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Consensus of Multi-Agent Systems by Distributed Event-Triggered Control ⋆

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Abstract: In this paper, the consensus problem for multi-agent systems with general linear dynamics is studied. A novel event-triggered control scheme with some desirable features, including distributed, asynchronous and independent, is proposed. It is shown that consensus of the controlled multi-agent system can be reached asymptotically. The feasibility of the event-triggered strategy is further verified by the exclusion of both singular triggering and Zeno behaviors. The proposed control scheme is applied to solve a spacecraft formation flying problem. Simulations illustrate the effectiveness of the control scheme.

1. INTRODUCTION

In the past few years, various control problems of a group of networked subsystems called agents have been extensively studied. These problems include formation, synchronization, flocking, swarming, rendezvous, to name just a few. In many circumstances, the states of all agents need to reach a common quantity of interest while each agent only has access to information of its neighboring agents. This is the so-called consensus problem of multi-agent systems under distributed framework, which plays a fundamental role in study of multi-agent systems. Typical results on this topic can be found in Cortés and Bullo [2005], Jadbabaie et al. [2003], Olfati-Saber [2006], Olfati-Saber et al. [2007], Ren [2007], Ren and Atkins [2007] and references therein.

It should be noted that each individual agent is usually equipped with simple embedded microprocessors, on-board communication modules, and actuation modules, which have limited energy resources to perform such functions as gathering information, communicating with neighboring agents, and driving the agent. For most existing control strategies in consensus problems, an agent needs to measure its state, send the state to its neighbors, and update its control signal continuously or in a fixed sampling rate. Such control laws might become infeasible or impractical in many applications due to their excessive consumption of on-board energy resources. It is thus desirable to design novel control schemes, such that the load of communication and controller update for each agent can be reduced significantly. In this way, limited energy resources of agents can be greatly saved and operational lifespan of multi-agent systems can be thus prolonged. To address this issue, Tabuada [2007] introduces an event-triggered strategy for a stabilization problem, where the control actuation is triggered whenever a defined error exceeds a threshold with respect to the norm of the state. Dimarogonas et al. [2012] apply the distributed event-driven scheme to a first order agreement problem in multi-agent systems. Most recently, a new combined measurement approach to event-based design is developed in Fan et al. [2013]. As a result, control of agents is only triggered at their own event time, which is a significant improvement. The authors in Seyboth et al. [2012] present an event-based broadcasting strategy for multi-agent systems with double-integrator dynamics, where a time-dependent threshold is used to bound each agent's measurement error. Some other relevant studies on the topic can be found in Hu et al. [2011], Liu et al. [2012], Wang and Lemmon [2011], Wang and Ni [2012].

In spite of the advances in event-triggered consensus of multi-agent systems, there are still some challenging issues to be addressed. One of the issues is to consider more general agent dynamics as most existing works focus on single or double integrators. Another issue is to develop more efficient triggering mechanisms that have the following features. First, each agent only needs the information of its neighbors and itself, termed as distributed. Second, all the agents are not required to be triggered at a synchronous way, termed as asynchronous. Third, triggering of an agent should not affect or be affected by triggering of other agents, termed as independent. However, the triggering mechanisms in most existing works only have some but not all of these features. For example, all the agents are triggered at the same time, namely, in a synchronous fashion in Hu et al. [2011], Liu et al. [2012], Wang and Ni [2012]. The controllers in Dimarogonas et al. [2012] are required to be triggered at the neighbors' event time.

Motivated by the above-mentioned considerations, in this paper, we propose a novel distributed event-triggered strategy with those desirable features for the consensus problem of multi-agent systems with general linear dynamics.

The rest of the paper is organized as follows. Section 2 introduces some preliminaries and problem formulation. In Section 3, we propose a novel distributed event-triggered

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control scheme for multi-agent systems with general linear dynamics and analyze its feasibility. The proposed control scheme is then applied to solve a spacecraft formation flying problem in Section 4 and conclusions are drawn in Section 5.

2. PROBLEM FORMULATION

2.1 Notations

We will use the notation $\|\cdot\|$ to denote the Euclidean norm for vectors or the induced 2-norm for matrices. Given a symmetric matrix M, $M > 0$ (or $M \geq 0$) means that M is a positive definite (or semi positive definite) matrix. The notation $A \otimes B$ represents the Kronecker product of matrices A and B, with the following properties: $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$. Denote $\mathcal{N} = \{1, \cdots, N\}$. I_n is the identity matrix with dimension n.

2.2 Algebraic Graph Theory

We will state some useful facts from algebraic graph theory. One can refer to Godsil and Royle [2001] for more details.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of a set of vertices or nodes $\mathcal{V} = \{1, \cdots, N\}$ and edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, if there is an edge $(i, j) \in \mathcal{E}$ between nodes i and j, then nodes i and j are called adjacent. Graph $\mathcal{G}$ is called undirected if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. The adjacency matrix $\mathcal{A} = \mathcal{A}(\mathcal{G}) = (a_{ij})_{N \times N}$ is an $N \times N$ matrix defined by $a_{ij} = 1$ if and only if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. A path from i to j is a sequence of distinct nodes, which starts with i and ends with j while each pair of consecutive nodes is adjacent. If there is a path between any two nodes of the graph $\mathcal{G}$, then $\mathcal{G}$ is called connected. The degree matrix D of $\mathcal{G}$ is a diagonal matrix with element d_i equaling the cardinality of node i's neighbor set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. The Laplacian matrix L of $\mathcal{G}$ is defined as $L = D - \mathcal{A}$. For undirected graphs, L satisfies $L = L^T \geq 0$ with the vector of ones $\mathbf{1}$ as an eigenvector corresponding to the eigenvalue zero. If an undirected graph is connected, the Laplacian has a single zero eigenvalue, and the other eigenvalues can be listed in an increasing order, $0 = \lambda_1(\mathcal{G}) < \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_N(\mathcal{G})$. The second smallest eigenvalue $\lambda_2(\mathcal{G})$ is called the algebraic connectivity or Fiedler eigenvalue (Fiedler [1973]).

2.3 Problem Formulation

Consider a multi-agent system with N agents. The dynamics of the i-th agent are described by

$$\dot{x}_i = Ax_i + Bu_i, \ \ i \in \mathcal{N}, \tag{1}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ is the control input, A and B are constant matrices with compatible dimensions. If agents i and j are adjacent, then the pair of agents can communicate with each other, and the communication topology among all the agents is represented by an undirected graph $\mathcal{G}$.

In order to develop our event-triggered strategy, we consider the combined measurement

$$q_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) \tag{2}$$

as in Fan et al. [2013]. In this case, the measurement error can be defined as

$$e_i(t) = q_i\left(t_k^i\right) - q_i(t). \tag{3}$$

We propose the following control law for agent i

$$u_i(t) = Kq_i\left(t_k^i\right), \ t \in \left[t_k^i, t_{k+1}^i\right), \tag{4}$$

where K is the feedback gain matrix to be designed. The event triggering time sequence $\left\{t_0^i, t_1^i, \cdots\right\}$ for agent i will be determined by the following triggering condition which is also to be developed,

$$h\left(e_i\left(t_k^i\right), q_i\left(t_k^i\right)\right) = 0. \tag{5}$$

Definition 2.1. The consensus problem for multi-agent systems described by (1) is said to be solved if and only if for any finite $x_i(0), \forall i \in \mathcal{N}$, the states of agents satisfy:

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \ \ \forall i, j \in \mathcal{N}. \tag{6}$$

In this paper, the objective is to develop, for each agent i, a control law and an event-triggered mechanism of the form (4) and (5), respectively, such that the consensus problem is solved.

To achieve this, we introduce the following assumptions and lemma.

Assumption 2.1. (A, B) is stabilizable.

Assumption 2.2. The undirected communication graph $\mathcal{G}$ is connected.

Lemma 2.1. (Kucera [1972]) Consider a linear system (A, B, C), if (A, B) is stabilizable and (C, A) is observable, then there is a unique solution $P > 0$ to the following so-called algebraic Riccati equation

$$PA + A^T P - PBB^T P + C^T C = 0. \tag{7}$$

∎

3. DISTRIBUTED EVENT-TRIGGERED CONTROL DESIGN

3.1 Distributed Event-Triggered Control Scheme

In this subsection, a distributed event-triggered control scheme for the multi-agent system with linear dynamics (1) will be developed.

Theorem 3.1. Under Assumptions 2.1 and 2.2, there always exists at least one solution $P > 0$ for the following inequality

$$PA + A^T P - \alpha PBB^T P + \beta I_n \leq 0, \tag{8}$$

where $0 < \alpha \leq 2\lambda_2$, $\beta \geq 2\lambda_N$, with λ_2 and λ_N the Fiedler eigenvalue and the largest eigenvalue of the Laplacian matrix associated with graph $\mathcal{G}$, respectively. Then, letting $K = B^T P$, the consensus problem of the multi-agent system (1) can be solved by the control law (4) with the triggering condition

$$h(e_i(t), q_i(t)) = \|e_i(t)\| - \eta_i \|q_i(t)\| = 0, \tag{9}$$

where $\eta_i = \sqrt{\frac{\sigma_i \cdot a(2 - a\rho)}{\rho}}$ with $\sigma_i \in (0, 1)$, $\rho = \left\|PBB^T P\right\|$, and a being a positive number satisfying $a < \frac{2}{\rho}$. ∎

The proof is provided in the Appendix A.

Remark 3.1. Our results contain several existing results as special cases. On one hand, if we let $A = 0$, $B = 1$, for any topology with N agents, inequality (8) always holds with $P \geq \sqrt{\frac{\beta}{\alpha}} I_n$, thus our results can be used to tackle the problem in Fan et al. [2013] where multi-agent systems with single integrator dynamics are considered. On the other hand, it can be seen that when $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, inequality (8) can be always solved, thus our results can also be used to handle the consensus problem for double-integrator agents in Seyboth et al. [2012]. ∎

Remark 3.2. The proposed event-triggered control scheme has the three desirable features mentioned in Introduction, that is, it is distributed, asynchronous, and independent. However, it is noted that continuous monitoring of the combined measurement is required to check condition (9), which is a disadvantage of this control scheme. How to avoid this continuous monitoring is an interesting and challenging problem, which we are still working on. ∎

3.2 Feasibility

Now we investigate the feasibility of this proposed triggering mechanism by excluding both scenarios of *singular triggering* and *Zeno behavior*.

Singular triggering means that there will be no more triggering after a single triggering. We will prove that such scenario will not happen for our proposed control scheme in the following theorem.

Theorem 3.2. Consider the multi-agent system with linear dynamics (1), controller (4) and triggering rule (9). No agent will exhibit singular triggering behavior. ∎

The proof is attached in the Appendix B.

As for Zeno behavior, which means that there are infinite number of triggering instants in a finite time (Tabuada [2007]), we have the following result.

Theorem 3.3. Consider the multi-agent system with linear dynamics (1), controller (4) and triggering rule (9). No agent will exhibit Zeno behavior. ∎

The proof is given in the Appendix C.

4. AN EXAMPLE

In this section, we will apply the proposed control scheme to solve a spacecraft formation flying problem in the low Earth orbit (Ren [2007], Li et al. [2010]).

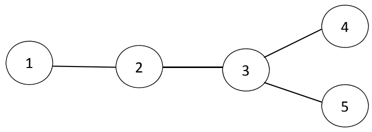

Fig. 1. Communication graph $\mathcal{G}$ of the multi-agent system

Consider the relative dynamics of the ith satellite with respect to the virtual spacecraft in the following linearized form,

$$\begin{aligned} \ddot{\widetilde{x}}_i - 2\omega_0 \dot{\widetilde{y}}_i &= u_i^x \\ \ddot{\widetilde{y}}_i + 2\omega_0 \dot{\widetilde{x}}_i - 3\omega_0^2 \widetilde{y}_i &= u_i^y \\ \ddot{\widetilde{z}}_i + \omega_0^2 \widetilde{z}_i &= u_i^z \end{aligned} \quad (10)$$

where $\widetilde{x}_i$, $\widetilde{y}_i$, and $\widetilde{z}_i$ represent the position of the ith satellite in the rotating coordinates, u_i^x, u_i^y, u_i^z are the control inputs, and ω_0 is the angular rate of the virtual satellite.

Rewrite the position components in vector form as $r_i = [\widetilde{x}_i, \widetilde{y}_i, \widetilde{z}_i]^T$, velocity vector as $\dot{r}_i = [v_i^x, v_i^y, v_i^z]^T$ and the control vector as $u_i = [u_i^x, u_i^y, u_i^z]^T$. Satellite formation flying is said to be reached if the velocity vectors of all satellites converge to the same value and they keep a prescribed distance with each other, that is, $r_i - h_i \to r_j - h_j$, $\dot{r}_i \to \dot{r}_j$, $\forall i, j \in \mathcal{N}$ as $t \to \infty$, where $h_i = [h_i^x, h_i^y, h_i^z]^T$, and $h_i - h_j \in R^3$ is the desired constant distance between satellite i and j, we define position error as $(r_i - h_i) - (r_j - h_j) = \left[e_{ij}^x, e_{ij}^y, e_{ij}^z\right]^T$. Let $x_i = \begin{bmatrix} r_i - h_i \\ \dot{r}_i \end{bmatrix}$, then (10) can be rewritten as,

$$\dot{x}_i = \begin{bmatrix} 0 & I_3 \\ A_1 & A_2 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} \bar{u}_i, \quad (11)$$

where $A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\omega_0^2 & 0 \\ 0 & 0 & -\omega_0^2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 2\omega_0 & 0 \\ -2\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\bar{u}_i = u_i + A_1 h_i$. It can be seen that (11) is in the form of (1). Therefore, the satellites formation problem can be viewed as a consensus problem.

Consider the scenario that the formation flying model consists of five satellites with a communication topology described by Fig. 1 and the virtual satellite moves in a circular orbit at rate $\omega_0 = 0.001$. It can be verified that the conditions of Theorem 3.1 are all satisfied, and thus the problem can be solved by utilizing the proposed control scheme. For all $i \in \mathcal{N}$, choose $\sigma_i = 0.999$, $a = \frac{1}{\rho} < \frac{2}{\rho}$. The time response of the satellites' position error and velocity on x-axis via event-triggered scheme are shown in Fig. 2 and Fig. 3, respectively.

To better demonstrate the triggering situations for each agent, we further present a figure describing the control inputs on x-axis for all agents with Theorem 3.1 applied, as shown in Fig. 4. It can be seen that, by the proposed event-triggered control scheme, consensus is achieved asymptotically.

5. CONCLUSIONS

In this paper, a novel event-triggered control scheme for the consensus problem of linear multi-agent systems is proposed. It is shown that with this event-triggered control scheme, consensus can be reached asymptotically, and singular triggering and Zeno behavior can be both excluded. As a result, the communication load among all agents can be significantly reduced.

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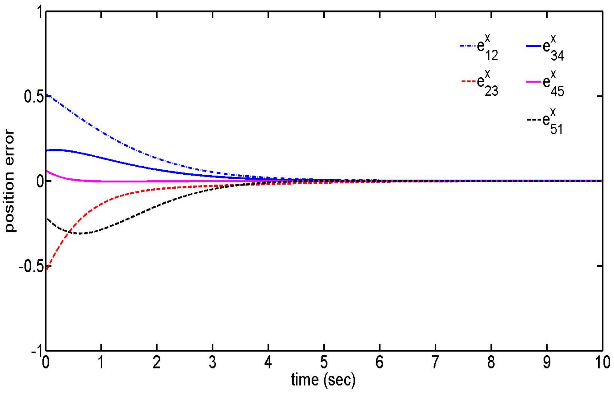

Fig. 2. The position error response on x-axis

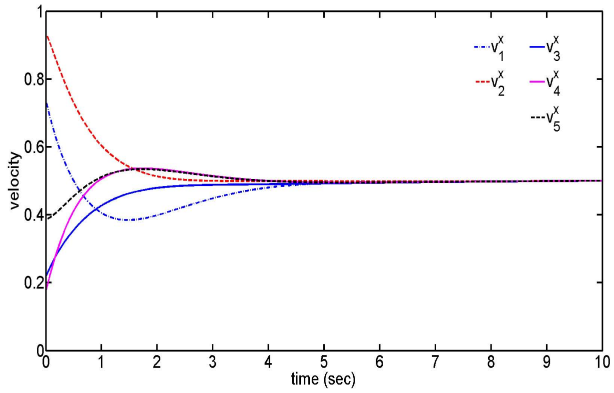

Fig. 3. The velocity response on x-axis

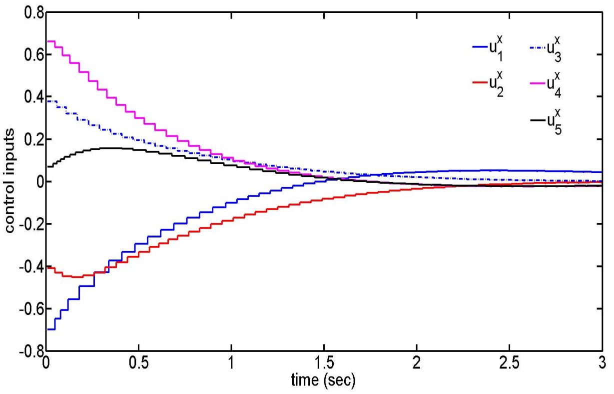

Fig. 4. The control inputs on x-axis for all agents

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Appendix A. PROOF OF THEOREM 3.1

Proof: With $K = B^T P$ and (3), the closed-loop system consisting of (1) and (4) can be expressed as

$$\dot{x}_i = Ax_i + BB^T P \left(e_i\left(t\right) + q_i\left(t\right)\right),\ i \in \mathcal{N}. \tag{A.1}$$

Let $x\left(t\right) = \mathrm{col}\left(x_1\left(t\right), \cdots, x_N\left(t\right)\right)$, $e\left(t\right) = \mathrm{col}\big(e_1\left(t\right), \cdots, e_N\left(t\right)\big)$ and $q\left(t\right) = \mathrm{col}\left(q_1\left(t\right), \cdots, q_N\left(t\right)\right)$. Then $q\left(t\right) = -\left(L \otimes I_n\right) x\left(t\right)$, and (A.1) can be rewritten in a compact form as follows,

$$\begin{aligned}\dot{x}\left(t\right) &= \left(I_N \otimes A - L \otimes BB^T P\right) x\left(t\right) \\ &\quad + \left(I_N \otimes BB^T P\right) e\left(t\right).\end{aligned} \tag{A.2}$$

Consider the following Lyapunov function candidate

$$V(t) = \frac{1}{2} x^T(t)(L \otimes P)x(t). \quad \text{(A.3)}$$

Denote $\hat{A} = \frac{PA+A^TP}{2}$, $\hat{B} = PBB^TP$, and the time derivative of $V(t)$ along the trajectory of (A.2) is

$$\begin{aligned}\dot{V}(t) = & x^T(t)\left(L \otimes \hat{A} - L^2 \otimes \hat{B}\right)x(t) \\ & + x^T(t)\left(L \otimes \hat{B}\right)e(t). \end{aligned} \quad \text{(A.4)}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be the eigenvalues of matrix L, satisfying $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$, and $U = [v_1, v_2, \cdots, v_N]$, where $v_i \in R^N$ is the eigenvector of matrix L associated with the eigenvalue λ_i. Since the graph is undirected, the corresponding Laplacian matrix L is symmetric. Then there exists a U such that

$$U^{-1}LU = U^TLU = J = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N). \quad \text{(A.5)}$$

One can observe that $U^TU = I_N$ and $L = UJU^T$. Define $y(t) = \left(U^T \otimes I_n\right)x(t) = \text{col}(y_1(t), \cdots, y_N(t))$, $\hat{e}(t) = \left(U^T \otimes I_n\right)e(t) = \text{col}(\hat{e}_1(t), \cdots, \hat{e}_N(t))$. (A.4) can be rewritten as

$$\begin{aligned}\dot{V}(t) = & y^T(t)\left(J \otimes \hat{A} - J^2 \otimes \hat{B}\right)y(t) \\ & + y^T(t)\left(J \otimes \hat{B}\right)\hat{e}(t) \\ = & \sum_{i=2}^{N} y_i^T(t)\left(\lambda_i \hat{A} - \lambda_i^2 \hat{B}\right)y_i(t) \\ & + \sum_{i=2}^{N} y_i^T(t)\left(\lambda_i \hat{B}\right)\hat{e}_i(t). \end{aligned} \quad \text{(A.6)}$$

Since $\hat{B} \geq 0$, it follows from (8) that for any $i \in \mathcal{N} - \{1\}$,

$$\hat{A} - \lambda_i \hat{B} \leq \hat{A} - \frac{\alpha}{2}\hat{B} \leq -\frac{\beta}{2}I_n \leq -\lambda_i I_n. \quad \text{(A.7)}$$

Then noting inequality $\|\xi\| \cdot \|\zeta\| \leq \frac{\kappa}{2}\|\xi\|^2 + \frac{1}{2\kappa}\|\zeta\|^2$ for any $\kappa > 0$ and any $\xi, \zeta \in R^n$, one has

$$\begin{aligned}\dot{V}(t) \leq & \sum_{i=2}^{N} -\lambda_i^2 y_i^T(t)y_i(t) + \sum_{i=2}^{N} \lambda_i y_i^T(t)\hat{B}\hat{e}_i(t) \\ \leq & -\left(1 - \frac{a\rho}{2}\right)\sum_{i=2}^{N} \lambda_i^2 \cdot \|y_i(t)\|^2 \\ & + \frac{\rho}{2a}\sum_{i=2}^{N} \|\hat{e}_i(t)\|^2 \end{aligned} \quad \text{(A.8)}$$

where $\rho = \left\|\hat{B}\right\|$ and a is a positive number.

Furthermore, noting $q(t) = -(L \otimes I_n)x(t)$,

$$\begin{aligned}\sum_{i=1}^{N} \|q_i(t)\|^2 = & x^T(t)\left(L^2 \otimes I_n\right)x(t) \\ = & \sum_{i=2}^{N} \lambda_i^2 \cdot \|y_i(t)\|^2, \end{aligned} \quad \text{(A.9)}$$

$$\sum_{i=1}^{N} \|\hat{e}_i(t)\|^2 = e^T(t)e(t) = \sum_{i=1}^{N} \|e_i(t)\|^2, \quad \text{(A.10)}$$

inequality (A.8) can be written as follows,

$$\begin{aligned}\dot{V}(t) \leq & -\left(1 - \frac{a\rho}{2}\right)\sum_{i=1}^{N} \|q_i(t)\|^2 \\ & + \frac{\rho}{2a}\sum_{i=1}^{N} \|e_i(t)\|^2. \end{aligned} \quad \text{(A.11)}$$

Then, by choosing $a < \frac{2}{\rho}$ and enforcing the condition

$$\|e_i(t)\| \leq \sqrt{\frac{\sigma_i \cdot a(2 - a\rho)}{\rho}}\|q_i(t)\| = \eta_i \|q_i(t)\|, \quad \text{(A.12)}$$

where $\sigma_i \in (0, 1)$ for all $i \in \mathcal{N}$, (A.11) becomes

$$\dot{V}(t) \leq -\left(1 - \frac{a\rho}{2}\right)\sum_{i=1}^{N}(1 - \sigma_i)\|q_i(t)\|^2 \leq 0. \quad \text{(A.13)}$$

Letting $e_{ij}(t) = x_i(t) - x_j(t)$, and denoting the set $O = \{e_{ij}(t) = 0 | i, j = 1, \cdots, N\}$, one has

$$V(x) = \frac{1}{2}\left[\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}e_{ij}^T P e_{ij}\right] \geq 0, \quad \text{(A.14)}$$

It follows from (A.14) that $e_{ij} = 0, i, j = 1, \cdots, N \Leftrightarrow V(x) = 0$. It can be further verified that

$$\|q_i(t)\| = 0, i = 1, \cdots, N \Leftrightarrow e_{ij} = 0, i, j = 1, \cdots, N. \quad \text{(A.15)}$$

It then can be concluded from (A.13) that $\dot{V}(x) = 0$ only when $e_{ij} = 0, i, j = 1, \cdots, N$.

Hence, according to the corollary of LaSalle's theorem (Khalil and Grizzle [2002]), it follows that $e_{ij}(t) \to 0, i, j = 1, \cdots, N$ as $t \to \infty$. Thus, the consensus problem is solved.

As for the existence of the solution P of (8), letting $B' = \sqrt{\alpha}B$, $C' = \sqrt{\beta}I_n$, inequality (8) can be rewritten as

$$PA + A^TP - PB'B'^TP + C'^TC' \leq 0, \quad \text{(A.16)}$$

which has the same form as the algebraic Riccati equation (7) if the equality sign is taken. It is noted that (C', A) is observable. Thus, by Lemma 1, one knows that if (A, B) is stabilizable, which implies that so is (A, B'), then at least one solution $P > 0$ is guaranteed for inequality (8). Thus, the proof is completed.

Appendix B. PROOF OF THEOREM 3.2

Proof: For any agent i, $i \in \mathcal{N}$, assume its current triggering time is t_k^i and $q_i\left(t_k^i\right) \neq 0$. We need to prove that the next triggering time after t_k^i, i.e., t_{k+1}^i exists, and $q_i\left(t_{k+1}^i\right) \neq 0$.

Since $\|e_i(t)\| \leq \eta_i \|q_i(t)\|$, by utilizing $\left|\|q_i(t_k^i)\| - \|q_i(t)\|\right| \leq \|e_i(t)\|$, one has

$$\frac{\left\|q_i\left(t_k^i\right)\right\|}{1 + \eta_i} \leq \|q_i(t)\| \leq \frac{\left\|q_i\left(t_k^i\right)\right\|}{1 - \eta_i}. \quad \text{(B.1)}$$

It is noted that $\sup \eta_i = \frac{\sqrt{\sigma_i}}{\rho}$ (when $a = \frac{1}{\rho}$), so $\eta_i < 1$ can be guaranteed when we choose appropriate a and σ_i. By defining $\gamma_1 = \frac{1}{(1+\eta_i)}\left\|q_i\left(t_k^i\right)\right\|$, $\gamma_2 = \frac{1}{(1-\eta_i)}\left\|q_i\left(t_k^i\right)\right\|$, one can conclude that $\|q_i(t)\|$ will always stay between γ_1 and γ_2, and events occur once $\|q_i(t)\|$ reaches the boundary values. Next, we will prove the existence of t^i_{k+1} such that $\left\|q_i\left(t^i_{k+1}\right)\right\| = \gamma_1$ or $\left\|q_i\left(t^i_{k+1}\right)\right\| = \gamma_2$ with the condition $q_i\left(t^i_{k+1}\right) \neq 0$. It follows from (A.14) that $V(t) \geq \frac{1}{4}\lambda_{\min}(P)\left\|x_i - x_j\right\|^2$. Then one has

$$\left\|q_i(t)\right\| \leq \sum_{j=1}^{N} a_{ij}\left\|x_j(t) - x_i(t)\right\| \leq d_i\sqrt{\frac{4V(t)}{\lambda_{\min}(P)}}. \quad \text{(B.2)}$$

It follows from (A.13) that $V(t)$ strictly decreases to zero. Thus, $\left\|q_i(t)\right\|$ will eventually decrease to γ_1 because of (B.2), then at least one event will be triggered at that instant, which can be assigned to be t^i_{k+1}. The proof is thus completed.

Appendix C. PROOF OF THEOREM 3.3

Proof: For any agent i, $i \in \mathcal{N}$, assume its current triggering time instant is t^i_k. We need to prove the length of its next inter-event interval is strictly positive. We first propose the following sufficient condition to guarantee that $h\left(e_i(t), q_i(t)\right) \leq 0$,

$$\left\|e_i(t)\right\| \leq \frac{\eta_i}{\sqrt{2+2\eta_i^2}}\left\|q_i\left(t^i_k\right)\right\|, \quad \text{(C.1)}$$

which follows directly from

$$\begin{aligned}\left\|e_i(t)\right\|^2 &\leq \frac{\eta_i^2}{2+2\eta_i^2}\left\|e_i(t) + q_i(t)\right\|^2 \\ &\leq \frac{\eta_i^2}{1+\eta_i^2}\left(\left\|e_i(t)\right\|^2 + \left\|q_i(t)\right\|^2\right). \quad \text{(C.2)}\end{aligned}$$

Then the time derivative of $\left\|e_i(t)\right\|$ over the interval $\left[t^i_k, t^i_{k+1}\right)$ is

$$\begin{aligned}\frac{d}{dt}\left\|e_i(t)\right\| \leq& \Bigg\| - Aq_i(t) + d_iBB^TPq_i\left(t^i_k\right) \\ & -BB^TP\sum_{j=1}^{N} a_{ij}q_j\left(t^j_{k'(t)}\right)\Bigg\| \\ \leq& \left\|A\right\|\left\|e_i(t)\right\| + \Bigg\| - Aq_i\left(t^i_k\right) + d_iBB^TPq_i\left(t^i_k\right) \\ & -BB^TP\sum_{j=1}^{N} a_{ij}q_j\left(t^j_{k'(t)}\right)\Bigg\| \\ <& \left\|A\right\|\left\|e_i(t)\right\| + \alpha^i_k, \quad \text{(C.3)}\end{aligned}$$

where $k'(t) = \arg\max_{k \in N}\left\{t^j_k \mid t^j_k \leq t,\, j \in \mathcal{N}_i\right\}$, $\frac{d}{dt}\left\|e_i(t)\right\|$ denotes the right-hand derivative of $\|e_i(t)\|$ when $t = t^i_k$, and $\alpha^i_k > \|(d_iBB^TP - A)q_i(t^i_k) - BB^TP\sum_{j=1}^{N} a_{ij}q_j(t^j_{k'(t)})\|$, $t \in \left[t^i_k, t^i_{k+1}\right]$. Then, it follows that

$$\left\|e_i(t)\right\| < \frac{\alpha^i_k}{\|A\|}\left(e^{\|A\|\left(t - t^i_k\right)} - 1\right). \quad \text{(C.4)}$$

Let $s^i_k = \frac{\eta_i}{\sqrt{2+2\eta_i^2}}\left\|q_i\left(t^i_k\right)\right\|$. Using (C.1) and (C.4) gives that

$$\left\|e_i\left(t^i_{k+1}\right)\right\| = s^i_k < \frac{\alpha^i_k}{\|A\|}\left(e^{\|A\|\left(t^i_{k+1} - t^i_k\right)} - 1\right), \quad \text{(C.5)}$$

which yields $t^i_{k+1} - t^i_k > \frac{1}{\|A\|}\ln\left(\frac{\|A\|s^i_k}{\alpha^i_k} + 1\right)$. Thus, one can conclude that the inter-event time for agent i is strictly positive. The proof is thus completed.