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$\mathscr{H}_2$-norm of fractional transfer functions of implicit type of the first kind

Mathieu Chevrié *,** **Rachid Malti** * **Christophe Farges** *
Jocelyn Sabatier *

* *Bordeaux University, IMS – UMR 5218 CNRS*
351 cours de la Libération, 33405 Talence cedex, France
(e-mail: firstname.lastname@ims-bordeaux.fr)
** *PSA Peugeot Citroën*
2 route de Gisy, 78943 Vélizy-Villacoublay Cedex, France
(e-mail: mathieu.chevrie@mpsa.com)

Abstract: This paper studies the $\mathscr{H}_2$-norm, or impulse response energy, of fractional transfer functions of implicit type. The analytical expression of the $\mathscr{H}_2$-norm is first derived for an elementary fractional transfer function of the first kind with a single real pole. Series connection of such a transfer function with a pure fractional integrator and with another implicit transfer function of the first kind are then studied. Results developed in the paper are finally used to derive a criterion to evaluate the quality of an integer order approximation for an implicit type fractional order model of the first kind.

Keywords: $\mathscr{H}_2$-norm, fractional order models, stability, recursive poles-zeros approximation

1. INTRODUCTION AND MATHEMATICAL BACKGROUND

Define a fractional transfer function of implicit type as:

$$F(s) = K \prod_{i=0}^{M} (s + \lambda_i')^{\nu_i'} \Big/ \prod_{j=0}^{N} (s + \lambda_j)^{\nu_j} , \qquad (1)$$

where $K \in \mathbb{R}$ is a gain, and where each $-\lambda_i' \in \mathbb{C}$ in the numerator is defined as an s-zero of multiplicity $\nu_i' \in \mathbb{R}_+^\star$. Similarly, each $-\lambda_j \in \mathbb{C}$ in the denominator is defined as an s-pole of multiplicity $\nu_j \in \mathbb{R}_+^\star$. Moreover, to have a real-valued impulse response of $F(s)$, complex s-zeros and s-poles must be conjugate, that is: if λ_i (similarly λ_i') is an s-pole (s-zero) of $F(s)$, then $\overline{\lambda_i}$ (similarly $\overline{\lambda_i'}$) is also an s-pole (s-zero) of $F(s)$. Additionally, numerator order must be less than denominator order

$$\sum_{i=0}^{M} \nu_i' < \sum_{j=0}^{N} \nu_j \qquad (2)$$

for the system to be strictly proper. Moreover, the following assumptions allow not to have any possible simplification in the numerator, in the denominator, and between the numerator and the denominator: $\lambda_i' \neq \lambda_k', \forall (i,k) \in \{0,1,\dots,M\}^2$, $i \neq k$; $\lambda_i \neq \lambda_k, \forall (i,k) \in \{0,1,\dots,N\}^2$, $i \neq k$; $\lambda_i' \neq \lambda_k, \forall i \in \{0,1,\dots,M\}$ and $\forall k \in \{0,1,\dots,N\}$.

Transfer function (1) was first introduced in the literature by Davidson and Cole in Davidson and Cole (1951) to model dielectric relaxation in glycerol, propylene glycol and n-propanol. It is thus known as Davidson-Cole transfer functions Khamzin and Popov (2012), Jeon and Grischkowsky (1998) but can also be denoted as implicit transfert function. The adjective implicit is introduced to highligh that the fractional order ν does not directly affect the Laplace variable s but a polynomial of s. Since Davidson and Cole work, implicit fractional transfer functions have been used many time to model dielectric relaxations Khamzin and Popov (2012). But transfer function (1) can be found in several other fields such as:

- materials Caponetto et al. (2012) in which it is used to model ionic polymer metal composites materials;
- thermal Nguyen (2013) to model themal behaviour of wires in automotive application;
- hydrodynamic Tarasov (2005) to describe dynamical fluid flow in the fractal medium;
- electrochemistry Sabatier et al. (2006) Sabatier et al. (2013) to model lead-acid and lithium ion batteries;
- bioengineering Sommacal et al. (2006) Melchior et al. (2012) for rat muscle modelling using fractional multimodels or to take into account thermal transfers inside lungs;
- signal processing Ortigueira and Trujillo (2012) to show the link between the fractional derivative and the Hilbert transform;
- robust control Oustaloup (1995) to define a robust control strategy.

If transfer function $F(s)$ has no s-zero, it can be written as a series connexion of elementary transfer functions of the first and second kind defined as follows.

A transfer function of implicit type of the first kind involves a single real s-pole:

$$F_1(s) = \frac{1}{(s+\lambda)^\nu}, \qquad (3)$$

where $\lambda \in \mathbb{R}$ and $\nu \in \mathbb{R}_+^\star$.

A transfer function of the second kind involves two complex conjugate s-poles of multiplicity ν:

$$F_2(s) = \frac{1}{(s+\lambda)^\nu (s+\bar{\lambda})^\nu} = \frac{1}{((s+\mathscr{Re}(\lambda))^2+\mathscr{Im}(\lambda)^2)^\nu} \quad (4)$$

where $\lambda \in \mathbb{C}$ and $\nu \in \mathbb{R}_+^\star$. Such a transfer function involves only real-valued parameters in its expended form (4).

Instead of tackling the most general case of implicit transfer functions defined by (1), this paper studies the $\mathscr{H}_2$-norm of elementary transfer functions of the first kind and their series connection.

1.1 Time domain response of implicit transfer functions of the first kind

Applying the frequency shift property of the Laplace transform, it is easy to show that the impulse response of the elementary transfer function of the first kind (3) is given by Oustaloup (1995):

$$f_1(t) = \mathscr{L}^{-1}\{F_1(s)\} = e^{-\lambda t}\mathscr{L}^{-1}\{s^{-\nu}\} = e^{-\lambda t}\frac{t^{1-\nu}}{\Gamma(\nu)} \quad (5)$$

where Γ is the Euler Gamma function (A.1). Note the presence of a singularity at $t = 0$ for all $0 < \nu < 1$.

1.2 Stability of implicit fractional transfer functions

Stability of fractional systems was treated in different contexts (linear, non linear, commensurate, non commensurate, time-variant, time invariant, delayed, non delayed, analytical, numerical) by different authors as presented in the state of the art by Sabatier et al. (2010) with more than 20 references on the subject.

The system $F(s)$ is $\mathscr{L}_p$-stable, $1 \leq p \leq \infty$, if and only if:

$$\sup_{u\in\mathscr{L}_p, u\neq 0} \frac{\|f \star u\|_p}{\|u\|_p} < \infty \quad (6)$$

where $\star$ stands for the convolution product and $u(t)$ the system input.

Regarding BIBO stability (defined as the $\mathscr{L}_\infty$-stability) of a system described by transfer function (1) nothing can be found in the literature. This may be due to that the stability condition is similar to classical integer system stability condition. The stability depends on the sign of the real part of the pole $-\lambda_j$.

Theorem 1. Transfer function (3) is BIBO stable if the λ is greater than 0.

Proof. Let $0 < \nu < 1$, the response of a system of the first kind to an input $u(t)$ is given by

$$y(t) = \int_0^t \frac{1}{\Gamma(\nu)}\tau^{\nu-1}e^{-\lambda\tau}u(t-\tau)d\tau. \quad (7)$$

It is now supposed that $|u(t)| < M$ and that $\lambda \in \mathbb{R}$, thus:

$$|y(t)| = \left|\int_0^t \frac{1}{\Gamma(\nu)}\tau^{\nu-1}e^{-\lambda\tau}u(t-\tau)d\tau\right| \quad (8)$$

$$\leq \frac{1}{\Gamma(\nu)}M\int_0^t \left|\tau^{\nu-1}e^{-\lambda\tau}\right| d\tau. \quad (9)$$

As $\tau^{\nu-1}e^{-\lambda\tau} \geq 0$ for $0 \leq \tau \leq t$, we have:

$$|y(t)| \leq \frac{1}{\Gamma(\nu)}M\int_0^t \tau^{\nu-1}e^{-\lambda\tau}d\tau. \quad (10)$$

For any $t_0 \in \mathbb{R}_+$, then

$$|y(t)| \leq \frac{M}{\Gamma(\nu)}\left(\int_0^{t_0} \tau^{\nu-1}e^{-\lambda\tau}d\tau + \int_{t_0}^t \tau^{\nu-1}e^{-\lambda\tau}d\tau\right) \quad (11)$$

and thus given that $\tau^{\nu-1}$ is a decreasing function on $\mathbb{R}$

$$|y(t)| \leq \frac{M}{\Gamma(\nu)}\left(\int_0^{t_0} \tau^{\nu-1}e^{-\lambda\tau}d\tau + \int_{t_0}^t t_0^{\nu-1}e^{-\lambda\tau}d\tau\right). \quad (12)$$

In the previous equation and according the Riemann criterion, the first integral converge as $0 < \nu < 1$. We thus suppose that this integral is bounded by a real I_0. The second integral can be computed and thus:

$$|y(t)| \leq \frac{M}{\Gamma(\nu)}\left(I_0 - \frac{t_0^{\nu-1}}{\lambda}e^{-\lambda(t-t_0)}\right). \quad (13)$$

Hence if $\lambda \geq 0$, $|y(t)|$ is bounded.

Considering now that $\lambda < 0$ and that the input $u(t)$ is the Heaviside function. Hence, $y(t)$ is given by:

$$|y(t)| = y(t) = \frac{1}{\Gamma(\nu)}\int_0^t \tau^{\nu-1}e^{-\lambda\tau}d\tau. \quad (14)$$

Thus,

$$y(t) = \frac{1}{\Gamma(\nu)}\left(\int_0^{t_0} \tau^{\nu-1}e^{-\lambda\tau}d\tau + \int_{t_0}^t t_0^{\nu-1}e^{-\lambda\tau}d\tau\right). \quad (15)$$

As before, we suppose that the first integral is bounded by a real scalar I_0. However, as $\lambda < 0$ the second integral tends to ∞ when t tends to ∞. Thus there exists a bounded input such that the output is not bounded for $\lambda < 0$. This complete the proof. ■

Remark 2. A general implicit transfer function $F(s)$ (given by (1)) can always be written as a series connection of transfer functions (3). As an example consider the function

$$H(s) = \frac{(s+\lambda_1)^{\nu_1}}{(s+\lambda_2)^{\nu_2}} \quad (16)$$

with $N < \nu_1 < N+1$ and $M < \nu_2 < M+1$, $N \in \mathbb{N}$ and $M \in \mathbb{N}$. Thus, (16) can be rewritten as:

$$H(s) = \frac{(s+\lambda_1)^{N+1}(s+\lambda_1)^{\nu_1-(N+1)}}{(s+\lambda_2)^M(s+\lambda_2)^{\nu_2-M}} \quad (17)$$

with $-1 < \nu_1 - (N+1) < 0$ and $0 < \nu_2 - M < 1$.

Result of theorem 1 can be proved in a different way using the following theorem.

Theorem 3. A transfer function is BIBO stable if and only if the $\mathscr{L}_1$ norm of its impulse response is bounded.

From (7),$f_1(t)$ is defined by:

$$f_1(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}e^{-\lambda t}. \quad (18)$$

Using the proof of theorem 1 with $M = 1$, it is easy to show that the $\mathscr{L}_1$ norm of $f_1(t)$ is bounded if λ is greater than 0.

1.3 Problem formulation

The $\mathscr{H}_2$-norm of a transfer function $F(s)$, namely $\|F\|_2$, which also corresponds to the impulse response energy, or

ℓ_2-norm of $f(t)$ namely $\|f\|_{\ell_2}$, can be computed either in the time-domain

$$\|f\|_{\ell_2}^2 = \int_0^\infty f(t)f(t)dt \tag{19}$$

or in the frequency-domain using Plancherel's theorem

$$\|F\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mathrm{j}\omega)\overline{F(\mathrm{j}\omega)}d\omega = \frac{1}{\pi}\int_0^\infty F(\mathrm{j}\omega)\overline{F(\mathrm{j}\omega)}d\omega. \tag{20}$$

Recently Malti et al. (2011) established analytical expressions for computing the $\mathscr{H}_2$-norm of fractional transfer functions of explicit type, i.e. with fractional powers on the s variable (not on $(s+\lambda)$), which requires only the knowledge of transfer function coefficients and differentiation orders. Further, $\mathscr{L}_p$-norm boundedness conditions of the impulse response of fractional transfer functions of explicit type has been established in Malti (2013).

The main concern of this paper is to extend these results to implicit transfer functions of the first kind. More precisely, the main objective is to derive analytical expressions of the $\mathscr{H}_2$-norm of elementary transfer functions of the first kind and their series connection. Hence, these analytical formulae are given in section 2 for an elementary fractional transfer function of the first kind. Then, the series connection of such a transfer function with a pure fractional integrator and with another transfer function of the first kind is studied in section 3. In section 4, these results are used to derive a criterion to appreciate the quality of an integer order approximation of an implicit type fractional order model of the first kind. Finally, some conclusions are drawn in section 5.

2. $\mathscr{H}_2$-NORM OF A FRACTIONAL TRANSFER FUNCTION OF THE FIRST KIND

In this section is first derived the analytical expression for the computation of the $\mathscr{H}_2$-norm of an implicit transfer function of the first kind. Then, the analytical formula of the $\mathscr{H}_2$-norm of such a transfer function in the special case when ν takes an integer value is deduced. Finally, the $\mathscr{H}_2$-norm computation versus differentiation order is presented for several values of the s-pole λ.

2.1 Analytical expression of the $\mathscr{H}_2$-norm of a fractional transfer function of the first kind

The first result regarding the computation of the $\mathscr{H}_2$-norm of fractional transfer functions of the first kind is presented in the following theorem.

Theorem 4. The $\mathscr{H}_2$-norm of the stable transfer function $F_1(s)$ defined in (3), is given by:

$$\|F_1\|_2^2 = \begin{cases} \infty & \text{if } \nu \le \dfrac{1}{2} \\ \dfrac{\mathrm{B}\left(\frac{1}{2}, \nu - \frac{1}{2}\right)}{2\pi\lambda^{2\nu-1}} & \text{if } \nu > \dfrac{1}{2} \end{cases} \tag{21}$$

where $\mathrm{B}(.,.)$ is the Euler Beta function whose definition is reminded (A.3).

Proof. According to (20):

$$\|F_1\|_2^2 = \frac{1}{\pi}\int_0^\infty \frac{d\omega}{(\lambda^2+\omega^2)^\nu}. \tag{22}$$

Define the following change of variable

$$\omega^2 = x \to d\omega = \frac{dx}{2\sqrt{x}}. \tag{23}$$

Hence,

$$\|F_1\|_2^2 = \frac{1}{2\pi\lambda^{2\nu}}\int_0^\infty \frac{x^{-\frac{1}{2}}dx}{(1+\frac{1}{\lambda^2}x)^\nu}. \tag{24}$$

Such an integral has been computed by Erdélyi et al. (1954) and is reported in formula 3.194.3 of Gradshteyn and Ryshik (2007), (see also (A.4)). Hence, provided $\nu > \frac{1}{2}$, one gets (21) which is also equivalent to:

$$\|F_1\|_2^2 = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu-\frac{1}{2}\right)}{2\pi\lambda^{2\nu-1}\Gamma(\nu)}. \tag{25}$$

In the other cases, when $\nu \le \frac{1}{2}$, the integral (24) is divergent. ■

Based on the Stirling asymptotic formula of the Gamma function (A.2), it can be shown that:

$$\lim_{\nu\to\infty}\|F_1\|_2 = \begin{cases} 0 & \text{if} \quad \lambda \ge 1 \\ \infty & \text{if} \quad 0 < \lambda < 1. \end{cases} \tag{26}$$

2.2 Analytical expression of the $\mathscr{H}_2$-norm of a fractional transfer function of the first kind when ν is an integer

It is well known that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Moreover, it can be shown easily that

$$\Gamma\left(n-\frac{1}{2}\right) = \frac{2^{1-2n}\Gamma\left(\frac{1}{2}\right)\Gamma(2n)}{\left(n-\frac{1}{2}\right)\Gamma(n)}. \tag{27}$$

Hence, for all integer values of $\nu = n \in \mathbb{N}$, equations (21) for $\nu > \frac{1}{2}$ and (25) reduce to :

$$\|F_1\|_2^2 = \frac{2^{-2n}(2n-1)!}{\lambda^{2n-1}\left(n-\frac{1}{2}\right)\left((n-1)!\right)^2}. \tag{28}$$

This equation can further be established by using the early work by Aström (1970).

2.3 $\mathscr{H}_2$-norm computation for different values of the differentiation order and of the s-pole

The $\mathscr{H}_2$-norm of $F_1(s)$ in (3) with different values of λ is plotted in figure 1. The $\mathscr{H}_2$-norm tends to ∞ as ν tends to $\frac{1}{2}$. On the other side, if $0 < \lambda < 1$, then the $\mathscr{H}_2$-norm tends to ∞ as ν tends to ∞, otherwise if $\lambda \ge 1$ the $\mathscr{H}_2$-norm tends to 0. Moreover, when ν is an integer the transfer function (3) reduces to a rational one. Hence analytical formulae established since the early work by Aström (1970), and implemented in the `norm` method of the dynamic system class `tf` in Matlab, are applied to compute the $\mathscr{H}_2$-norm when $\nu = 1$ and $\nu = 2$ (respectively represented by the squares and the diamonds in figure 1). These values show to coincide with (21).

3. $\mathscr{H}_2$-NORM OF SOME SERIES CONNECTION OF FRACTIONAL TRANSFER FUNCTION OF THE FIRST KIND

Result of previous section is now extended to handle the series connection of a fractional transfer function of the first kind, firstly with a pure fractional integrator and then with another implicit transfer function of the first kind.

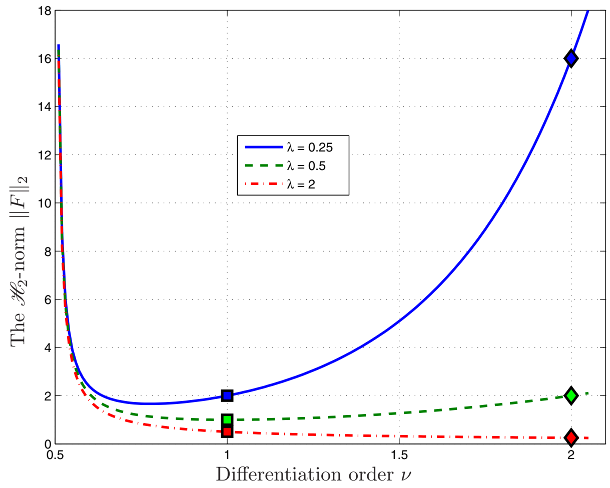

Fig. 1. The $\mathscr{H}_2$-norm squared $\|F_1\|_2^2$ versus differentiation order ν, for different values of λ.

3.1 $\mathscr{H}_2$-norm of a fractional transfer function of the first kind cascaded with a pure fractional integrator

The $\mathscr{H}_2$-norm of the series connection of a fractional transfer function of the first kind and a pure fractional integrator is given in the following theorem.

Theorem 5. The $\mathscr{H}_2$-norm of the stable transfer function $F_1(s)$ defined in (3), cascaded with a pure fractional integrator of order $\mu > 0$, namely,

$$\tilde{F}_1(s) = \frac{F_1(s)}{s^\mu} = \frac{1}{s^\mu\,(s+\lambda)^\nu}, \tag{29}$$

is given by:

$$\left\|\tilde{F}_1\right\|_2^2 = \begin{cases} \infty & \text{if } \nu+\mu \leq \dfrac{1}{2} \text{ or } \mu \geq \dfrac{1}{2} \\ \dfrac{\mathrm{B}\left(\frac{1}{2}-\mu, \nu+\mu-\frac{1}{2}\right)}{2\pi\lambda^{2\nu+2\mu-1}} & \text{if } \nu+\mu > \dfrac{1}{2} \text{ and } \mu < \dfrac{1}{2} \end{cases} \tag{30}$$

Proof. According to (20):

$$\left\|\tilde{F}_1\right\|_2^2 = \frac{1}{\pi}\int_0^\infty \frac{\omega^{-2\mu}d\omega}{(\lambda^2+\omega^2)^\nu}. \tag{31}$$

Using the change of variable (23) leads to,

$$\left\|\tilde{F}_1\right\|_2^2 = \frac{1}{2\pi\lambda^{2\nu}}\int_0^\infty \frac{x^{-\mu-\frac{1}{2}}dx}{(1+\frac{1}{\lambda^2}x)^\nu}. \tag{32}$$

As before, the solution of (32) is given by formula 3.194.3 of Gradshteyn and Ryshik (2007). Hence, provided $\nu + \mu > \frac{1}{2}$ and $\mu < \frac{1}{2}$, one gets (30). ∎

3.2 $\mathscr{H}_2$-norm of two cascaded fractional transfer functions of the first kind

The series connection of two fractional transfer functions of the first kind is now considered.

Theorem 6. Let $F_{12}(s)$ resulting in the series connection of two transfer functions of the first kind:

$$F_{12}(s) = \frac{1}{(s+\lambda_1)^{\nu_1}\,(s+\lambda_2)^{\nu_2}}. \tag{33}$$

The $\mathscr{H}_2$ norm of F_{12} is given by (34) (see at the top of the next page).

Proof. According to (20):

$$\|F_{12}\|_2^2 = \frac{1}{\pi}\int_0^\infty \frac{d\omega}{(\omega^2+\lambda_1^2)^{\nu_1}(\omega^2+\lambda_2^2)^{\nu_2}} \tag{35}$$

Using once again the change of variable (23), one gets

$$\|F_{12}\|_2^2 = \frac{1}{\pi}\int_0^\infty \frac{x^{-\frac{1}{2}}dx}{(x+\lambda_1^2)^{\nu_1}(x+\lambda_2^2)^{\nu_2}} \tag{36}$$

Such an integral is reported in formula 3.197.1 of Gradshteyn and Ryshik (2007) then provided $\nu_1+\nu_2 > \frac{1}{2}$, one gets (34). ∎

4. APPLICATION

Results developed in the paper are now used to derive a criterion to evaluate the quality of an integer order approximation of an implicit type fractional order model of the first kind.

Such a model denoted $F_1(s)$ as given in equation (3) can be approximated by an integer order model using several methods. Using the well-known recursive poles-zeros method Oustaloup (1995), $F_1(s)$ can be approximated by the integer order transfer function:

$$\bar{F}_N(s) = C_0 \prod_{k=1}^{N-1}(s+\bar{z}_k) \Bigg/ \prod_{k=1}^{N}(s+\bar{\lambda}_k) \tag{37}$$

where $C_0 \in \mathbb{R}$ is a constant and where $\bar{z}_k \in \mathbb{R}$ and $\bar{\lambda}_k \in \mathbb{R}$ denote respectively the zeros and poles of $\bar{F}_N(s)$. $N \in \mathbb{N}$ is the number of poles in the approximation.

Using partial fraction decomposition, transfer function $\bar{F}_N(s)$ also writes:

$$\bar{F}_N(s) = \sum_{k=1}^{N} \frac{a_k}{s+\bar{\lambda}_k} \tag{38}$$

where $a_k \in \mathbb{R}$.

In order to evaluate the quality of such an approximation, let us consider the block diagram of figure 2.

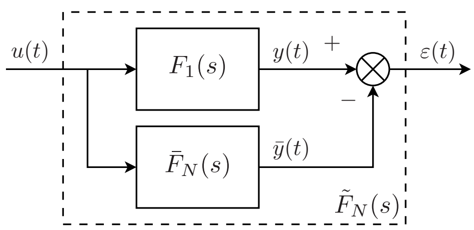

Fig. 2. Block diagram of output error

Same input $u(t)$ is applied to transfer functions $F_1(s)$ and $\bar{F}_N(s)$, leading to outputs denoted $y(t)$ and $\bar{y}(t)$ respectively. Output error $\varepsilon(t) = y(t) - \bar{y}(t)$ is studied.

Transfer function between $u(t)$ and $\varepsilon(t)$ is given by:

$$\tilde{F}_N(s) = F_1(s) - \bar{F}_N(s) \tag{39}$$

Computing $\mathscr{H}_2$-norm of $\tilde{F}_N(s)$ allows to appreciate the quality of the approximation according to the following property given in Zhou et al. (1996):

$$\|\tilde{F}_N(s)\|_2 = \sup_{\|u(t)\|_2=1} \|\varepsilon(t)\|_\infty. \tag{40}$$

$$\|F_{12}\|_2^2 = \begin{cases} \infty & \text{if} \quad \nu_1+\nu_2 \leq \frac{1}{2} \\ \frac{\lambda_1^{-2\nu_1}\lambda_2^{1-2\nu_2}}{2\pi} \mathrm{B}\left(\frac{1}{2}, \nu_1+\nu_2-\frac{1}{2}\right) {}_2\mathrm{F}_1\left(\nu_1, \frac{1}{2}; \nu_1+\nu_2; 1-\left(\frac{\lambda_2}{\lambda_1}\right)^2\right) & \text{if} \quad \nu_1+\nu_2 > \frac{1}{2} \end{cases} \tag{34}$$

where ${}_2\mathrm{F}_1(a,b;c,z)$ is the Gauss hypergeometric function defined in (A.5). Functions in (34) can be easily numerically evaluated with softwares such as Matlab, Mathematica or Maple. This remark also holds for functions in (21) and (30).

Indeed, last formula indicates that maximum value of the approximation error $\varepsilon(t)$ in response to any input $u(t)$ of unitary energy is $\|\tilde{F}_N(s)\|_2$. According to equation (20), the calculation of $\|\tilde{F}_N(s)\|_2^2$ leads to the following integral:

$$\|\tilde{F}_N(s)\|_2^2 = \frac{1}{\pi}\int_0^\infty \left(\frac{1}{(\mathrm{j}\omega+\lambda)^\nu} - \sum_{k=1}^N \frac{a_k}{s+\bar{\lambda}_k}\right) \times \overline{\left(\frac{1}{(\mathrm{j}\omega+\lambda)^\nu} - \sum_{k=1}^N \frac{a_k}{s+\bar{\lambda}_k}\right)} d\omega. \tag{41}$$

Developing the integrand of equation (41) integral, and noting that the integral of the sum is equal to the sum of the integrals, one has to solve 3 kinds of integrals.

The first type of integral is:

$$\frac{1}{\pi}\int_0^\infty \frac{1}{(\mathrm{j}\omega+\lambda)^\nu\overline{(\mathrm{j}\omega+\lambda)^\nu}} d\omega \tag{42}$$

which is the purpose of the paper (see (22) and its solution given by (21)).

The second kind of integral is:

$$\frac{a_k^2}{\pi}\int_0^\infty \frac{1}{(\mathrm{j}\omega+\lambda_k)\overline{(\mathrm{j}\omega+\lambda_k)}} d\omega \tag{43}$$

which is the squared $\mathscr{H}_2$-norm of a first order rational transfer function of static gain a_k and s-pole λ_k. The analytic expression of this $\mathscr{H}_2$-norm can be deduced from equation (28) for $n=1$, multiplying the latter result by the scalar factor a_k^2.

The remaining terms are given by integrals of the form:

$$\begin{aligned} I_{kl} &= \frac{a_k a_l}{\pi}\int_0^\infty \frac{1}{(\mathrm{j}\omega+\lambda_k)^{\nu_k}\overline{(\mathrm{j}\omega+\lambda_l)^{\nu_l}}} d\omega \\ &= (\mathrm{j})^{\nu_l-\nu_k}\frac{a_k a_l}{\pi}\int_0^\infty \frac{1}{(\omega+(-\mathrm{j}\lambda_k))^{\nu_k}\,\overline{(\omega+\mathrm{j}\lambda_l)^{\nu_l}}} d\omega. \end{aligned} \tag{44}$$

Last expression is similar to (36). Thus, using formula 3.197.1 of Gradshteyn and Ryshik (2007), the analytical expression of (44) is expressed as:

$$I_{kl} = \begin{cases} \infty & \text{if } \nu_k+\nu_l \leq 1 \\ \mathrm{j}\frac{a_k a_l \lambda_l^{1-\nu_l}}{\pi\lambda_k^{\nu_k}} \times \mathrm{B}(1, \nu_k+\nu_l-1) & \\ \times {}_2\mathrm{F}_1\left(\nu_k, 1; \nu_k+\nu_l; 1+\frac{\lambda_l}{\lambda_k}\right) & \text{if } \nu_k+\nu_l > 1 \end{cases} \tag{45}$$

Based on equations (21), (28) and (45), the analytical expression of $\mathscr{H}_2$-norm of $\tilde{F}_N(s)$ can be determined. Thanks to this result, and given that the $\mathscr{H}_2$-norm of $F_1(s)$ can be also determined using (21), the relative error between $\mathscr{H}_2$-norm of implicit fractional order model and its integer order approximation, namely

$$\epsilon(N) = \frac{\|F_1(s) - \bar{F}_N(s)\|_2}{\|F_1(s)\|_2} = \frac{\|\tilde{F}_N(s)\|_2}{\|F_1(s)\|_2} \tag{46}$$

can be obtained analytically.

For different values of N, that imposes the number of poles and zeros involved in the integer order approximation, Figure 3 shows the influence of N on the relative error when $\lambda = 1$ and $\nu = 0.75$. Value of $\epsilon(N)$ gives a numerical criterion to appreciate formally the quality of the approximation.

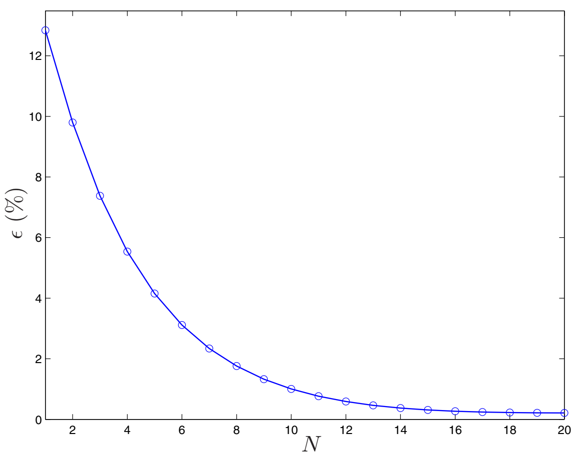

Fig. 3. Influence of N on the relative error between $\mathscr{H}_2$-norms of implicit fractional order model and its integer order approximation

5. CONCLUSION

In this paper, analytical expressions are given for the $\mathscr{H}_2$-norm of several implicit type transfer functions. Specifically, the authors are interested in the scope of this paper to transfer functions of the first kind, of the first kind plus a fractional integrator and of the series connection of two transfer functions of the first kind. These transfer functions are widely used in the literature to model real systems or phenomena, but also to define filters or control strategies. However, the properties of these transfer functions have been little studied since their introduction in 1951. Given the frequency of use of these functions, it is important to study their properties and this article thus contributes to fill this gap. The authors show the interest of the obtained $\mathscr{H}_2$-norm analytical expressions for approximating an implicit type transfer functions with a classical rational transfer function. This approximation is required for the implementation of a fractional controller or to simulate a filter designed with an implicit type transfer function. It is thus really important to have a bound on the approximation error. Such a bound is proposed in term of the infinity norm of the time response error. In future works, the authors intend to extend the number of implicit type transfer functions studied and to propose several other applications to demonstrate the interest of the $\mathscr{H}_2$-norm computation for the considered class of systems.

Appendix A. APPENDIX

- According to formula 8.310.1 Gradshteyn and Ryshik (2007), the Euler Gamma function Γ is defined as:

$$\Gamma(\nu) = \int_0^\infty e^{-x} x^{\nu-1} dx \quad , \quad \nu > 0. \tag{A.1}$$

- The Stirling asymptotique formula for $\nu \gg 0$ writes (Gradshteyn and Ryshik, 2007, formula 8.337.2):

$$\Gamma(\nu+1) \sim \sqrt{2\pi\nu}\,(\nu/e)^\nu \tag{A.2}$$

- According to formula 8.384.1 Gradshteyn and Ryshik (2007), the Euler Beta function $\mathrm{B}(x, y)$ is defined as:

$$\mathrm{B}(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)\ . \tag{A.3}$$

- The following integral is reported in formula 3.194.3 of Gradshteyn and Ryshik (2007):

$$\int_0^\infty \frac{x^{\rho-1} dx}{(1 + \beta x)^\nu} = \beta^{-\rho}\mathrm{B}(\rho, \nu - \rho). \tag{A.4}$$

where the conditions $|\arg(\beta)| < \pi$ and $\nu > \mu > 0$ must hold and where $\mathrm{B}(.,.)$ is the Euler's Beta function.

- According to formula 8.14.1 Gradshteyn and Ryshik (2007), the Gauss hypergeometric function ${}_2\mathrm{F}_1$ is:

$${}_2\mathrm{F}_1(a, b; c; \mathsf{z}) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{\mathsf{z}^k}{k!}. \tag{A.5}$$

where $(q)_k$ is the rising Pochhammer symbol:

$$(q)_k = \begin{cases} \Gamma(q + k)/\Gamma(q) & q > 0 \\ (-1)^k \Gamma(1 - q)/\Gamma(1 - q - k) & q \leq 0. \end{cases} \tag{A.6}$$

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