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# On Networked Evolutionary Games Part 2: Dynamics and Control ⋆

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**Abstract:** As Part 2 of the paper "on networked evolutionary games", this paper uses the framework presented in Part 1 (Qi et al., 2014) to explore further issues about networked evolutionary games (NEGs). First, the strategy profile dynamics (SPD) is constructed from the fundamental evolutionary equations (FEEs). Using SPD, the control of NEGs are investigated. Detailed mathematical models are obtained for both deterministic and dynamic cases respectively. Then certain more complicated NEGs are explored. They are: (i) NEG with strategies of different length information, which allows some players use longer history information such as the information at $t$ and $t-1$ or so; (ii) NEG with Multi-Species, which allows an NEG with various kinds of players, they play several different fundamental network games according to their identities. (iii) NEG with time-varying payoffs. Since payoffs determine the evolution, the network profile dynamics will be a time-varying one. These more complicated NEGs can cover more general evolutions and they generalized the method proposed in Cheng et al. (Preprint2013).

*Keywords:* Networked evolutionary game, fundamental evolutionary equation, network profile dynamics, heterogeneous NEG, semi-tensor product of matrices

## 1. INTRODUCTION

In Part 1 of this paper an NEG is defined as following, which was firstly proposed in Cheng et al. (Preprint2013).

*Definition 1.* An NEG, game, denoted by $((N,E),G,\Pi)$, consists of three ingredients as:

(i) a network (graph) $(N,E)$;
(ii) a fundamental network game (FNG), $G$, such that if $(i,j)\in E$, then $i$ and $j$ play the FNG with strategies $x_i(t)$ and $x_j(t)$ respectively.
(iii) a local information based strategy updating rule (SUR).

It was proved that the fundamental evolutionary equation (FEE) for each player can be obtained as

$$x_i(t+1)=f_i\left(\{x_k(t)|k\in U_2(i)\}\right),\quad i=1,\cdots,n. \tag{1}$$

Then the network profile dynamics is uniquely determined by FEEs.

We refer to Qi et al. (2014) and Cheng et al. (Preprint2013) for details.

Part 2 of the paper considers several advanced problems about NEGs. In Section 2 the SPD is constructed from FEEs. Using SPD, the control problems of NEGs are investigated. A detailed mathematical framework is presented in Section 3 as a standard $k$-valued logical control networks. Then all the techniques for the control of $k$-valued logical networks can be used. Section 4 considers the NEGs where players can use different length of historical information to update their strategies. In Section 5 we consider the NEGs with multi-species. That is, the players are classified into several species, and players of different species play different roles in the networked games. Section 6 considers when the fundamental network game has time-varying payoff functions. Section 7 is a brief conclusion.

## 2. FROM FEE TO NPD

The NPD is used to describe the evolution of the overall networked games. This section consider how to construct the NPD of an NEG using its nodes' FEEs. We consider two cases: (i) the FEEs are deterministic model; (ii) the FEEs are probabilistic model.

### 2.1 Deterministic Model

Assume

$$\begin{cases}x_1(t+1)=M_1x(t),\\ \vdots\\ x_n(t+1)=M_nx(t),\end{cases} \tag{2}$$

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where $x(t) = \ltimes_{i=1}^{n} x_i(t)$ and $M_i \in \mathcal{L}_{k\times k^n}$. Then we have the NPD as

$$x(t+1) = Mx(t), \tag{3}$$

where

$$M = M_1 * M_2 * \cdots * M_n \in \mathcal{L}_{k^n\times k^n}. \tag{4}$$

*Example 2.* Recall Example 12 of Part 1. We have

$$\begin{aligned}
x_1(t+1) &= M_f x_4(t)x_5(t)x_1(t)x_2(t)x_3(t)\\
&= M_f W_{[2^3,2^2]}x(t) := M_1x(t),\\
x_2(t+1) &= M_f x_5(t)x_1(t)x_2(t)x_3(t)x_4(t)\\
&= M_f W_{[2^4,2]}x(t) := M_2x(t),\\
x_3(t+1) &= M_f x(t) := M_3x(t),\\
x_4(t+1) &= M_f x_2(t)x_3(t)x_4(t)x_5(t)x_1(t)\\
&= M_f W_{[2,2^4]}x(t) := M_4x(t),\\
x_5(t+1) &= M_f x_3(t)x_4(t)x_5(t)x_1(t)x_2(t)\\
&= M_f W_{[2^2,2^3]}x(t) := M_5x(t).
\end{aligned}$$

Finally, we have the NPD as

$$x(t+1) = Mx(t), \tag{5}$$

where

$$\begin{aligned}
M &= M_1 * M_2 * M_3 * M_4 * M_5\\
&= \delta_{32}[1,20,8,24,15,32,16,32,29,32,32,32,31,32,32,32\\
&\qquad 26,28,32,32,32,32,32,32,30,32,32,32,32,32,32,32].
\end{aligned} \tag{6}$$

### 2.2 Probabilistic Model

Assume the strategies have the probabilistic $k$-valued logical form as

$$\begin{aligned}
x_i(t+1) = M_1^j x(t), \quad \text{with} \quad Pr = p_i^j,\\
j = 1,\cdots,s_i;\ i = 1,\cdots,n.
\end{aligned} \tag{7}$$

Then we have

$$x(t+1) = Mx(t), \tag{8}$$

where $M \in \Upsilon_{k^n\times k^n}$ can be calculated as

$$M = \sum_{j_1=1}^{s_1}\sum_{j_2=1}^{s_2}\cdots\sum_{j_n=1}^{s_n}\left[\left(\prod_{i=1}^{n} p_i^{j_i}\right) M_1^{j_1} * M_2^{j_2} * \cdots * M_n^{j_n}\right]. \tag{9}$$

We use an example to depict it.

*Example 3.* Recall Example 13 of Part 1. In fact, we can use Table 5 there to calculate $M$ row by row. For instance, it is obvious that

$$\mathrm{Col}_1(M) = \mathrm{Col}_2(M) = \mathrm{Col}_3(M) = \delta_{32}^1.$$

As for $\mathrm{Col}_4(M)$, with probability 1/4 it could be $\delta_{32}^3$ or $\delta_{32}^4$ or $\delta_{32}^7$ or $\delta_{32}^8$. That is,

$$\begin{aligned}
\mathrm{Col}_4(M) = [&0,0,\tfrac{1}{4},\tfrac{1}{4},0,0,\tfrac{1}{4},\tfrac{1}{4},0,0,0,0,0,0,0,0,\\
&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]^T.
\end{aligned}$$

We simply express it as

$$\delta_{32}\left[3/\tfrac{1}{4} + 4/\tfrac{1}{4} + 7/\tfrac{1}{4} + 8/\tfrac{1}{4}\right].$$

Using this notation and a similar computation, we have

$$\begin{aligned}
M = \delta_{32}[&1,1,1,\alpha,1,\alpha,\beta,\gamma,\mu,\lambda,11,32,\lambda,32,32,32\\
&1,1,1,\alpha,1,22,\alpha,p,\mu,q,r,32,s,32,32,32],
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\alpha &= 3/\tfrac{1}{4} + 4/\tfrac{1}{4} + 7/\tfrac{1}{4} + 8/\tfrac{1}{4},\\
\beta &= 3/\tfrac{1}{2} + 7/\tfrac{1}{2},\\
\gamma &= 8/\tfrac{1}{3} + 16/\tfrac{2}{3},\\
\mu &= 1/\tfrac{2}{3} + 9/\tfrac{1}{3},\\
\lambda &= 18/\tfrac{1}{6} + 20/\tfrac{1}{6} + 26/\tfrac{1}{3} + 28/\tfrac{1}{3},\\
p &= 24/\tfrac{1}{3} + 32/\tfrac{2}{3},\\
q &= 26/\tfrac{1}{2} + 28/\tfrac{1}{2},\\
r &= 27/\tfrac{1}{4} + 28/\tfrac{1}{4} + 31/\tfrac{1}{4} + 32/\tfrac{1}{4},\\
s &= 29/\tfrac{1}{2} + 31/\tfrac{1}{2}.
\end{aligned}$$

## 3. MODELING CONTROLLED NEGS

*Definition 4.* Let $((N,E),G,\Pi)$ be an NEG, and $N = U \cup Z$ be a partition of $N$. We call $((U\cup Z),E),G,\Pi)$ a controlled NEG, if the strategies of $u \in U$ can be chosen arbitrarily. As a result, $z \in Z$ is called a state and $u \in U$ is called a control.

Using FEE, the strategy evolutionary equations can be expressed as (Cheng et al., Preprint2013)

$$x_i(t+1) = M_i x(t), \quad i = 1,\cdots,n, \tag{11}$$

where $x(t) = \ltimes_{j=1}^{n} x_j(t)$. Assume $U = \{i_1,\cdots,i_q\}$ with $1 \le i_1 < i_2 < \cdots < i_q \le n$, and $Z = \{j_1,j_2,\cdots,j_p\}$ with $1 \le j_1 < j_2 < \cdots < j_p \le n$, where $p+q = n$. Define $u_r = x_{i_r}$, $r = 1,\cdots,q$, and $z_s = x_{j_s}$, $s = 1,\cdots,p$.

We consider the deterministic case and the probabilistic case separately.

(1) (Deterministic Case) Assume $M_i \in \mathcal{L}_{k\times k^n}$. Then we have

$$\begin{aligned}
z_s(t+1) &= x_{j_s}(t+1) = M_{j_s} \ltimes_{i=1}^{n} x_i(t)\\
&= M_{j_s} W_{[k,k^{i_q-1}]} u_q(t) x_1(t) \ltimes x_2(t) \ltimes \cdots\\
&\quad \hat{x}_{i_q} \ltimes \cdots \ltimes x_n(t)\\
&= M_{j_s} W_{[k,k^{i_q-1}]} W_{[k,k^{i_q-1}]} u_{q-1}(t) u_q(t)\\
&\quad x_1(t) \ltimes x_2(t) \ltimes \cdots \ltimes \hat{x}_{i_{q-1}} \ltimes \cdots\\
&\quad \hat{x}_{i_q} \ltimes \cdots \ltimes x_n(t)\\
&= \cdots\\
&= M_{j_s} \ltimes_{r=m}^{1} W_{[k,k^{i_r+m-r-1}]} u(t) z(t),
\end{aligned}$$

where $u(t) = \ltimes_{i=1}^{q} u_i(t)$, and $z(t) = \ltimes_{i=1}^{p} z_i(t)$. The notation $\hat{x}_s$ means this factor is removed.

Define

$$\Psi_s := M_{j_s} \ltimes_{r=m}^{1} W_{[k,k^{i_r+m-r-1}]} \in \mathcal{L}_{k\times k^n}, \tag{12}$$

then we have

$$z_s(t+1) = \Psi_s u(t) z(t), \quad s = 1,\cdots,p. \tag{13}$$

Set

$$\Psi := \Psi_1 * \Psi_2 * \cdots * \Psi_p \in \mathcal{L}_{k^p\times k^n}. \tag{14}$$

The controlled network profile evolutionary equation is expressed as

$$z(t+1) = \Psi u(t) z(t). \tag{15}$$

This is a standard $k$-valued logical control network.

(2) (Probabilistic Case) Assume

$$\begin{aligned}
M_i = M_i^{j_i} \in \mathcal{L}_{k\times k^n}, \text{ with } Pr = p_i^{j_i}\\
j = 1,\cdots,r_i,\ i = 1,\cdots,n.
\end{aligned} \tag{16}$$

Then for each choice: $\{j_1,\cdots,j_n | 1 \le j_i \le r_i\}$ we can use $\{M_i^{j_i} | i = 1,\cdots,n\}$ to construct $\Psi^{j_1,\cdots,j_n}$, using the technique developed for deterministic case. Finally we have (15) again with

$$\Psi = \sum_{j_1=1}^{r_1}\sum_{j_2=1}^{r_2}\cdots\sum_{j_n=1}^{r_n}\prod_{i=1}^{n} p_i^{j_i}\Psi^{j_1,\cdots,j_n} \in \Upsilon_{k^p\times k^n}. \quad (17)$$

Note that a general procedure is provided above. But for a particular NEG, the process may be simplified. We use some examples to depict this.

*Example 5.* Recall Example 12 in Part 1. Assume players 2 and 4 are controls and the others are states. That is,

$$u_1 = x_2,\ u_2 = x_4,\ z_1 = x_1,\ z_2 = x_3,\ z_3 = x_5.$$

Then we have

$$\begin{aligned} x_1(t+1) &= M_f x_4(t)x_5(t)x_1(t)x_2(t)x_3(t)\\ &= M_f W_{[2^3,2^2]} x_1(t)x_2(t)x_3(t)x_4(t)x_5(t)\\ &= M_f W_{[2^3,2^2]} W_{[2,2^3]} x_4(t)x_1(t)x_2(t)x_3(t)x_5(t)\\ &= M_f W_{[2^3,2^2]} W_{[2,2^3]} W_{[2,2^2]} u_1(t)u_2(t)z_1(t)z_2(t)z_3(t)\\ &:= L_1 u(t)z(t), \end{aligned}$$

where $L_1 = M_f W_{[2^3,2^2]} W_{[2,2^3]} W_{[2,2^2]}$, $u(t) = u_1(t)u_2(t)$, $z(t) = z_1(t)z_2(t)z_3(t)$. Similarly, we can have

$$z_i(t+1) = L_i u(t)z(t), \quad i = 1, 2, 3, \quad (18)$$

where

$$\begin{aligned} L_1 &= M_f W_{[2^3,2^2]} W_{[2,2^3]} W_{[2,2^2]}\\ &= \delta_2[1,2,1,2,2,2,2,2,1,2,1,2,2,2,2,2,\\ &\qquad 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2];\\ L_2 &= M_f W_{[2,2^3]} W_{[2,2^2]}\\ &= \delta_2[1,1,2,2,1,1,2,2,2,2,2,2,2,2,2,2,\\ &\qquad 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2];\\ L_3 &= M_f W_{[2^2,2^3]} W_{[2,2^3]} W_{[2,2^2]}\\ &= \delta_2[1,2,1,2,2,2,2,2,2,2,2,2,2,2,2,2,\\ &\qquad 1,2,1,2,2,2,2,2,2,2,2,2,2,2,2,2]. \end{aligned}$$

Finally, we have controlled NEG as

$$z(t+1) = Lu(t)z(t), \quad (19)$$

where

$$\begin{aligned} L &= L_1 * L_2 * L_3\\ &= \delta_8[1,6,3,8,6,6,8,8,4,8,4,8,8,8,8,8,\\ &\qquad 7,8,7,8,8,8,8,8,8,8,8,8,8,8,8,8]. \end{aligned}$$

*Example 6.* Recall Example 13 in Part 1. Assume players 3 and 4 are controls and the others are states. That is,

$$u_1 = x_3,\ u_2 = x_4,\ z_1 = x_1,\ z_2 = x_2,\ z_3 = x_5.$$

Then we have

$$\begin{aligned} L_1 &= M_1 W^2_{[2,2^3]}\\ &= \delta_2[1,1,1,2,1,1,2,2,1,1,1,2,1,1,2,2,\\ &\qquad 1,1,2,2,1,2,2,2,1,1,2,2,1,2,2,2];\\ L_2 &= M_2 W^2_{[2,2^3]}\\ &= \delta_2[1,1,1/\tfrac{2}{3}+2/\tfrac{1}{3},1/\tfrac{1}{3}+2/\tfrac{2}{3},1,1,1/\tfrac{2}{3}+2/\tfrac{1}{3},2,\\ &\qquad 1,1,2,2,1,1,2,2,1,1,1/\tfrac{1}{3}+2/\tfrac{2}{3},2,1,1,\\ &\qquad 2,2,1,1/\tfrac{1}{3}+2/\tfrac{2}{3},2,2,1,,1/\tfrac{1}{3}+2/\tfrac{2}{3},2,2];\\ L_3 &= M_5 W^2_{[2,2^3]}\\ &= \delta_2[1,1,1,2,1,1,1,2,1,1/\tfrac{1}{2}+2/\tfrac{1}{2},1,2,1,\\ &\qquad 1/\tfrac{1}{2}+2/\tfrac{1}{2},1/\tfrac{1}{2}+2/\tfrac{1}{2},2,1,1/\tfrac{1}{2}+2/\tfrac{1}{2},\\ &\qquad 1,2,1,2,1,2,1,2,2,2,1/\tfrac{1}{2}+2/\tfrac{1}{2},2,2,2]. \end{aligned}$$

Finally, we have the networked profile evolutionary equation as

$$z(t+1) = Lu(t)z(t), \quad (20)$$

where $L$ can be calculated by using the technique proposed in Example 17 in Part 1 (Qi et al., 2014) as

$$\begin{aligned} L = \delta_8[&1,1,1/\tfrac{2}{3}+3/\tfrac{1}{3},6/\tfrac{1}{3}+7/\tfrac{2}{3},1,1,5/\tfrac{2}{3}+7/\tfrac{1}{3},8,1,\\ &1/\tfrac{1}{2}+2/\tfrac{1}{2},3,8,1,1/\tfrac{1}{2}+2/\tfrac{1}{2},7/\tfrac{1}{2}+8/\tfrac{1}{2},8,\\ &1,1/\tfrac{1}{2}+2/\tfrac{1}{2},5/\tfrac{1}{3}+7/\tfrac{2}{3},8,1,6,7,8,\\ &1,2/\tfrac{1}{3}+4/\tfrac{2}{3},8,8,1/\tfrac{1}{2}+2/\tfrac{1}{2},7/\tfrac{1}{3}+8/\tfrac{2}{3},8,8]. \end{aligned}$$

## 4. NEG WITH STRATEGIES OF DIFFERENT LENGTH INFORMATION

*Definition 7.* Given an NEG $((N,E),G,\Pi)$.

(i) A player, say, $i$, is said to use length-$r$ (historic) information, if

$$\begin{aligned} x_i(t+1) = f_i\big(\{&x_j(t),x_j(t-1),\cdots,x_j(\ell),\\ &c_j(t),\cdots,c_j(\ell)\big|j\in U(i)\}\big), \end{aligned} \quad (21)$$

where $\ell = \max\{0, t-r+1\}$.

(ii) The NEG is said to have strategies of different length (historic) information, if there is a partition $N = N_1 \cup N_2 \cup \cdots \cup N_s$, $N_i \cap N_j = \emptyset$ $(i \neq j)$ such that a player $j \in N_r$ implies that $j$ is with $r$-length information.

Now assume $i$ uses length-$r$ information and let $t \geq r-1$. Then we have

$$\begin{aligned} &x_i(t+1)\\ =&f_i\big(\{x_j(t),x_j(t-1),\cdots,x_j(t-r+1),\\ &\qquad c_j(t),\cdots,c_j(t-r+1)\big|j\in U(i)\}\big)\\ =&f_i\left(\{x_j(t),x_j(t-1),\cdots,x_j(t-r+1)\big|j\in U_2(i)\}\right). \end{aligned} \quad (22)$$

Note that in the above equation the $f_i$ in the first equality is different from the $f_i$ in the second equation. To avoid notational complexity, we use the same notation. Now for each $j$ we define

$$\begin{aligned} z_1^j(t+1) &:= x_j(t)\\ z_2^j(t+1) &:= z_1^j(t) = x_j(t-1)\\ &\ \vdots\\ z_{r-1}^j(t+1) &:= z_{r-2}^j(t) = x_j(t-r+2). \end{aligned} \quad (23)$$

Using this set of new variables, we can express (22) into a normal form as

$$\begin{cases} z_1^j(t+1) = x_j(t)\\ z_2^j(t+1) = z_1^j(t)\\ \vdots\\ z_{r-1}^j(t+1) = z_{r-2}^j(t), \quad j\in U_2(i)\\ \\ x_i(t+1) = f_i\left(\{x_j(t),z_1^j(t),z_2^j(t),\cdots,z_{r-1}^j(t)\big|j\in U_2(i)\}\right). \end{cases} \quad (24)$$

Define

$$y^i = \left\{z_1^j,\cdots,z_{r-1}^j\big|j\in U_2(i)\right\}\cup\{x_i\},$$

then we have

$$y^i(t+1) = F_i\left(\{y^j(t)\big|j\in U_2(i)\}\right). \quad (25)$$

Then the technique developed in Part 1 for standard NEGs is applicable for this case. Finally, we consider the initial values. We consider $\{x_i(0),\cdots,x_i(r-1)\big|i=1,\cdots,n\}$ as the initial values. In fact, only $\{x_i(0) \big| i = 1, \cdots, n\}$ are real initial values. Then we can use the following equation

$$
\begin{aligned}
x_i(1) &= F_i\left(\{x_j(0) \big| j \in U_2(i)\}\right)\\
x_i(2) &= F_i\left(\{x_j(0), x_j(1) \big| j \in U_2(i)\}\right)\\
&\ \ \vdots\\
x_i(r-1) &= F_i\left(\{x_j(0), x_j(1), \cdots, x_j(r-2) \big| j \in U_2(i)\}\right)
\end{aligned} \tag{26}
$$

and the method similar to (23)–(25) to calculate all other initial values.

To calculate the network profile dynamics of this kind of networks, we need the following lemma (Cheng et al., Preprint2013)

*Lemma 8.* Assume $X \in \Delta_p$ and $Y \in \Delta_q$. We define two dummy matrices, named by "front-maintaining operator" (FMO) and "rear-maintaining operator"(RMO) respectively, as:

$$
D_f^{[p,q]} = \delta_p[\underbrace{1 \cdots 1}_{q}\ \underbrace{2 \cdots 2}_{q}\ \cdots\ \underbrace{p \cdots p}_{q}],
$$

$$
D_r^{[p,q]} = \delta_q[\underbrace{\underbrace{1\,2 \cdots q}\ \underbrace{1\,2 \cdots q}\ \cdots\ \underbrace{1\,2 \cdots q}}_{p}].
$$

Then we have

$$
D_f^{[p,q]} XY = X. \tag{27}
$$

$$
D_r^{[p,q]} XY = Y. \tag{28}
$$

We give an example to depict this.

*Example 9.* Consider an NEG $((N,E),G,\Pi)$, where the graph is a cycle of $n = 6$ nodes, and the FNG is the same as in Example 12 of Part 1. Assume players 2, 3, 4, 5, 6 use length-1 information, and the SUR, $\Pi$, is the same as in Example 13 in Part 1; and the player 1 uses length-2 information; and the SUR for $t = 1$ it is $\Pi$, for $t > 1$ the SUR for player 1 is as follows: using $\Pi$ to get $x_{j^*(t)}(t)$ and $x_{j^*(t-1)}(t)$. Then we assume

$$
x_1(t+1) = \begin{cases} x_{j^*(t)}(t), & Pr = 0.8,\\ x_{j^*(t-1)}(t), & Pr = 0.2. \end{cases}
$$

Then the strategy dynamics for players 2, 3, 4, 5, 6 are the same as in Example 12 of Part 1. The strategy dynamics for player 1 is

$$
\begin{aligned}
z_1(t+1) &= x_1(t)\\
z_2(t+1) &= x_2(t)\\
z_3(t+1) &= x_3(t)\\
z_4(t+1) &= x_5(t)\\
z_5(t+1) &= x_6(t)\\
x_1(t+1) &= \begin{cases} f_1(x_5(t),x_6(t),x_1(t),x_2(t),x_3(t)), & Pr = 0.8\\ f_1(z_4(t),z_5(t),z_1(t),z_2(t),z_3(t)), & Pr = 0.2 \end{cases}\\
&= \begin{cases} M_f D_r^{[2,2^5]} W_{[2^3,2^3]} x(t), & Pr = 0.8\\ M_f W_{[2^3,2^2]} z(t), & Pr = 0.2. \end{cases}
\end{aligned} \tag{29}
$$

where $x(t) = \ltimes_{i=1}^6 x_i(t)$, $z(t) = \ltimes_{i=1}^5 z_i(t)$, $M_f$ is the same as in Example 13 in Part 1. Denoted by

$$
y_i(t) = \begin{cases} z_i(t), & i = 1, \cdots, 5;\\ x_{i-5}(t), & i = 6, \cdots, 11. \end{cases}
$$

Then

$$
\begin{aligned}
y_1(t+1) &= x_1(t)\\
&= D_r^{[2^5,2]} z(t) x_1(t)\\
&= D_r^{[2^5,2]} z(t) D_f^{[2,2^5]} x(t)\\
&= D_r^{[2^5,2]} \left(I_{2^5} \otimes D_f^{[2,2^5]}\right) z(t) x(t)\\
&:= M_1 y(t),
\end{aligned}
$$

where

$$
M_1 = D_r^{[2^5,2]} \left(I_{2^5} \otimes D_f^{[2,2^5]}\right).
$$

Similarly, we can calculate that

$$
y_i(t+1) = M_i y(t), \quad i = 2, \cdots, 11,
$$

where

$$
\begin{aligned}
M_2 &= D_r^{[2^5,2]} \left(I_{2^5} \otimes D_r^{[2,2]} D_f^{[2^2,2^4]}\right)\\
M_3 &= D_r^{[2^5,2]} \left(I_{2^5} \otimes D_r^{[2,2^2]} D_f^{[2^3,2^3]}\right)\\
M_4 &= D_r^{[2^5,2]} \left(I_{2^5} \otimes D_r^{[2,2^4]} D_f^{[2^5,2]}\right)\\
M_5 &= D_r^{[2^5,2]} \left(I_{2^5} \otimes D_r^{[2,2^5]}\right)\\
M_6 &= \begin{cases} M_f D_r^{[2,2^5]} W_{[2^3,2^3]} D_r^{[2^5,2^6]} := M_6^1, & Pr = 0.8\\ M_f W_{[2^3,2^2]} D_f^{[2^5,2^6]} := M_6^2, & Pr = 0.2 \end{cases}\\
M_7 &= M_f D_r^{[2,2^5]} W_{[2^4,2^2]} D_r^{[2^5,2^6]}\\
M_8 &= M_f D_f^{[2^5,2]} D_r^{[2^5,2^6]}\\
M_9 &= M_f D_r^{[2^6,2^5]}\\
M_{10} &= M_f D_f^{[2^5,2]} W_{[2^2,2^4]} D_r^{[2^5,2^6]}\\
M_{11} &= M_f D_f^{[2^5,2]} W_{[2^3,2^3]} D_r^{[2^5,2^6]}.
\end{aligned}
$$

Then

$$
y(t+1) = Ly(t),
$$

where

$$
L = \begin{cases} L_1 := M_1 * M_2 * M_3 * M_4 * M_5 * M_6^1 * M_7 * M_8 * M_9 * M_{10} * M_{11}, & Pr = 0.8,\\ L_2 := M_1 * M_2 * M_3 * M_4 * M_5 * M_6^2 * M_7 * M_8 * M_9 * M_{10} * M_{11}, & Pr = 0.2. \end{cases}
$$

As for the initial value, we have

$$
\begin{aligned}
y(1) = (&x_1(0), x_2(0), x_3(0), x_5(0), x_6(0),\\
&(x_1(1), x_2(1), x_3(1), x_4(1), x_5(1), x_6(1))),
\end{aligned} \tag{30}
$$

with $(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0))$ are free values, and

$$
x(1) = M^0 x(0), \tag{31}
$$

where

$$
\begin{aligned}
M^0 &= M_1^0 * M_2^0 * M_3^0 * M_4^0 * M_5^0 * M_6^0\\
&= (M_f D_f^{[2^5,2]} W_{[2^4,2^2]}) * (M_f D_f^{[2^5,2]} W_{[2^5,2]})\\
&\quad *(M_f D_f^{[2^5,2]}) * (M_f D_r^{[2,2^5]} W_{[2,2^5]})\\
&\quad *(M_f D_r^{[2,2^5]} W_{[2^2,2^4]}) * (M_f D_r^{[2,2^5]} W_{[2^3,2^3]}).
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
L &= 0.8 L_1 + 0.2 L_2\\
&= \delta_{2048}[1, 68/0.2 + 100/0.8, 136, 200/0.2 + 232/0.8, 15,\\
&\qquad 80/0.2 + 112/0.8, 144, 208/0.2 + 240/0.8, 285,\\
&\qquad \ldots, 1984, 2048].
\end{aligned}
$$

Then, after 18 times iterations $L$ converges to the following matrix

$$
\delta_{2048}[1, 2048, 2048, \ldots, 2048].
$$

According to this matrix (the whole set of data is omitted), we splite $\Delta_{2048}$ into three subsets:

$$
D_1 = \delta_{2048}\{1, 129, 257, 385\};
$$

$$
\begin{aligned}
D_2 = \delta_{2048}\ \{&65\ \ 193\ \ 321\ \ 449\ \ 513\ \ 577\ \ 641\\
&705\ \ 769\ \ 833\ \ 897\ \ 961\ \ 1025\ 1089\\
&1153\ 1217\ 1281\ 1345\ 1409\ 1473\ 1537\\
&1601\ 1665\ 1729\ 1793\ 1857\ 1921\ 1985\};
\end{aligned}
$$

$$D_3 = \Delta_{2048} \backslash (D_1 \cup D_2).$$

If initial state $x_0 \in D_1$, then $x(t) \to \delta_{2048}^1$ as $t \to \infty$. Else if $x_0 \in D_2$, then $x(t) \to 0.8 * \delta_{2048}^1 + 0.2 * \delta_{2048}^{2048}$ as $t \to \infty$. Else where $x_0 \in D_3$, then $x(t)\delta_{2048}^{2048}$ as $t \to \infty$.

Note that not all $x \in \Delta_{2048}$ can be chosen as the initial value, because the initial value should satisfy (30)–(31).

## 5. NEG WITH MULTI-SPECIES

*Definition 10.* An NEG is said to have $s$ species, if there is a partition $N = N_1 \cup N_2 \cup \cdots \cup N_s$, a set of fundamental games $\{G_{i,j} \big| 1 \le i, j \le s\}$

To avoid the notational complexity, we assume $s = 2$. We call these two kinds of players (nodes) white $(W)$ and black $(B)$ respectively. Then there are three different NEGs: $G_w$, $G_b$, and $G_m$. It is reasonable to assume that $G_w$ and $G_b$, which are the games between two white and two black players respectively, are symmetric, and $G_m$, which is the game between a white and a black players, is asymmetric. Assume in all the three games there are $k$ strategies for each player. Then each player has $k \times k$ possible strategies, denoted by $z_i(t) = x_i(t) \ltimes y_i(t)$, where $x_i$ is the strategy against white neighbors and $y_i$ is the strategy against black neighbors. We give an example to depict this.

*Example 11.* A game with its graph depicted in Fig. 1, where 4 nodes are white and 2 others are black. Assume $k = 2$ and the payoff bi-matrices for three FNGs are described by

(i) $G_w$ is $S$-2 with parameters as (Snowdrift Game)
$$R = -1; S = -1; T = 2; P = 0;$$
(ii) $G_b$ is $S$-2 with parameters as (Hawk-Dove Game)
$$R = -1; S = 2; T = 0; P = 1;$$
(iii) $G_m$ is $A$-2 with parameters as
$$A = 2, B = 1, C = 0, D = 0, E = 0, F = 0, G = 1, H = 2.$$

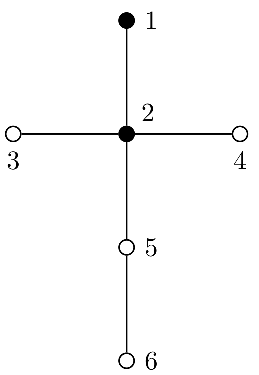


Fig. 1. Graph for Example 11

We can calculate $f_i$ as in Tables 1-4.

Using SUR UI-1, We have the NEG as
$$x(t+1) = L \ltimes_{i=1}^{6} x_i$$

Table 1. Payoffs → Dynamics (Example 11)

| Profile | 11111 | 11112 | 11121 | 11122 | 11211 | 11212 | 11221 | 11222 |
|---|---|---|---|---|---|---|---|---|
| $c_1$ | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| $c_2$ | 5/4 | 3/4 | 3/4 | 1/4 | 1/4 | 1/4 | 1/4 | -1/4 |
| $c_3$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $c_4$ | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| $f_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $f_3$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |
| $f_4$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |
| ⋮ | | | | | | | | |
| Profile | 22111 | 22112 | 22121 | 22122 | 22211 | 22212 | 22221 | 22222 |
| $c_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $c_2$ | 1/4 | 1/2 | 1/2 | 3/4 | 1/2 | 3/4 | 3/4 | 1 |
| $c_3$ | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 |
| $c_4$ | 0 | 0 | 2 | 2 | 0 | 0 | 2 | 2 |
| $f_1$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $f_3$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $f_4$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

Table 2. Payoffs → Dynamics (Example 11)

| Profile | 111111 | 111112 | 111121 | 111122 | 111211 | 111212 | 111221 | 111222 |
|---|---|---|---|---|---|---|---|---|
| $c_1$ | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| $c_2$ | 5/4 | 5/4 | 3/4 | 3/4 | 3/4 | 3/4 | 1/4 | 1/4 |
| $c_3$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $c_4$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $c_5$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| $c_6$ | -1 | 2 | -1 | 0 | -1 | 2 | -1 | 0 |
| $f_2$ | 1 | 1 | (2/3,1/3) | 1 | 1 | 1 | (1/2,1/2) | 1 |
| $f_5$ | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 |
| ⋮ | | | | | | | | |
| Profile | 122111 | 122112 | 122121 | 122122 | 122211 | 122212 | 122221 | 122222 |
| $c_1$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $c_2$ | 1/4 | 1/4 | 1/2 | 1/2 | 1/2 | 1/2 | 3/4 | 3/4 |
| $c_3$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $c_4$ | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 |
| $c_5$ | -1/2 | -1/2 | 2 | 1 | -1/2 | -1/2 | 2 | 1 |
| $c_6$ | -1 | 2 | -1 | 0 | -1 | 2 | -1 | 0 |
| $f_2$ | (1/2,1/2) | (1/2,1/2) | (1/3,2/3) | (1/2,1/2) | (1/3,2/3) | (1/3,2/3) | (1/4,3/4) | (1/3,2/3) |
| $f_5$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

Table 3. Payoffs → Dynamics (Example 11)

| Profile | 211111 | 211112 | 211121 | 211122 | 211211 | 211212 | 211221 | 211222 |
|---|---|---|---|---|---|---|---|---|
| $c_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $c_2$ | 2 | 2 | 3/2 | 3/2 | 3/2 | 3/2 | 1 | 1 |
| $c_3$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $c_4$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $c_5$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| $c_6$ | -1 | 2 | -1 | 0 | -1 | 2 | -1 | 0 |
| $f_2$ | 1 | 1 | 1 | 1 | 1 | 1 | (2/3,1/3) | 1 |
| $f_5$ | 1 | (1/2,1/2) | 1 | 1 | 1 | 2 | (1/2,1/2) | 1 |
| ⋮ | | | | | | | | |
| Profile | 222111 | 222112 | 222121 | 222122 | 222211 | 222212 | 222221 | 222222 |
| $c_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $c_2$ | 1/2 | 1/2 | 3/4 | 3/4 | 3/4 | 3/4 | 1 | 1 |
| $c_3$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $c_4$ | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 |
| $c_5$ | -1/2 | -1/2 | 2 | 1 | -1/2 | -1/2 | 2 | 1 |
| $c_6$ | -1 | 2 | -1 | 0 | -1 | 2 | -1 | 0 |
| $f_2$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $f_5$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

Table 4. Payoffs → Dynamics (Example 11)

| Profile | 111 | 112 | 121 | 122 | 211 | 212 | 221 | 222 |
|---|---|---|---|---|---|---|---|---|
| $c_5$ | 0 | 3/2 | -1/2 | 0 | -1/2 | 1 | 1/2 | 1 |
| $c_6$ | -1 | 2 | -1 | 0 | -1 | 2 | -1 | 0 |
| $f_6$ | 1 | 2 | 2 | 2 | 1 | 2 | 2 | 2 |

where

$$
\begin{aligned}
L = \delta_{64}[&1, 4, 4, 2, 1, 4, 4, 2, 1, 4, 4, 2, 1, 4, 32, 32, 15, 16,\\
&16, 16, 15, 16, 16, 16, 15, 16, 16, 16, 15, 16, 16, 16,\\
&1, 2, 2, 2, 1, 4, 2, 2, 1, 4, 2, 2, 1, 4, 20, 2, 63, 64,\\
&64, 64, 63, 64, 64, 64, 63, 64, 64, 64, 63, 64, 64, 64].
\end{aligned}
$$

This NEG has two fixed points $(1, 1, 1, 1, 1, 1)$ and $(2, 2, 2, 2, 2, 2)$. Besides, it has two cycles with length 2, i.e.,
$$(2, 2, 2, 2, 2, 1) \to (2, 2, 2, 2, 1, 1) \to (2, 2, 2, 2, 2, 1)$$
and
$$(2, 2, 1, 1, 1, 1) \to (2, 1, 1, 1, 1, 1) \to (2, 2, 1, 1, 1, 1).$$

## 6. NEG WITH TIME-VARYING PAYOFFS

*Definition 12.* An NEG is said to have varying payoffs, if the parameters in the payoff bi-matrix of the NEG are time-varying.

*Example 13.* Recall Example 12 of Part 1, where network graph is Fig. 1 (a) and the SUR is UI-1. As for the FNG, we let the non-zero parameters be flexible, that is: FNG is S-2 with constrains: $R = S$, $P = 0$ (the generalized game of Snowdrift).

Similar to Example 12 of Part 1, the FEE can be determined via Table 5. The parameters in Table 5 are as follows:

$$\alpha = \begin{cases} 1, & R \geq T \\ 2, & R < T, \end{cases} \quad \beta = \begin{cases} 1, & R \geq \frac{T}{2} \\ 2, & R < \frac{T}{2}, \end{cases}$$

$$\gamma = \begin{cases} 1, & R \geq \max\{\frac{T}{2}, 0\} \\ 2, & \text{otherwise}, \end{cases} \quad \theta = \begin{cases} 1, & R \geq \max\{T, \frac{T}{2}\} \\ 2, & \text{otherwise}. \end{cases}$$

Table 5. (Parameter-Depending) Payoffs $\to$ Dynamics

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Profile | 11111 | 11112 | 11121 | 11122 | 11211 | 11212 | 11221 | 11222 |
| $f$ | 1 | 1 | $\alpha$ | $\beta$ | $\alpha$ | $\alpha$ | $\beta$ | $\gamma$ |
| Profile | 12111 | 12112 | 12121 | 12122 | 12211 | 12212 | 12221 | 12222 |
| $f$ | $\alpha$ | $\alpha$ | $\alpha$ | $\theta$ | $\beta$ | $\beta$ | 2 | 2 |
| Profile | 21111 | 21112 | 21121 | 21122 | 21211 | 21212 | 21221 | 21222 |
| $f$ | 1 | 1 | $\alpha$ | $\beta$ | $\alpha$ | $\alpha$ | $\beta$ | $\gamma$ |
| Profile | 22111 | 22112 | 22121 | 22122 | 22211 | 22212 | 22221 | 22222 |
| $f$ | $\beta$ | $\beta$ | $\theta$ | $\beta$ | $\gamma$ | $\gamma$ | 2 | 2 |

Hence, the FEE is

$$\begin{aligned} f = \delta_2[&1,1,\alpha,\beta,\alpha,\alpha,\beta,\gamma,\alpha,\alpha,\alpha,\theta,\beta,\beta,2,2, \\ &1,1,\alpha,\beta,\alpha,\alpha,\beta,\gamma,\beta,\beta,\theta,\beta,\gamma,\gamma,2,2] \ltimes_{j=i-2}^{i+2} x_j. \end{aligned} \tag{32}$$

Denote by

$$\begin{aligned} A &= \{R \geq T\}; & B &= \{R \geq \tfrac{T}{2}\}; \\ C &= \left\{R \geq \max\{\tfrac{T}{2}, 0\}\right\}; & D &= \left\{R \geq \max\{T, \tfrac{T}{2}\}\right\}. \end{aligned}$$

Let $\Theta$ (with $0 \leq \Theta < 2\pi$) be defined by

$$\sin(\Theta) = \frac{R}{\sqrt{R^2+T^2}}; \quad \cos(\Theta) = \frac{T}{\sqrt{R^2+T^2}}.$$

Then the parameter space can be decomposed into 5 parts as shown in Fig. 2, where

$$\begin{aligned} I &= A \cup B \cup C \cup D &&= \left\{0 \leq \Theta \leq \frac{\pi}{4}\right\} \cup \left\{\frac{3}{2}\pi \leq \Theta < 2\pi\right\} \\ II &= A \cup B \cup C^c \cup D &&= \left\{\pi + \arctan(2) \leq \Theta < \frac{3}{2}\pi\right\} \\ III &= A \cup B^c \cup C^c \cup D^c &&= \left\{\pi + \frac{1}{4}\pi \leq \Theta < \pi + \arctan(2)\right\} \\ VI &= A^c \cup B^c \cup C^c \cup D^c &&= \left\{\arctan(2) < \Theta < \pi + \frac{1}{4}\pi\right\} \\ V &= A^c \cup B \cup C \cup D^c &&= \left\{\pi + \frac{1}{4}\pi < \Theta \leq \arctan(2)\right\}. \end{aligned}$$

It follows that

(i) When $(R, T) \in I$, the FEE (32) is specified as $f_1$ where
$$\alpha = 1; \quad \beta = 1; \quad \gamma = 1; \quad \theta = 1.$$

(ii) When $(R, T) \in II$, the FEE (32) is specified as $f_2$ where
$$\alpha = 1; \quad \beta = 1; \quad \gamma = 2; \quad \theta = 1.$$

(iii) When $(R, T) \in III$, the FEE (32) is specified as $f_3$ where
$$\alpha = 1; \quad \beta = 2; \quad \gamma = 2; \quad \theta = 2.$$

(iv) When $(R, T) \in VI$, the FEE (32) is specified as $f_4$ where
$$\alpha = 2; \quad \beta = 2; \quad \gamma = 2; \quad \theta = 2.$$

(v) When $(R, T) \in V$, the FEE (32) is specified as $f_5$ where
$$\alpha = 2; \quad \beta = 1; \quad \gamma = 1; \quad \theta = 2.$$

Finally, we consider the NEG with time-varying payoff parameters as

$$\begin{cases} R = \sin\left(\frac{\pi}{6}t\right) \\ T = \cos\left(\frac{\pi}{6}t\right). \end{cases}$$

Then it is clear that the FEE becomes a periodic function with period 12, precisely,

$$x_i(t+1) = f(t) \ltimes_{j=i-2}^{i+2} x_i(t), \quad \forall i, \tag{33}$$

where

$$f(t) = \begin{cases} f_1, & t \in \{12m, 12m+1, 12m+9, 12m+10, 12m+11\} \\ f_3, & t = 12m+8 \\ f_4, & t \in \{12m+2, 12m+4, 12m+5, 12m+6, 12m+7\} \\ f_5, & t = 12m+2, \qquad m \in \mathbb{Z}_+. \end{cases}$$

Note that this is correct for $S_n$ and $R_\infty$ (Graph of $\mathbb{R}$ with all integers as nodes).

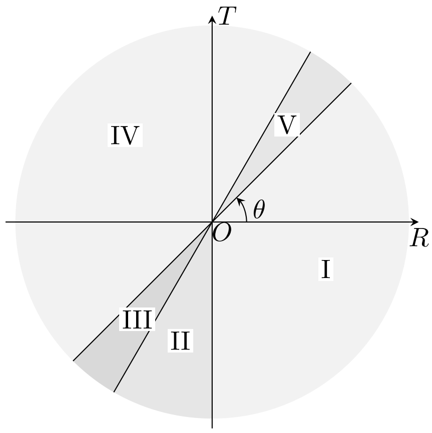


Fig. 2. A Partition of Parameter Space

## 7. CONCLUSION

Based on the FEEs of all players, the SPD of overall NEG is constructed. Using SPD, the controlled NEGs are introduced and converted into standard $k$-valued logical control networks. Then some more complicated kinds of NEGs are investigated. They are (i) players using different length historical information; (ii) players of multi-species; and (iii) the fundamental network game with time-varying payoffs. The formulations for these three more complicated kinds of NEGs are obtained, some interesting results are investigated.

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