamics (3) is a convex function, and we consider polyhedral sets as $(k, 1)$–contractive sets candidates, then an easier verification method can be derived which does not involve solving an optimization problem.

Solution for Φ *convex and polyhedral sets*:

Proposition 4.11. Let the map Φ *which describes the system* (3) *be a convex function and consider the polyhedral proper* C*–set* $\mathcal{S}$*. Then* $\mathrm{g}(\mathcal{S}, \Phi^k(v_i)) \le 1$ *implies that* $\mathrm{g}(\mathcal{S}, \Phi^k(x)) \le 1$*, for all* $x \in \mathcal{S}$*, where* $v_i \in \mathbb{R}^n$*,* $i = 1, \ldots, q$ *are the vertices of the set* $\mathcal{S}$*.*

Proof. The proof is straightforward and exploits the homogeneity of the Minkowski function. ■

Let the proper C–sets $\mathcal{S}$ and $\mathbb{X}$ be polyhedral, $\mathcal{S} \subseteq \mathbb{X} \subseteq \mathbb{R}^n$, and let $\mathcal{S}$ be described by vertices v_i, $i = 1, \ldots q$ and $\mathbb{X}$ be described by vertices u_j, $j = 1, \ldots p$. Based on Proposition 4.11, if the inequalities

$$\max_i \mathrm{g}(\mathcal{S}, \Phi^k(v_i)) \le 1 \tag{12}$$

$$\max_j \mathrm{g}(\mathcal{S}, \Phi^l(u_j)) \le 1, \quad l = 1, \ldots, k-1 \tag{13}$$

are satisfied for some finite $k \in \mathbb{N}_+$, then the polyhedral set $\mathcal{S}$ is $(k, 1)$–contractive with respect to $\mathbb{X}$.

Finite–time Lyapunov function verification: solution to (ii):

Consider the $(k, 1)$–contractive proper C–set $\mathcal{S} \subseteq \mathbb{X} \subseteq \mathbb{R}^n$ and a real valued function $V : \mathbb{X} \to \mathbb{R}_+$. For some $\rho \in (0, 1)$ define the cost function $F : \mathcal{S} \to \mathbb{R}$,

$$F_k(x) := -V(\Phi^k(x)) + \rho V(x) \tag{14}$$

and the problem

$$\inf_{x \in \mathcal{S}} F_k(x). \tag{15}$$

Proposition 4.12. Let x^* *denote the global optimum of the problem described in* (15) *and let* $\mathcal{S}$ *be a* $(k, 1)$*–contractive proper* C*–set. If the value function* $F_k(x^*)$ *satisfies* $F_k(x^*) \ge 0$*, then the function* $V(\cdot)$ *is a FTLF associated with* $\mathcal{S}$*, relative to* $\mathcal{S}$ *and with respect to* $\mathbb{X}$ *and by Theorem 4.6, the system* (3) *is ES in* $\mathcal{S}$*.*

Alternative verification method:

Problems (i) and (ii) can also be answered by making use of the result in Theorem 4.5. Therefore, we check if some proper C–set is (k, λ)–contractive for the considered system. This means that for a given proper C–set $\mathcal{S} \subseteq \mathbb{X} \subseteq \mathbb{R}^n$, where $\mathbb{X}$ is also a proper C–set, the problem

$$\inf_{x \in \mathcal{S}} F_k(x), \tag{16}$$

$$F_k(x) := -\,\mathrm{g}(\mathcal{S}, \Phi^k(x)) + \lambda, \quad \forall x \in \mathcal{S}, \tag{17}$$

has to be solved for some chosen $\lambda \in (0, 1)$ and finite $k \in \mathbb{N}_+$. Additionally, from Definition 4.1, also the condition $\overline{\Phi}^i(\mathcal{S}) \subseteq \mathbb{X}$, $i \in \mathbb{N}_{[1,k-1]}$ needs to be verified, i.e. by

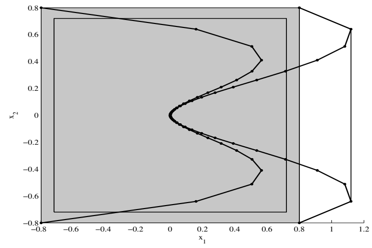

Fig. 1. (k, λ)–contractive set for the system in Example 1.

solving the minimization problem (10). Similarly as in Proposition 4.9, if the cost function (17) evaluated at the global optimum of (16) is positive, then the set $\mathcal{S}$ is (k, λ)–contractive for (3). Due to the equivalence theorem result in Theorem 4.5, if $\mathcal{S}$ is (k, λ)–contractive, then the function $V(x) = \mathrm{g}(\mathcal{S}, x)$, for any $x \in \mathcal{S}$ is a FTLF for the dynamics (3), i.e. it is a solution to the problem in (ii).

This method provides a more direct stability test since it only involves testing that a given set is (k, λ)–contractive and provides immediately a FTLF. However, when the scope is to enlarge the domain of attraction of a system, by the first approach, we get more flexibility in choosing the set and the type of the FTLF candidate, which enables a tradeoff between the values of λ and k. Furthermore, when the map describing the considered dynamics is convex, then it is possible to completely avoid solving an optimization problem, since the vertex conditions can be applied to check if a given set is (k, λ)–contractive and a FTLF is then provided for free.

The proposed approach is systematic compared to standard Lyapunov function approaches, where it is not known how to choose the candidate function. In (Lazar et al., 2013), scalable verification methods were developed for switched linear systems only. Therefore, the methods presented in this paper can be employed to verify GES for any homogeneous dynamics of order zero w.r.t. the standard dilation matrix, as regional ES implies GES for such dynamics.

4.3 Illustrative example

Consider the dynamics (3) where the map $\Phi(\cdot)$ is defined as $\Phi(x) = \begin{pmatrix} x_2^2+0.6x_1 \\ 0.8x_2 \end{pmatrix}$. Then the system is homogeneous of order $\tau = 0$, w.r.t. the dilation matrix $\Lambda_\alpha = \mathrm{diag}(\alpha^2, \alpha)$, for any $\alpha \in \mathbb{R}_+$. For this system, consider the proper C–set $\mathcal{S}$ defined as the convex hull of the vertex matrix: $U_{\mathcal{S}} = \begin{pmatrix} 0.8 & 0.8 & -0.8 & -0.8 \\ 0.8 & -0.8 & -0.8 & 0.8 \end{pmatrix}$. Here, the map Φ is convex and we can test the vertex conditions to check if $\mathcal{S}$ is $(k, 1)$–contractive with respect to the set $\mathbb{X}$ plotted in Figure 1 with white interior and black contour, and defined as the convex hull of the vertex matrix: $U_{\mathbb{X}} = \begin{pmatrix} 0.8 & 0.8 & -0.8 & -0.8 & 1.1200 & 1.1200 \\ 0.8 & -0.8 & -0.8 & 0.8 & 0.6400 & -0.6400 \end{pmatrix}$. Thus, it results that for $k = 4$ the conditions in (12) and (13) are verified for the vertices of $\mathcal{S}$, as shown in Figure 1. This means that $\mathcal{S}$ is a $(k, 1)$–contractive set for the map Φ. Consider the FTLF candidate $V(x) = x^\top x$. For $\rho = 0.9$ and $k = 4$, the global minimum x^* of (10) for the cost function (14) satisfies $F_4(x^*) \geq 0$, where $x^* = (0.0896 \quad 0.8000)^\top$ and $F_4(x^*) = 0.0885$. From Theorem 4.4 and Corollary 4.7 it results that the system is ES in $\mathcal{S}$. However this can be verified also using the cost function (17). For $\lambda = 0.9$ and $k = 4$ the set $\mathcal{S}$ resulted to be (k, λ)–contractive. Then, the function $V(x) = \mathrm{g}(\mathcal{S}, x) = \|x\|_\infty$, where $\|\cdot\|_\infty$ denotes the infinity norm, is a FTLF for the system and, thus, the system is ES in $\mathcal{S}$.

5. CONCLUSIONS

This paper dealt with stability analysis of discrete–time homogeneous nonlinear dynamics. In particular, semiglobal $\mathcal{KL}$–stability was considered, for which necessary and sufficient conditions were established. These conditions lead to equivalence between semiglobal $\mathcal{KL}$–stability and semiglobal exponential stability of general homogeneous dynamics. Furthermore, they allowed for deriving systematic stability analysis verification tests, based on optimization and was illustrated by an example.

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