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Sufficient Conditions for Exponential Stabilization of Linear Distributed Parameter Systems with Time Delays

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Abstract: The framework of Lyapunov–Krasovskii functions is the most popular approach for studying the problem of stability analysis and synthesis for time-delay systems. In this framework, this paper addresses the exponential stabilization problem of a class of linear distributed parameter systems with time delays described by partial differential-difference equations (PDdEs). Both delay-independent and delay-dependent design methods are subsequently developed by constructing two integral types of Lyapunov–Krasovskii functions and introducing a new form of Jensen's inequality, and presented in terms of a standard linear matrix inequality (LMI). Finally, the effectiveness and merit of the proposed design method are demonstrated by a numerical example.

Keywords: Jensen's inequality; Distributed parameter systems; Time-delay systems; Partial differential-difference equation (PDdE); Linear matrix inequalities (LMIs)

1. INTRODUCTION

In many cases, time delay is a source of instability and performance deterioration. Hence, time-delay systems have been studied extensively over the past decades (see e.g., Nilsson (1998), Gu, Kharitonov, & Chen (2003), Zhong (2006), Hespanha, Naghshtabrizi, & Xu (2007), and the references therein). The most popular approach remains the use of the framework of Lyapunov–Krasovskii functions, since this framework allows studying a large class of time-delay systems. The main challenge is to propose new functional and techniques for deriving less and less conservative results. To reduce the conservativeness of Lyapunov–Krasovskii techniques, many techniques have been proposed over the past two decades, such as the free-weighting matrix approach (Wu, He, She, & Liu, 2004), delay-decomposition approach (Gouaisbaut & Peaucelle, 2006a, 2006b), Jensen's inequality (Nilsson, 1998; Han, 2005), Wirtinger's inequality (Liu, Suplin, & Fridman, 2010; Seuret & Gouaisbaut, 2013a, 2013b) and combinations of these techniques. However, most of these results only focus on the class of systems modeled by ordinary differential-difference equations (ODdEs).

In fact, significant physical systems are spatiotemporal in nature so that their behavior must depend on time as well as spatial position, for example, thermal diffusion, fluid heat exchangers, chemical engineering, to name a few (Ray, 1981; Christofides, 2001; Deng, Li & Chen 2005; Padhi & Ali, 2009). These spatiotemporal processes are referred to as distributed parameter systems (DPSs). The mathematical models describing this type system are typically derived from the dynamic conservation laws and take the form of partial differential equations (PDEs). Thus, the study of stability analysis and controller synthesis for time-delay DPSs described by partial differential-difference equations (PDdEs) is essentially more complicated and of practical importance. However, there are only a few works on stability analysis and synthesis of time-delay DPSs. For example, Wang (1964) utilized the dynamic programming technique to deal with the optimum control of DPSs with time delays modeled by a set of PDdEs. A variational calculus approach was used in Kim (1974) to study the quadratic optimization of linear time-delay PDE systems with distributed and boundary control functions. The second Lyapunov method was extended to nonlinear time-delay DPSs represented by abstract evolution equation in the Banach space (Wang, 1994a), and was applied to stability analysis of some scalar heat/wave equations with constant delays and with the Dirichlet boundary conditions (Wang, 1994b). In Nicaise & Pignotti (2006), stability and instability conditions for delay wave equations were presented by introducing suitable energy functions and using some observability inequalities. Fridman and Orlov (2009) extended the Lyapunov–Krasovkii method to exponential stability analysis of linear time-delay DPSs described by abstract evolution equation in Hilbert space. However, the results (Wang, 1994a, 1994b; Nicaise & Pignotti, 2006; Fridman & Orlov, 2009) only focus on the stability analysis of time-delay DPSs. To the best of authors' knowledge, few results are available on the control synthesis of time-delay DPSs via the Lyapunov–Krasovkii method with the exception of Luo *et al* (2009) .

In this paper, we will follow the Lyapunov–Krasovskii function approach to deal with the problem of exponential stabilization via distributed feedback controllers for linear DPSs with time delays modeled by PDdEs. To this end, a new form of Jensen's inequality and two novel integral types of Lyapunov–Krasovskii functions are introduced. By using these Lyapunov–Krasovskii functions and this new form Jensen's inequality, both delay-independent and delay-dependent sufficient stabilization conditions are presented in terms of a standard linear matrix inequality (LMI), which can be directly verified via the existing LMI techniques (Boyd, Ghaoui. Feron, & Balakrishnan, 1994; Gahinet, Nemirovskii, Laub, & Chilali, 1995). Finally, the effectiveness and merit of the proposed design methods are illustrated by a numerical example.

Notations: $\Re$, $\Re^n$ and $\Re^{m\times n}$ denote the set of all real numbers, n-dimensional Euclidean space and the set of all $m\times n$ matrices, respectively. Identity matrix, of appropriate dimension, will be denoted by $\boldsymbol{I}$. For a symmetric matrix $\boldsymbol{M}$, $\boldsymbol{M}>(\geq,<,\leq)0$ means that it is positive definite (positive semi-definite, negative definite, negative semi-definite, respectively). $\mathcal{H}^n \triangleq \mathcal{L}_2([l_1,l_2];\Re^n)$ is a Hilbert space of n-dimensional square integrable vector functions $\boldsymbol{\omega}(x)\in\Re^n$, $x\in[l_1,l_2]\subset\Re$, $\forall t\geq 0$ with the inner product and norm:

$$\langle \boldsymbol{\omega}_1(\cdot),\boldsymbol{\omega}_2(\cdot)\rangle = \int_{l_1}^{l_2}\boldsymbol{\omega}_1^T(x)\boldsymbol{\omega}_2(x)dx,$$

$$\|\boldsymbol{\omega}_1(\cdot)\|_2 = \langle \boldsymbol{\omega}_1(\cdot),\boldsymbol{\omega}_1(\cdot)\rangle^{1/2},$$

where $\boldsymbol{\omega}_1(\cdot)$, $\boldsymbol{\omega}_2(\cdot)\in\mathcal{H}^n$. The superscript '$T$' is used for the transpose of a vector or a matrix. The symbol '$*$' is used as an ellipsis in matrix expressions that are induced by symmetry, e.g.,

$$\begin{bmatrix} \boldsymbol{S}+[\boldsymbol{M}+\boldsymbol{N}+*] & \boldsymbol{X} \\ * & \boldsymbol{Y}\end{bmatrix} \triangleq \begin{bmatrix} \boldsymbol{S}+[\boldsymbol{M}+\boldsymbol{N}+\boldsymbol{M}^T+\boldsymbol{N}^T] & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{Y}\end{bmatrix}.$$

2. PROBLEM FORMULATION AND PRELIMINARIES

We consider a class of linear time-delay DPSs in one spatial dimension described by the following PDdEs:

$$\begin{aligned}\boldsymbol{y}_t(x,t) &= \Theta\boldsymbol{y}_{xx}(x,t)+\boldsymbol{A}\boldsymbol{y}(x,t)\\ &\quad+\boldsymbol{A}_d\boldsymbol{y}(x,t-\tau)+\boldsymbol{B}\boldsymbol{u}(x,t),\quad (x,t)\in[l_1,l_2]\times(0,\infty),\end{aligned} \tag{1}$$

subject to the homogeneous Neumann boundary conditions:

$$\boldsymbol{y}_x(x,t)|_{x=l_1} = \boldsymbol{y}_x(x,t)|_{x=l_2} = 0,\ t\in[-\tau,\infty), \tag{2}$$

and the initial condition:

$$\boldsymbol{y}(x,t)=\boldsymbol{y}_0(x,t),\ (x,t)\in[l_1,l_2]\times[-\tau,0], \tag{3}$$

where $\boldsymbol{y}(x,t)\in\Re^n$ is the state, $\boldsymbol{u}(x,t)\in\Re^m$ is the distributed control input, $x\in[l_1,l_2]\subset\Re$ and $t\in[-\tau,\infty)$ are the spatial position and time, respectively, and τ is a known time delay. The subscripts x and t stand for the partial derivatives with respect to x, t, respectively. $0\leq\Theta\in\Re^{n\times n}$, $\boldsymbol{A}\in\Re^{n\times n}$, $\boldsymbol{A}_d\in\Re^{n\times n}$, and $\boldsymbol{B}\in\Re^{n\times m}$ are known matrices. $\boldsymbol{y}_0(x,t)\in\Re^n$, $(x,t)\in[l_1,l_2]\times[-\tau,0]$ is the initial value.

In this paper, we consider the following distributed feedback controller:

$$\boldsymbol{u}(x,t)=\boldsymbol{K}\boldsymbol{y}(x,t) \tag{4}$$

where $\boldsymbol{K}\in\Re^{m\times n}$ is the control gain matrix to be determined. Substitution of (4) into the equation (1)-(3), yields

$$\begin{cases}\boldsymbol{y}_t(x,t)=\Theta\boldsymbol{y}_{xx}(x,t)+\boldsymbol{A}_c\boldsymbol{y}(x,t)\\ \qquad +\boldsymbol{A}_d\boldsymbol{y}(x,t-\tau),\quad (x,t)\in[l_1,l_2]\times(0,\infty)\\ \boldsymbol{y}_x(x,t)|_{x=l_1}=\boldsymbol{y}_x(x,t)|_{x=l_2}=0,\quad t\in[-\tau,\infty)\\ \boldsymbol{y}(x,t)=\boldsymbol{y}_0(x,t),\quad (x,t)\in[l_1,l_2]\times[-\tau,0],\end{cases} \tag{5}$$

where $\boldsymbol{A}_c\triangleq\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$.

This study will develop simple LMI-based methods to design a distributed feedback controller (4) for the linear time-delay PDE system (1)-(3) such that the closed-loop PDdE system (5) is exponentially stable in the sense of norm $\|\cdot\|_2$, i.e., $\|\boldsymbol{y}(\cdot,t)\|_2^2\leq\rho\sup_{t\in[-\tau,0]}\|\boldsymbol{y}_0(\cdot,t)\|_2^2\exp(-\overline{\chi}t)$, where ρ are $\overline{\chi}$ two given positive constants. To do this, the following lemma is useful:

Lemma 1 (Spatial Integration form of Jensen's inequality). For any constant matrix $0<\boldsymbol{R}\in\Re^{n\times n}$, a scalar $h>0$, and vector valued function $\boldsymbol{y}(\cdot,t):[-h,0]\to\mathcal{H}^n$ such that the following integration is well defined, then

$$\begin{aligned}&-h\int_{l_1}^{l_2}\int_{t-h}^{t}\boldsymbol{y}_s^T(x,s)\boldsymbol{R}\boldsymbol{y}_s(x,s)dsdx\\ &\leq\int_{l_1}^{l_2}\begin{bmatrix}\boldsymbol{y}(x,t)\\ \boldsymbol{y}(x,t-h)\end{bmatrix}^T\begin{bmatrix}-\boldsymbol{R} & \boldsymbol{R}\\ \boldsymbol{R} & -\boldsymbol{R}\end{bmatrix}\begin{bmatrix}\boldsymbol{y}(x,t)\\ \boldsymbol{y}(x,t-h)\end{bmatrix}dx.\end{aligned} \tag{6}$$

Notice that this lemma can be easily proved on the basis of Lemma 2 (Han, 2005). An inner product form of Jensen's inequality has been reported in Fridman & Orlov (2009).

3. STABILIZATION CONDITIONS

3.1 Delay-independent Stabilization Condition

This subsection first gives the delay-independent stabilization condition of the system (1)-(3). Let us consider the following Lyapunov–Krasovskii function:

$$V(t)=\int_{l_1}^{l_2}\boldsymbol{y}^T(x,t)\boldsymbol{P}\boldsymbol{y}(x,t)dx+\int_{l_1}^{l_2}\int_{t-\tau}^{t}\boldsymbol{y}^T(x,s)\boldsymbol{Q}\boldsymbol{y}(x,s)dsdx, \tag{7}$$

where $0<\boldsymbol{P}\in\Re^{n\times n}$ and $0<\boldsymbol{Q}\in\Re^{n\times n}$ are Lyapunov matrices to be determined.

Differentiating $V(t)$ in (7) along the solution to the PDdE system (5), we obtain

$$\begin{aligned}\dot{V}(t) &= 2\int_{l_1}^{l_2} \boldsymbol{y}^T(x,t)\boldsymbol{P}\Theta\boldsymbol{y}_{xx}(x,t)dx \\ &+\int_{l_1}^{l_2} \boldsymbol{y}^T(x,t)[\boldsymbol{PA}_c+*]\boldsymbol{y}(x,t)dx \\ &+\int_{l_1}^{l_2} \boldsymbol{y}^T(x,t)\boldsymbol{Q}\boldsymbol{y}(x,t)dx - \int_{l_1}^{l_2} \boldsymbol{y}^T(x,t-\tau)\boldsymbol{Q}\boldsymbol{y}(x,t-\tau)dx \\ &+2\int_{l_1}^{l_2} \boldsymbol{y}^T(x,t)\boldsymbol{PA}_d\boldsymbol{y}(x,t-\tau)dx.\end{aligned} \quad (8)$$

Integrating by parts and taking into account (2), we have

$$\int_{l_1}^{l_2} \boldsymbol{y}^T(x,t)\boldsymbol{P}\Theta\boldsymbol{y}_{xx}(x,t)dx = -\int_{l_1}^{l_2} \boldsymbol{y}_x^T(x,t)\boldsymbol{P}\Theta\boldsymbol{y}_x(x,t)dx \quad (9)$$

which means

$$\begin{aligned}&2\int_{l_1}^{l_2} \boldsymbol{y}^T(x,t)\boldsymbol{P}\Theta\boldsymbol{y}_{xx}(x,t)dx \\ &\qquad = -\int_{l_1}^{l_2} \boldsymbol{y}_x^T(x,t)[\boldsymbol{P}\Theta+*]\boldsymbol{y}_x(x,t)dx.\end{aligned} \quad (10)$$

Since $0 \le \Theta \in \Re^{n\times n}$, we have

$$[\boldsymbol{P}\Theta+*] \ge 0\,. \quad (11)$$

Using the inequality (11), we can obtain

$$-\int_{l_1}^{l_2} \boldsymbol{y}_x^T(x,t)[\boldsymbol{P}\Theta+*]\boldsymbol{y}_x(x,t)dx \le 0\,. \quad (12)$$

Substituting (10) into (8) and considering (12), we get

$$\dot{V}(t) \le \int_{l_1}^{l_2} \tilde{\boldsymbol{y}}^T(x,t)\tilde{\Xi}\tilde{\boldsymbol{y}}(x,t)dx \quad (13)$$

where $\tilde{\boldsymbol{y}}(x,t) \triangleq [\boldsymbol{y}^T(x,t) \quad \boldsymbol{y}^T(x,t-\tau)]^T$ and

$$\tilde{\Xi} \triangleq \begin{bmatrix} \boldsymbol{Q}+[\boldsymbol{PA}_c+*] & \boldsymbol{PA}_d \\ * & -\boldsymbol{Q}\end{bmatrix}.$$

Henceforth, from the above analysis, the following theorem is obtained in terms of standard LMI:

Theorem 1. Consider the linear time-delay PDE system (1)-(3). If there exist matrices $\boldsymbol{X}>0$, $\tilde{\boldsymbol{Q}}>0$, and $\boldsymbol{Z}$ such that the following LMI is satisfied:

$$\Xi \triangleq \begin{bmatrix} \tilde{\boldsymbol{Q}}+[\boldsymbol{AX}+\boldsymbol{BZ}+*] & \boldsymbol{A}_d\boldsymbol{X} \\ * & -\tilde{\boldsymbol{Q}}\end{bmatrix} < 0\,, \quad (14)$$

then there exists a distributed feedback controller (4) such that the closed-loop PDdE system (5) is exponentially stable. In this case, the control gain matrix $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = \boldsymbol{ZX}^{-1}\,. \quad (15)$$

Proof. Assume that the LMI (14) is fulfilled. Set

$$\boldsymbol{X} = \boldsymbol{P}^{-1}, \quad \boldsymbol{Z} = \boldsymbol{KX}, \quad \tilde{\boldsymbol{Q}} = \boldsymbol{XQX}\,. \quad (16)$$

Pre- and post-multiplying the matrix Ξ by $\mathcal{P} \triangleq \mathrm{diag}\{\boldsymbol{P},\boldsymbol{P}\}$, respectively and considering (16), we can obtain

$$\tilde{\Xi} = \mathcal{P}\Xi\mathcal{P}\,. \quad (17)$$

Using the LMI (14) and $\mathcal{P}>0$, we have

$$\tilde{\Xi} < 0\,. \quad (18)$$

On the other hand, there exists a scalar $\chi>0$ such that the inequality (18) can be written as

$$\tilde{\Xi} + \chi\boldsymbol{I} \le 0\,. \quad (19)$$

Substitution of (19) into (13) yields

$$\dot{V}(t) \le -\chi\|\tilde{\boldsymbol{y}}(\cdot,t)\|_2^2 \le -\chi\|\boldsymbol{y}(\cdot,t)\|_2^2\,. \quad (20)$$

By using the technique reported in Mao, Koroleva, & Rodkina (1998), it can be concluded from the inequality (20) that the closed-loop PDdE system (5) is exponentially stable if LMI (14) is fulfilled. From (16), we have (15). □

From Theorem 1, the outcome of stabilization via distributed feedback controller (4) for the linear time-delay PDE system (1)-(3) is presented in terms of standard LMI (14), which can be easily solved via the existing LMI optimization techniques (Boyd, Ghaoui, Feron, & Balakrishnan, 1994; Gahinet, Nemirovskii, Laub, & Chilali, 1995). Notice that the proposed design in this subsection is a delay-independent one, since LMI (14) does not depends on the time-delay scalar τ.

Remark 1. Notice that the non-positive term $-\int_{l_1}^{l_2} \boldsymbol{y}_x^T(x,t)[\boldsymbol{P}\Theta+*]\boldsymbol{y}_x(x,t)dx$ in (12) is directly neglected in the derivation process of the LMI-based stabilization condition, i.e., LMI (14). Obviously, this inevitably brings some conservativeness. Therefore, it remains a challenge to avoid or reduce the conservativeness resulted from this non-positive term, which we leave for future study.

3.2 Delay-dependent Stabilization Condition

It is well-known that for the case of small time delays, delay-independent conditions are more conservative than delay-dependent ones. This subsection will give an LMI-based delay-dependent stabilization condition for the linear time-delay PDE system (1)-(3). To this end, we consider the following Lyapunov–Krasovskii function:

$$\begin{aligned}V(t) &= \int_{l_1}^{l_2} \boldsymbol{y}^T(x,t)\boldsymbol{P}_1\boldsymbol{y}(x,t)dx + \int_{l_1}^{l_2} \boldsymbol{y}_x^T(x,t)\boldsymbol{P}_2\boldsymbol{y}_x(x,t)dx \\ &+\int_{l_1}^{l_2}\int_{t-\tau}^{t} \boldsymbol{y}^T(x,s)\boldsymbol{R}_1\boldsymbol{y}(x,s)dsdx \\ &+\tau\int_{l_1}^{l_2}\int_{-\tau}^{0}\int_{t+\theta}^{t} \boldsymbol{y}_s^T(x,s)\boldsymbol{R}_2\boldsymbol{y}_s(x,s)dsd\theta dx\end{aligned} \quad (21)$$

where $0<\boldsymbol{P}_1\in\Re^{n\times n}$, $0<\boldsymbol{P}_2\in\Re^{n\times n}$, $0<\boldsymbol{R}_1\in\Re^{n\times n}$, and $0<\boldsymbol{R}_2\in\Re^{n\times n}$ are Lyapunov matrices to be determined.

Using Lemma 1 and considering $\boldsymbol{R}_2>0$, we can derive

$$-\tau\int_{l_1}^{l_2}\int_{t-\tau}^{t} \boldsymbol{y}_s^T(x,s)\boldsymbol{R}_2\boldsymbol{y}_s(x,s)dsdx$$

$$\leq \int_{l_1}^{l_2} \begin{bmatrix} y(x,t) \\ y(x,t-\tau) \end{bmatrix}^T \begin{bmatrix} -R_2 & R_2 \\ R_2 & -R_2 \end{bmatrix} \begin{bmatrix} y(x,t) \\ y(x,t-\tau) \end{bmatrix} dx. \quad (22)$$

Integrating by parts and taking into account (2), we have

$$\int_{l_1}^{l_2} y_x^T(x,t) P_2 y_{xt}(x,t) dx = -\int_{l_1}^{l_2} y_{xx}^T(x,t) P_2 y_t(x,t) dx . \quad (23)$$

Considering (22) and (23), the time derivative of $V(t)$ in (21) along the solution of the system (5) is given as

$$\dot{V}(t) \leq \int_{l_1}^{l_2} \xi^T(x,t) \Psi \xi(x,t) dx \quad (24)$$

where $\xi(x,t) \triangleq [y^T(x,t) \quad y_{xx}^T(x,t) \quad y^T(x,t-\tau)]^T$,

$$\Psi \triangleq \begin{bmatrix} \Psi_{11} & P_1\Theta - A_c^T P_2^T & R_2 + P_1 A_d \\ * & -[P_2\Theta + *] & P_2 A_d \\ * & * & -R_1 - R_2 \end{bmatrix} + \tau^2 [A_c \quad \Theta \quad A_d]^T R_2 [A_c \quad \Theta \quad A_d]$$

in which $\Psi_{11} \triangleq R_1 - R_2 + [P_1 A_c + *]$.

To obtain a delay-dependent design method in term of LMI, we restrict ourselves to the case of $P_2 = \varpi P_1$, where $\varpi > 0$ is a real known constant. In this case, let

$$X_1 = P_1^{-1}, \; Z = KX_1, \; \tilde{R}_1 = X_1 R_1 X_1, \text{ and } \tilde{R}_2 = X_1 R_2 X_1 . \quad (25)$$

Pre- and post-multiplying both sides of the matrix Ψ with the matrix $\mathcal{X} \triangleq \operatorname{diag}\{X_1, X_1, X_1, X_1\}$, respectively, we have

$$\mathcal{X}\Psi\mathcal{X} \triangleq \begin{bmatrix} \tilde{\Psi}_{11} & \tilde{\Psi}_{12} & \tilde{R}_2 + A_d X_1 \\ * & -\varpi[\Theta X_1 + *] & \varpi A_d X_1 \\ * & * & -\tilde{R}_1 - \tilde{R}_2 \end{bmatrix} + \tau^2 \begin{bmatrix} (AX_1 + BZ)^T \\ X_1\Theta \\ X_1 A_d^T \end{bmatrix} R_2 \begin{bmatrix} (AX_1 + BZ)^T \\ X_1\Theta \\ X_1 A_d^T \end{bmatrix}^T , \quad (26)$$

where

$$\tilde{\Psi}_{11} \triangleq \tilde{R}_1 - \tilde{R}_2 + [AX_1 + BZ + *],$$
$$\tilde{\Psi}_{12} \triangleq \Theta X_1 - \varpi(AX_1 + BZ)^T .$$

Hence, we have the following theorem:

Theorem 2. For a given constant $\chi > 0$, consider the linear PDE system with time delay of the form (1)-(3). Given a scalar $\varpi > 0$, if there exist matrices $X_1 > 0$, $\tilde{R}_1 > 0$, $\tilde{R}_2 > 0$, and Z such that the following LMI is feasible:

$$\begin{bmatrix} \tilde{\Psi}_{11} & \tilde{\Psi}_{12} & \tilde{R}_2 + A_d X_1 & (AX_1 + BZ)^T \\ * & -\varpi[\Theta X_1 + *] & \varpi A_d X_1 & X_1\Theta \\ * & * & -\tilde{R}_1 - \tilde{R}_2 & X_1 A_d^T \\ * & * & * & -2\tau^{-2} X_1 + \tau^{-2}\tilde{R}_2 \end{bmatrix} < 0 \quad (27)$$

then there exists a distributed feedback controller (4) such that the closed-loop PDdE system (5) is exponentially stable. In this case, the control gain matrix K is given by

$$K = ZX_1^{-1} . \quad (28)$$

Proof. Assume that LMI (27) holds. Considering $\tilde{R}_2 = X_1 R_2 X_1$ and using the matrix theory, we have

$$R_2^{-1} = X_1 \tilde{R}_2^{-1} X_1 . \quad (29)$$

Using $X_1 > 0$ and $R_2 > 0$ yields $(X_1 - \tilde{R}_2)\tilde{R}_2^{-1}(X_1 - \tilde{R}_2) \geq 0$, which implies

$$X_1 \tilde{R}_2^{-1} X_1 \geq 2X_1 - \tilde{R}_2 . \quad (30)$$

From (29) and (30), we give

$$-R_2^{-1} \leq -2X_1 + \tilde{R}_2 . \quad (31)$$

It is clear from (31) that the following inequality holds:

$$\begin{bmatrix} \tilde{\Psi}_{11} & \tilde{\Psi}_{12} & \tilde{R}_2 + A_d X_1 & (AX_1 + BZ)^T \\ * & -\varpi[\Theta X_1 + *] & \varpi A_d X_1 & X_1\Theta \\ * & * & -\tilde{R}_1 - \tilde{R}_2 & X_1 A_d^T \\ * & * & * & -\tau^{-2} R_2^{-1} \end{bmatrix} < 0, \quad (32)$$

if LMI (27) is fulfilled.

By employing the Schur complement, the inequality (32) is equivalent to $\mathcal{X}\Psi\mathcal{X} < 0$. We can obtain the inequality $\Psi < 0$ because of $\mathcal{X} > 0$. Similar to the proof of Theorem 1, we can easily show from the inequality $\Psi < 0$ that the closed-loop system (5) is exponentially stable. From (25), we can obtain (28). □

Theorem 2 provides an LMI-based delay-dependent design method of an exponentially stabilizing distributed feedback controller (4) for the time-delay PDE system (1)-(3). The corresponding control gain matrix can be constructed as (28) via the feasible solution to LMI (27), which can be directly solved by employing the existing LMI toolbox in MATLAB (Gahinet, Nemirovskii, Laub, & Chilali, 1995).

Remark 2. Luo *et al* (2009) has proposed delay-dependent exponentially stabilizing state feedback control designs for a class of linear parabolic PDE with constant, varying delay, and multi-varying-delays. If we set $k = 0$ ($k \geq 0$ denotes the exponential decay rate (Luo *et al*, 2009)), these results can be easily simplified into delay-independent ones. Moreover, different from the exponential stability analysis in Fridman & Orlov (2009), this paper provides both delay-independent and delay-dependent conditions for exponential stabilization of the linear PDdE system (1)-(3).

4. NUMERICAL EXAMPLE

In this section, to illustrate the effectiveness and merit of the proposed design methods, we consider the following linear PDE system with time delay:

$$y_{1,t}(x,t) = y_{1,xx}(x,t) + 1.5y_1(x,t) - y_2(x,t) + y_1(x,t-\tau) + u(x,t) \quad (33)$$

$$y_{2,t}(x,t) = y_{2,xx}(x,t) + 0.45y_1(x,t) - 1.3y_2(x,t) + y_2(x,t-\tau) \quad (34)$$

subject to the homogeneous Neumann boundary conditions

$$y_{1,x}(x,t)\big|_{x=0} = y_{2,x}(x,t)\big|_{x=0} = 0,$$

$$y_{1,x}(x,t)\big|_{x=1} = y_{2,x}(x,t)\big|_{x=1} = 0, \ t \in [-\tau, \infty) \quad (35)$$

and the initial conditions

$$y_1(x,t) = y_{1,0}(x,t), \ y_2(x,t) = y_{2,0}(x,t),$$

$$(x,t) \in [0,1]\times[-\tau,0], \quad (36)$$

where $y_i(x,t) \in \Re$, $i \in \{1,2\}$ are the state variables, $u(x,t) \in \Re$ is the distributed input. t and x denote the independent time and spatial position variables, respectively. $y_{1,0}(x,t)$ and $y_{2,0}(x,t)$, $(x,t) \in [0,1]\times[-\tau,0]$ are the initial conditions. Let $\boldsymbol{y}(x,t) \triangleq [y_1(x,t) \quad y_2(x,t)]^T$, the system (33)-(36) can be rewritten as the form (1)-(3), where $\boldsymbol{\Theta} = \boldsymbol{I}$, $l_1 = 0$, $l_2 = 1$ and

$$\boldsymbol{A} = \begin{bmatrix} 1.5 & -1 \\ 0.45 & -1.3 \end{bmatrix}, \ \boldsymbol{A}_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (37)$$

Set $\tau = 0.8$. Since the operator $\mathcal{A}\overline{\boldsymbol{y}}(x) \triangleq d^2\overline{\boldsymbol{y}}(x)/dx^2 + \boldsymbol{A}\overline{\boldsymbol{y}}(x)$ has a positive eigenvalue in the linear time-delay PDE system (33)-(36) is unstable. Let $y_{1,0}(x,t) = 5\cos(\pi x)$ and $y_{2,0}(x,t) = 0.2 + \cos(\pi x)$, $(x,t) \in [0,1]\times[-0.8,0]$, Fig. 1 shows the open-loop trajectories of $\|y_1(\cdot,t)\|_2$ and $\|y_2(\cdot,t)\|_2$.

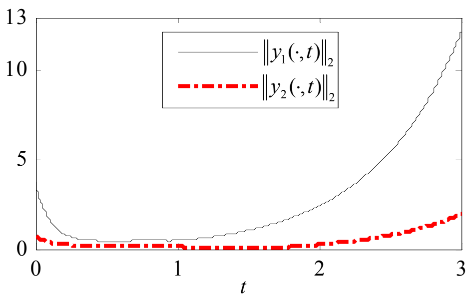

Fig. 1 Open-loop trajectories of $\|y_1(\cdot,t)\|_2$ and $\|y_2(\cdot,t)\|_2$.

4.1 Delay-independent Stabilization Condition

Solving the LMI (14) and using the (15), the control gain matrix $\boldsymbol{K}$ can be derived as

$$\boldsymbol{K} = [-4.5764 \quad -1.5825].$$

Applying the distributed feedback controller (4) with above control gain matrix to the linear PDdE system (33)-(36), Fig. 2 indicates the closed-loop trajectories of $\|y_1(\cdot,t)\|_2$ and $\|y_2(\cdot,t)\|_2$. The trajectory of $\|u(\cdot,t)\|_2$ is also shown in Fig. 2. Obviously, the proposed controller can stabilize the system (33)-(36).

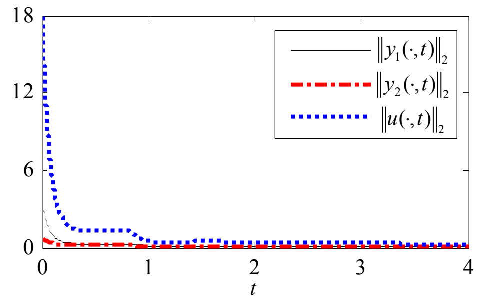

Fig. 2 Closed-loop trajectories of $\|y_1(\cdot,t)\|_2$ and $\|y_2(\cdot,t)\|_2$ as well as $\|u(\cdot,t)\|_2$ via Theorem 1.

4.2 Delay-dependent Stabilization Condition

Let $\varpi = 1.8$. Solving the LMI (27) and using the (28), the control gain matrix $\boldsymbol{K}$ can be derived as

$$\boldsymbol{K} = [-2.4251 \quad -1.0781].$$

Applying the distributed feedback controller (4) with above control gain matrix to the linear PDdE system (33)-(36), trajectories of $\|y_1(\cdot,t)\|_2$, $\|y_2(\cdot,t)\|_2$, and $\|u(\cdot,t)\|_2$ are given in Fig. 3, respectively. It is clearly seen from Fig. 3 that the proposed controller can stabilize the system (33)-(36).

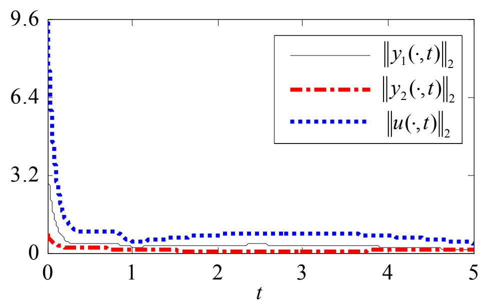

Fig. 3 Closed-loop trajectories of $\|y_1(\cdot,t)\|_2$ and $\|y_2(\cdot,t)\|_2$ as well as $\|u(\cdot,t)\|_2$ via Theorem 2.

On the other hand, we further give a comparison between the proposed design method and the exponential stabilizations reported in Luo *et al* (2009). To do this, in (37), we set

$$\boldsymbol{\Theta} = \boldsymbol{I}, \ \boldsymbol{A}_d = \boldsymbol{I}, \ \boldsymbol{A} = \begin{bmatrix} 1.5 & 1 \\ 0.45 & -1 \end{bmatrix},$$

$$\boldsymbol{B} = [1 \quad 0]^T, \ l_1 = 0 \text{ and } l_2 = 1.$$

The method proposed in Luo *et al* (2009) is infeasible for the case when $k = 0.05$ and $\beta = 0.5$ or $k = 0$ and $\beta = 0.01$

($\beta > 0$ is the parameter used in the Lyapunov function candidate Luo *et al* (2009)), whereas the maximum delay is derived via Theorem 2 as $\tau = 0.9037$ for the case when $\varpi = 1$. It is obvious that Theorem 2 is less conservative than the results reported in Luo *et al* (2009).

5. CONCLUSIONS

This paper has investigated the problem of exponential stabilization via distributed feedback controller for a class of linear PDE systems with time delays in the framework of Lyapunov–Krasovskii functions. Both delay-independent and delay-dependent design methods are developed for the distributed state feedback controller by constructing two integral types of Lyapunov–Krasovskii functions and introducing a new form of Jensen's inequality, and presented in terms of standard LMIs. Finally, simulation results of a numerical example indicate the effectiveness and merit of the proposed design method.

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