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Robust stabilizing simple multi-period repetitive controllers for multiple-input/ multiple-output time-delay plants

Tatsuya Sakanushi * **Kou Yamada** *

* *Gunma University, 1-5-1 Tenjincho, Kiryu, Japan (e-mail: {sakanushi; yamada}@gunma-u.ac.jp).*

Abstract: The simple multi-period repetitive control system proposed by Yamada and Takenaga is a type of servomechanism for a periodic reference input. That is, the simple multi-period repetitive control system follows the periodic reference input with a small steady state error, even if a periodic disturbance or uncertainty exists in the plant. In addition, simple multi-period repetitive control systems ensure that the transfer functions from the periodic reference input to the output and from the disturbance to the output have a finite number of poles. Yamada et al. clarified the parameterization of all stabilizing simple multi-period repetitive controllers. Recently, Sakanushi et al. proposed the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output plants with uncertainty. However, their method cannot be applied to time-delay plants. In this paper, we propose the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output time-delay plants.

Keywords: repetitive control; time-delay; uncertainty; robust stability; parameterization; finite number of poles; multiple-input/multiple-output plants.

1. INTRODUCTION

Multi-period repetitive controllers improve the disturbance attenuation characteristics of the modified repetitive control system that follows the periodic reference input with a small steady state error (Gotou et al., 1987; Okuyama et al., 2002; Yamada et al., 2003). Yamada et al. point out that the disturbance attenuation characteristic for frequency component, which is same as that of the periodic reference input, of multi-period repetitive control systems are not very good and propose a design method for multi-period repetitive controllers to attenuate wide-frequency disturbance which is the same as the frequency of periodic reference input based on the idea of changing time-delay (Yamada et al., 2003).

Using the multi-period repetitive controllers in Gotou et al. (1987); Okuyama et al. (2002); Yamada et al. (2003), even if the plant does not include time delays, transfer functions from the periodic reference input to the output and from the disturbance to the output have infinite numbers of poles. This makes it difficult to specify the input–output characteristic and the disturbance attenuation characteristic. However, from a practical point of view, it is desirable that these characteristics be easy to specify, which would require these transfer functions to have finite numbers of poles. To overcome this problem, Yamada and Takenaga propose simple multi-period repetitive control systems in which the controller works as a multi-period repetitive controller and the transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles (Yamada and Takenaga, 2008). In addition, they clarify the parameterization of all stabilizing simple multi-period repetitive controllers.

When we apply simple multi-period repetitive control design methods to real systems, the influence of uncertainties in the plant must be considered. In some cases, uncertainties in the plant make the control system unstable, even though the controller was designed to stabilize the nominal plant. The stability problem with uncertainty is known as the robust stability problem (Doyle et al., 1989). Yamada et al. propose the parameterization of all robust stabilizing simple multi-period repetitive controllers for plants with uncertainties (Yamada et al., 2008). However, because the method in Yamada et al. (2008) uses the characteristics of single-input/single-output systems, this method cannot be applied to multiple-input/multiple-output plants. To solve this problem, Sakanushi et al. examine the parameterization for multiple-input/multiple-output plants (Sakanushi et al., 2012). Many real plants have multiple-input and multiple-output, and include uncertainties and time-delays. In addition, the parameterization is useful to design stabilizing controllers (Youla et al., 1976; Kucera, 1979; Glaria and Goodwin, 1994; Vidyasagar, 1985). Therefore, the problem of obtaining the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output time-delay plants is important.

In this paper, we propose the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output time-delay plants such that the controller works as a robust stabilizing multi-period repetitive controller for time-delay plants and

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transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles when the uncettainty does not exist.

Notation

R	the set of real numbers.
R_+	$R \cup \{\infty\}$.
$R(s)$	the set of real rational functions with s.
RH_∞	the set of stable proper real rational functions.
H_∞	the set of stable causal functions.
$D^\perp$	orthogonal complement of D, i.e., $\left[\, D \ D^\perp \,\right]$ or $\begin{bmatrix} D \\ D^\perp \end{bmatrix}$ is unitary.
A^T	transpose of A.
$A^\dagger$	pseudo inverse of A.
$\rho(\{\cdot\})$	spectral radius of $\{\cdot\}$.
$\bar{\sigma}(\{\cdot\})$	largest singular value of $\{\cdot\}$.
$\\|\{\cdot\}\\|_\infty$	H_∞ norm of $\{\cdot\}$.
$\mathcal{L}\{\cdot\}$	the Laplace transformation of $\{\cdot\}$.
$\mathcal{L}^{-1}\{\cdot\}$	the inverse Laplace transformation of $\{\cdot\}$.
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	represents the state space description $C(sI-A)^{-1}B+D$.

2. PROBLEM FORMULATION

Consider the unity feedback control system given by

$$\begin{cases} y = G(s)e^{-sL}u + d \\ u = C(s)(r-y) \end{cases}, \tag{1}$$

where $G(s)e^{-sL}$ is the multiple-input/multiple-output time-delay plant, $L>0$ is the time-delay, $G(s) \in R^{m\times p}(s)$ is assumed to be stabilizable and detectable. $C(s)$ is the multi-period repetitive controller with m-th input and p-th output defined later, $u \in R^p$ is the control input, $d \in R^m$ is the disturbance, $y \in R^m$ is the output and $r \in R^m$ is the periodic reference input with period $T>0$ satisfying

$$r(t+T) = r(t) \quad (\forall t \geq 0). \tag{2}$$

It is assumed that $m \leq p$ and $\operatorname{rank} G(s) = m$. The nominal plant of $G(s)e^{-sL}$ is denoted by $G_m(s)e^{-sL_m}$, where $G_m(s) \in R^{m\times p}(s)$. Both $G(s)$ and $G_m(s)$ are assumed to have no zero or pole on the imaginary axis (Doyle et al., 1989). In addition, it is assumed that the number of poles of $G(s)$ in the closed right half plane is equal to that of $G_m(s)$ (Doyle et al., 1989). The relation between the plant $G(s)e^{-sL}$ and the nominal plant $G_m(s)e^{-sL_m}$ is written as

$$G(s)e^{-sL} = (e^{-sL_m}I + \Delta(s))G_m(s), \tag{3}$$

where $\Delta(s)$ is an uncertainty. The set of $\Delta(s)$ is all functions satisfying

$$\bar{\sigma}\{\Delta(j\omega)\} < |W_T(j\omega)| \quad (\forall \omega \in R_+), \tag{4}$$

where $W_T(s)$ is a stable rational function.

The robust stability condition for the plant with uncertainty $\Delta(s)$ satisfying (4) is given by

$$\|T(s)W_T(s)\|_\infty < 1, \tag{5}$$

where $T(s)$ is given by

$$T(s) = \left(I + G_m(s)e^{-sL_m}C(s)\right)^{-1} G_m(s)e^{-sL_m}C(s). \tag{6}$$

According to Gotou et al. (1987); Okuyama et al. (2002); Yamada et al. (2003), the general form of multi-period repetitive controller $C(s)$ which makes the output y to follow the periodic reference input r with period T in (1) with small steady state error, is written in the form

$$C(s) = C_0(s) + \sum_{i=1}^{N} C_i(s)e^{-sT_i}\left(I - \sum_{i=1}^{N} q_i(s)e^{-sT_i}\right)^{-1} \tag{7}$$

where N is an arbitrary positive integer, $T_i > 0 \in R$ $(i=1,\ldots,N)$, $C_0(s) \in R^{p\times m}(s)$, $C_i(s) \in R^{p\times m}(s)(i=1,\ldots,N)$ satisfying $\operatorname{rank} C_i(s) = m(i=1,\ldots,N)$, $q_i(s) \in R^{m\times m}(s)(i=1,\ldots,N)$ are low-pass filters satisfying $\sum_{i=1}^{N} q_i(0) = I$. In the following, $e^{-sT_i}(I - \sum_{i=1}^{N} q_i(s)e^{-sT_i})^{-1}$ defines the internal model for the periodic signal with period T. According to Gotou et al. (1987); Okuyama et al. (2002); Yamada et al. (2003), if the low-pass filters $q_i(s)(i=1,\ldots,N)$ satisfy

$$\bar{\sigma}\left\{I - \sum_{i=1}^{N} q_i(j\omega_k)\right\} \simeq 0 \quad (k=0,\ldots,N_{max}), \tag{8}$$

where $\omega_k(k=0,\ldots,N_{max})$ are frequency components of the periodic reference input r written by

$$\omega_k = \frac{2\pi}{T}k \quad (k=0,\ldots,N_{max}), \tag{9}$$

and $\omega_{N_{max}}$ is the maximum frequency component of the periodic reference input r, then the output y in (1) follows the periodic reference input r with small steady state error.

Using the multi-period repetitive controller $C(s)$ in (7), transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y in (1), respectively, are written as

$$\begin{aligned} y &= \left(I + G(s)e^{-sL}C(s)\right)^{-1} G(s)e^{-sL}C(s)r \\ &= (e^{-sL_m}I + \Delta(s))G_m(s)\left\{C_0(s) + \sum_{i=1}^{N}(C_i(s) - C_0(s)\right. \\ &\quad \left. q_i(s))\, e^{-sT_i}\right\}\left[I + \left(e^{-sL_m}I + \Delta(s)\right)G_m(s)C_0(s)\right. \\ &\quad - \sum_{i=1}^{N}\left[\{I + (e^{-sL_m}I + \Delta(s))G_m(s)C_0(s)\}\, q_i(s)\right. \\ &\quad \left.\left. -(e^{-sL_m}I + \Delta(s))G_m(s)C_i(s)\right] e^{-sT_i}\right]^{-1} r \end{aligned} \tag{10}$$

and

$$\begin{aligned} y &= \left(I + G(s)e^{-sL}C(s)\right)^{-1} d \\ &= \left(I - \sum_{i=1}^{N} q_i(s)e^{-sT_i}\right)\left[I + \left(e^{-sL_m}I + \Delta(s)\right)G_m(s)\right. \\ &\quad C_0(s) - \sum_{i=1}^{N}\left[\{I + (e^{-sL_m}I + \Delta(s))G_m(s)C_0(s)\}\, q_i(s)\right. \\ &\quad \left.\left. -(e^{-sL_m}I + \Delta(s))G_m(s)C_i(s)\right] e^{-sT_i}\right]^{-1} d. \end{aligned} \tag{11}$$

Generally, transfer functions from the periodic reference input r to the output y in (10) and from the disturbance

d to the output y in (11) have infinite numbers of poles, even if $\Delta(s) = 0$. In this situation, it is difficult to specify the input-output characteristic and the disturbance attenuation characteristic. From the practical point of view, it is desirable that the input-output characteristic and the disturbance attenuation characteristic are easily specified. To do this, it is desirable for transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y to have finite numbers of poles.

From above practical requirement, we define a robust stabilizing simple multi-period repetitive controller for multiple-input/multiple-output time-delay plants as Definition 1 and clarify the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output time-delay plants.

Definition 1. (robust stabilizing simple multi-period repetitive controller for multiple-input/multiple-output time-delay plants.)
We call the controller $C(s)$ a "robust stabilizing simple multi-period repetitive controller for multiple-input/multiple-output time-delay plants", if following expressions hold true:

(1) The controller $C(s)$ works as a modified repetitive controller. That is, the controller $C(s)$ is described by (7).
(2) When $\Delta(s) = 0$, the controller $C(s)$ makes transfer functions from the periodic reference input r to the output y in (1) and from the disturbance d to the output y in (1) have finite numbers of poles.
(3) The controller $C(s)$ satisfies the robust stability condition in (5).

3. THE PARAMETERIZATION

In this section, we clarify the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output time-delay plants defined in Definition 1.

In order to obtain the parameterization of all robust stabilizing simple multi-period repetitive controllers, we must see that controllers $C(s)$ ensure that (5) holds. The problem of obtaining the controller $C(s)$, which is not necessarily a simple multi-period repetitive controller, satisfying (5) is equivalent to the following H_∞ control problem. In order to obtain the controller $C(s)$ satisfying (5), we consider the control system shown in Fig. 1. $P(s)$

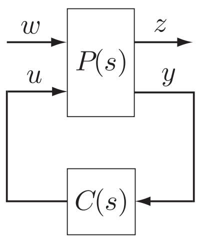

Fig. 1. Block diagram of H_∞ control problem

is selected such that the transfer function from w to z in Fig. 1 is equal to $T(s)W_T(s)$. The state space description of $P(s)$ is, in general,

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t - L_m) \\ z(t) = C_1 x(t) + D_{12} u(t) \\ y(t) = C_2 x(t) + D_{21} w(t) \end{cases}, \tag{12}$$

where $A \in R^{n\times n}$, $B_1 \in R^{n\times m}$, $B_2 \in R^{n\times p}$, $C_1 \in R^{m\times n}$, $C_2 \in R^{m\times n}$, $D_{12} \in R^{m\times p}$, $D_{21} \in R^{m\times m}$, $x(t) \in R^n$, $w(t) \in R^m$, $z(t) \in R^m$, $u(t) \in R^p$ and $y(t) \in R^m$. $P(s)$ is called the generalized plant. $P(s)$ is assumed to satisfy following assumptions:

(1) (C_2, A) is detectable, (A, B_2) is stabilizable.
(2) D_{12} has full column rank, and D_{21} has full row rank.
(3) $\text{rank}\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + p \quad (\forall\omega \in R_+)$,
$\text{rank}\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + m \quad (\forall\omega \in R_+)$.
(4) $C_1 A^i B_2 = 0 \quad (i = 0, 1, 2, \ldots)$.

Under these assumptions, from Abe and Kojima (2007), following lemma holds true.

Lemma 1. There exists an H_∞ controller $C(s)$ for the generalized plant $P(s)$ in (12) if and only if there exists an H_∞ controller $C(s)$ for the generalized plant $\tilde{P}(s)$ written by

$$\begin{cases} \dot{q}(t) = Aq(t) + B_1 w(t) + \tilde{B}_2 u(t) \\ \tilde{z}(t) = C_1 q(t) + D_{12} u(t) \\ \tilde{y}(t) = C_2 q(t) + D_{21} w(t) \end{cases}, \tag{13}$$

where $\tilde{B}_2 = e^{-AL_m} B_2$. When $u(s) = C(s)\tilde{y}(s)$ is an H_∞ control input for the generalized plant $\tilde{P}(s)$ in (13),

$$u(t) = \mathcal{L}^{-1}\left\{C(s)\tilde{y}(s)\right\} \tag{14}$$

is an H_∞ control input for the generalized plant $P(s)$ in (12), where

$$\tilde{y}(s) = \mathcal{L}\left\{ y(t) + C_2 \int_{-L_m}^{0} e^{-A(\tau + L_m)} B_2 u(t + \tau) d\tau \right\}. \tag{15}$$

From Lemma 1 and Doyle et al. (1989), the following lemma holds true.

Lemma 2. If controllers satisfying (5) exist, both

$$\begin{aligned} &X\left(A - \tilde{B}_2 D_{12}^{\dagger} C_1\right) + \left(A - \tilde{B}_2 D_{12}^{\dagger} C_1\right)^T X \\ &+X\left\{B_1 B_1^T - \tilde{B}_2 \left(D_{12}^T D_{12}\right)^{-1} \tilde{B}_2^T\right\} X \\ &+\left(D_{12}^{\perp} C_1\right)^T D_{12}^{\perp} C_1 = 0 \end{aligned} \tag{16}$$

and

$$\begin{aligned} &Y\left(A - B_1 D_{21}^{\dagger} C_2\right)^T + \left(A - B_1 D_{21}^{\dagger} C_2\right) Y \\ &+Y\left\{C_1^T C_1 - C_2^T \left(D_{21} D_{21}^T\right)^{-1} C_2\right\} Y \\ &+B_1 D_{21}^{\perp} \left(B_1 D_{21}^{\perp}\right)^T = 0 \end{aligned} \tag{17}$$

have solutions $X \geq 0$ and $Y \geq 0$ such that

$$\rho\left(XY\right) < 1 \tag{18}$$

and both

$$A - \tilde{B}_2 D_{12}^{\dagger} C_1 + \left\{ B_1 B_1^T - \tilde{B}_2 \left(D_{12}^T D_{12} \right)^{-1} \tilde{B}_2^T \right\} X \quad (19)$$

and

$$A - B_1 D_{21}^{\dagger} C_2 + Y \left\{ C_1^T C_1 - C_2^T \left(D_{21} D_{21}^T \right)^{-1} C_2 \right\} \quad (20)$$

have no eigenvalue in the closed right half plane. Using X and Y, the parameterization of all controllers satisfying (5) is given by

$$C(s) = C_{11}(s) + C_{12}(s) Q(s) (I - C_{22}(s) Q(s))^{-1} C_{21}(s), \quad (21)$$

where

$$\begin{bmatrix} C_{11}(s) & C_{12}(s) \\ C_{21}(s) & C_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & D_{c22} \end{array} \right], \quad (22)$$

$$\begin{aligned} A_c &= A + B_1 B_1^T X - \tilde{B}_2 \left(D_{12}^{\dagger} C_1 + E_{12}^{-1} \tilde{B}_2^T X \right) \\ &\quad - (I - YX)^{-1} \left(B_1 D_{21}^{\dagger} + Y C_2^T E_{21}^{-1} \right) \\ &\quad \left(C_2 + D_{21} B_1^T X \right), \end{aligned}$$

$$\begin{aligned} B_{c1} &= (I - YX)^{-1} \left(B_1 D_{21}^{\dagger} + Y C_2^T E_{21}^{-1} \right), \\ B_{c2} &= (I - YX)^{-1} \left(\tilde{B}_2 + Y C_1^T D_{12} \right) E_{12}^{-1/2}, \end{aligned}$$

$$\begin{aligned} C_{c1} &= -D_{12}^{\dagger} C_1 - E_{12}^{-1} \tilde{B}_2^T X, \\ C_{c2} &= -E_{21}^{-1/2} \left(C_2 + D_{21} B_1^T X \right), \end{aligned}$$

$$D_{c11} = 0, \quad D_{c12} = E_{12}^{-1/2}, \quad D_{c21} = E_{21}^{-1/2}, \quad D_{c22} = 0,$$

$$E_{12} = D_{12}^T D_{12}, \quad E_{21} = D_{21} D_{21}^T$$

and $Q(s) \in H_{\infty}^{p \times m}$ is any function satisfying $\|Q(s)\|_{\infty} < 1$ (Doyle et al., 1989).

Remark 1. $C(s)$ in (21) is written using Linear Fractional Transformation(LFT). Using homogeneous transformation, (21) is rewritten by

$$\begin{aligned} &C(s) \\ &= (Z_{11}(s) Q(s) + Z_{12}(s)) (Z_{21}(s) Q(s) + Z_{22}(s))^{-1} \\ &= \left(Q(s) \tilde{Z}_{21}(s) + \tilde{Z}_{22}(s) \right)^{-1} \left(Q(s) \tilde{Z}_{11}(s) + \tilde{Z}_{12}(s) \right), \end{aligned} \quad (23)$$

where $Z_{ij}(s)(i = 1,2; j = 1,2)$ and $\tilde{Z}_{ij}(s)(i = 1,2; j = 1,2)$ are defined by

$$\begin{aligned} &\begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \end{bmatrix} \\ &= \begin{bmatrix} C_{12}(s) - C_{11}(s) C_{21}^{-1}(s) C_{22}(s) & C_{11}(s) C_{21}^{-1}(s) \\ -C_{21}^{-1}(s) C_{22}(s) & C_{21}^{-1}(s) \end{bmatrix} \end{aligned} \quad (24)$$

and

$$\begin{aligned} &\begin{bmatrix} \tilde{Z}_{11}(s) & \tilde{Z}_{12}(s) \\ \tilde{Z}_{21}(s) & \tilde{Z}_{22}(s) \end{bmatrix} \\ &= \begin{bmatrix} C_{21}(s) - C_{22}(s) C_{12}^{-1}(s) C_{11}(s) & C_{12}^{-1}(s) C_{11}(s) \\ -C_{22}(s) C_{12}^{-1}(s) & C_{12}^{-1}(s) \end{bmatrix} \end{aligned} \quad (25)$$

and satisfying

$$\begin{aligned} &\begin{bmatrix} \tilde{Z}_{22}(s) & \tilde{Z}_{12}(s) \\ \tilde{Z}_{21}(s) & \tilde{Z}_{11}(s) \end{bmatrix} \begin{bmatrix} Z_{11}(s) & -Z_{12}(s) \\ -Z_{21}(s) & Z_{22}(s) \end{bmatrix} = I \\ &= \begin{bmatrix} Z_{11}(s) & -Z_{12}(s) \\ -Z_{21}(s) & Z_{22}(s) \end{bmatrix} \begin{bmatrix} \tilde{Z}_{22}(s) & \tilde{Z}_{12}(s) \\ \tilde{Z}_{21}(s) & \tilde{Z}_{11}(s) \end{bmatrix}. \end{aligned} \quad (26)$$

Using Lemma 1, Lemma 2 and Remark 1, the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output time-delay plants is given by following theorem.

Theorem 1. If simple multi-period repetitive controllers satisfying (5) exist, both (16) and (17) have solutions $X \geq 0$ and $Y \geq 0$ such that (18) holds and both (19) and (20) have no eigenvalue in the closed right half plane. Using X and Y, the parameterization of all robust stabilizing simple multi-period repetitive control laws satisfying (5) is given by

$$u(t) = \mathcal{L}^{-1} \left\{ C(s) \tilde{y}(s) \right\}, \quad (27)$$

where

$$\tilde{y}(s) = \mathcal{L} \left\{ y(t) + C_2 \int_{-L_m}^{0} e^{-A(\tau + L_m)} B_2 u(t+\tau) d\tau \right\} \quad (28)$$

and

$$\begin{aligned} &C(s) \\ &= (Z_{11}(s) Q(s) + Z_{12}(s)) (Z_{21}(s) Q(s) + Z_{22}(s))^{-1} \\ &= \left(Q(s) \tilde{Z}_{21}(s) + \tilde{Z}_{22}(s) \right)^{-1} \left(Q(s) \tilde{Z}_{11}(s) + \tilde{Z}_{12}(s) \right), \end{aligned} \quad (29)$$

where $Z_{ij}(s)(i = 1,2; j = 1,2)$ and $\tilde{Z}_{ij}(s)(i = 1,2; j = 1,2)$ are defined by (24) and (25), $C_{ij}(s)(i = 1,2; j = 1,2)$ are given by (22) and $Q(s) \in H_{\infty}^{p \times m}$ is any function satisfying $\|Q(s)\|_{\infty} < 1$ and takes the form

$$\begin{aligned} Q(s) = &\left(Q_{n0}(s) + \sum_{i=1}^{N} Q_{ni}(s) e^{-sT_i} \right) \\ &\left(Q_{d0}(s) + \sum_{i=1}^{N} Q_{di}(s) e^{-sT_i} \right)^{-1}, \end{aligned} \quad (30)$$

$$Q_{ni}(s) = G_{2d}(s) \bar{Q}_i(s) \in RH_{\infty}^{p \times m} \quad (i = 1, \dots, N) \quad (31)$$

and

$$Q_{di}(s) = -G_{1d}(s) G_{2n}(s) \bar{Q}_i(s) \in RH_{\infty}^{m \times m} \quad (i = 1, \dots, N). \quad (32)$$

Here, $G_{1n}(s) \in RH_{\infty}^{m \times m}$, $G_{1d}(s) \in RH_{\infty}^{m \times m}$, $G_{2n}(s) \in RH_{\infty}^{m \times p}$ and $G_{2d}(s) \in RH_{\infty}^{p \times p}$ are coprime factors satisfying

$$Z_{22}(s) + G_m(s)Z_{12}(s) = G_{1n}(s)G_{1d}^{-1}(s) \quad (33)$$

and

$$G_{1n}^{-1}(s)\left(Z_{21}(s) + G_m(s)Z_{11}(s)\right) = G_{2n}(s)G_{2d}^{-1}(s). \quad (34)$$

$Q_{n0}(s) \in RH_\infty^{p\times m}$, $Q_{d0}(s) \in RH_\infty^{m\times m}$ and $\bar{Q}_i(s) \in RH_\infty^{p\times m}$ $(i = 1, \ldots, N)$ are any functions satisfying

$$\bar{\sigma}\left\{Z_{22}(0)\left(Q_{d0}(0) + \sum_{i=1}^{N} Q_{di}(0)\right) + Z_{21}(0)\left(Q_{n0}(0) + \sum_{i=1}^{N} Q_{ni}(0)\right)\right\} = 0, \quad (35)$$

$$\text{rank}\ \left(Q_{ni}(s) - Q_{n0}(s)Q_{d0}^{-1}(s)Q_{di}(s)\right) = m \quad (i = 1, \ldots, N) \quad (36)$$

and rank $\bar{Q}_i(s) = m$ $(i = 1, \ldots, N)$.

Proof. First, the necessity is shown. That is, we show that if the multi-period repetitive controller written by (7) stabilizes the control system in (1) robustly and makes transfer functions from the periodic reference input r to the output y in (10) and from the disturbance d to the output y in (11) have finite numbers of poles, when $\Delta(s) = 0$, then $C(s)$ and $Q(s)$ are written by (29) and (30), respectively. From Lemma 2, the parameterization of all robust stabilizing controllers $C(s)$ for $G(s)$ is written by (29), where $Q(s) \in H_\infty^{p\times m}$ is any function satisfying $\|Q(s)\|_\infty < 1$. In order to prove the necessity, we will show that if the controller $C(s)$ written by (29) works as a multi-period repetitive controller, then $Q(s) \in H_\infty^{p\times m}$ in (29) is written by (30). Substituting $C(s)$ in (7) into (29), we have (30), where

$$Q_{n0}(s) = N_{0n}(s)N_d(s), \quad (37)$$

$$Q_{ni}(s) = N_{in}(s) \ \ (i = 1, \ldots, N), \quad (38)$$

$$Q_{d0}(s) = D_{0n}(s)D_d(s)N_{0d}(s)N_d(s) \quad (39)$$

and

$$Q_{di}(s) = D_{in}(s)N_{0d}(s)N_d(s) \ \ (i = 1, \ldots, N). \quad (40)$$

Here, $N_{0n}(s) \in RH_\infty^{p\times m}$, $N_{in}(s) \in RH_\infty^{p\times m}$ $(i = 1, \ldots, N)$, $N_{0d}(s) \in RH_\infty^{m\times m}$, $N_d(s) \in RH_\infty^{m\times m}$, $D_{0n}(s) \in RH_\infty^{m\times m}$, $D_{in}(s) \in RH_\infty^{m\times m}$ $(i = 1, \ldots, N)$, $D_{0d}(s) \in RH_\infty^{m\times m}$ and $D_d(s) \in RH_\infty^{m\times m}$ are coprime factors satisfying

$$\tilde{Z}_{21}(s)C_0(s) - \tilde{Z}_{11}(s) = D_{0n}(s)D_{0d}^{-1}(s), \quad (41)$$

$$\left(\tilde{Z}_{21}(s)C_i(s) - \tilde{Z}_{21}(s)C_0(s)q_i(s) + \tilde{Z}_{11}(s)q_i(s)\right)D_{0d}(s) = D_{in}(s)D_d^{-1}(s), \quad (42)$$

$$\left(\tilde{Z}_{12}(s) - \tilde{Z}_{22}(s)C_0(s)\right)D_{0d}(s)D_d(s) = N_{0n}(s)N_{0d}^{-1}(s) \quad (43)$$

and

$$-\left(\tilde{Z}_{22}(s)C_i(s) - \tilde{Z}_{22}(s)C_1(s)q(s) + \tilde{Z}_{12}(s)q_i(s)\right)D_{0d}(s)$$
$$D_d(s)N_{0d}(s) = N_{in}(s)N_d^{-1}(s) \ \ (i = 1, \ldots, N). \quad (44)$$

From (37) $\sim$ (40), all of $Q_{n0}(s)$, $Q_{ni}(s)$ $(i = 1, \ldots, N)$, $Q_{d0}(s)$ and $Q_{di}(s)$ $(i = 1, \ldots, N)$ are included in RH_∞. Thus, we have shown that if $C(s)$ written by (7) stabilizes the control system in (1) robustly, $Q(s)$ in (29) is written by (30). Since $\sum_{i=1}^{N} q_i(0) = I$, from (37) $\sim$ (40) and (26), (35) holds true. In addition, from the assumption of rank $C_i(s) = m$ $(i = 1, \ldots, N)$ and from (42) and (44),

$$\text{rank}\ D_{in}(s) = m \ \ (i = 1, \ldots, N) \quad (45)$$

and

$$\text{rank}\ N_{in}(s) = m \ \ (i = 1, \ldots, N) \quad (46)$$

hold true. From (45), (46), (38) and (40), (36) is satisfied.

The rest to prove the necessity is to show that when $\Delta(s) = 0$, if $C(s)$ in (7) makes transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y have finite numbers of poles, then $Q_{ni}(s)$ and $Q_{di}(s)$ are written by (31) and (32), respectively. From (30), when $\Delta(s) = 0$, transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y are written by

$$y = G_{ryn}(s)G_{ryd}^{-1}(s)r \quad (47)$$

and

$$y = G_{dyn}(s)G_{dyd}^{-1}(s)d, \quad (48)$$

respectively, where

$$G_{ryn}(s) = G_m(s)\left\{Z_{12}(s)Q_{d0}(s) + Z_{11}(s)Q_{n0}(s) + \sum_{i=1}^{N} \left(Z_{12}(s)Q_{di}(s) + Z_{11}(s)Q_{ni}(s)\right)e^{-sT_i}\right\}, \quad (49)$$

$$G_{ryd}(s) = \left(Z_{22}(s)+G_m(s)Z_{12}(s)\right)Q_{d0}(s)+\left(Z_{21}(s)+G_m(s)Z_{11}(s)\right)Q_{n0}(s) + \sum_{i=1}^{N}\left\{\left(Z_{22}(s) + G_m(s)Z_{12}(s)\right)Q_{di}(s) + \left(Z_{21}(s) + G_m(s)Z_{11}(s)\right)Q_{ni}(s)\right\}e^{-sT_i}, \quad (50)$$

$$G_{dyn}(s) = Z_{22}(s)Q_{d0}(s) + Z_{21}(s)Q_{n0}(s) + \sum_{i=1}^{N}\left(Z_{22}(s)Q_{di}(s) + Z_{21}(s)Q_{ni}(s)\right)e^{-sT_i} \quad (51)$$

and

$$G_{dyd}(s) = \left(Z_{22}(s)+G_m(s)Z_{12}(s)\right)Q_{d0}(s)+\left(Z_{21}(s)+G_m(s)Z_{11}(s)\right)Q_{n0}(s) + \sum_{i=1}^{N}\left\{\left(Z_{22}(s) + G_m(s)Z_{12}(s)\right)Q_{di}(s) + \left(Z_{21}(s) + G_m(s)Z_{11}(s)\right)Q_{ni}(s)\right\}e^{-sT_i}, \quad (52)$$

From the assumption that transfer functions from the periodic reference input r to the output y in (47) and from the disturbance d to the output y in (48) have finite numbers of poles, (50) and (52),

$$(Z_{22}(s)+G_m(s)Z_{12}(s))\,Q_{di}(s) + (Z_{21}(s)+G_m(s)Z_{11}(s))\,Q_{ni}(s)=0 \quad (53)$$

holds. Using (33) and (34), this equation is rewritten by

$$Q_{di}(s) = -G_{1d}(s)G_{2n}(s)G_{2d}^{-1}(s)Q_{ni}(s). \quad (54)$$

Since $Q_{ni}(s) \in RH_\infty^{p\times m}$ and $Q_{di}(s) \in RH_\infty^{m\times m}$, $Q_{ni}(s)$ and $Q_{di}(s)$ are written by (31) and (32), respectively, where $\bar{Q}_i(s) \in RH_\infty^{p\times m}(i=1,\ldots,N)$. From (45), (46), (38) and (40), rank $\bar{Q}_i(s) = m(i=1,\ldots,N)$ holds true. We have thus proved the necessity.

Next, the sufficiency is shown. That is, it is shown that if $C(s)$ and $Q(s) \in H_\infty^{p\times m}$ are given by (29) and (30), respectively, then the controller $C(s)$ is written by the form in (7), $\sum_{i=1}^N q_i(0) = I$ holds true and transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y have finite numbers of poles. Substituting (30) into (29), we have (7), where, $C_0(s)$, $C_i(s)(i=1,\ldots,N)$ and $q_i(s)(i=1,\ldots,N)$ are written by

$$C_0(s) = (Z_{11}(s)Q_{n0}(s)+Z_{12}(s)Q_{d0}(s)) (Z_{21}(s)Q_{n0}(s)+Z_{22}(s)Q_{d0}(s))^{-1}, \quad (55)$$

$$C_i(s) = \left(Q_{n0}(s)Q_{d0}^{-1}(s)\tilde{Z}_{21}(s)+\tilde{Z}_{22}(s)\right)^{-1} \left(Q_{ni}(s)-Q_{n0}(s)Q_{d0}^{-1}(s)Q_{di}(s)\right) (Z_{21}(s)Q_{n0}(s)+Z_{22}(s)Q_{d0}(s))^{-1} \quad (i=1,\ldots,N) \quad (56)$$

and

$$q_i(s) = -(Z_{21}(s)Q_{ni}(s)+Z_{22}(s)Q_{di}(s)) (Z_{21}(s)Q_{n0}(s)+Z_{22}(s)Q_{d0}(s))^{-1} \quad (i=1,\ldots,N). \quad (57)$$

We find that if $C(s)$ and $Q(s)$ are given by (29) and (30), respectively, then the controller $C(s)$ is written by the form in (7). From rank $\bar{Q}_i(s) = m(i=1,\ldots,N)$ and (56), rank $C_i(s) = m(i=1,\ldots,N)$ holds true. Substituting (35) into (57), we have $\sum_{i=1}^N q_i(0) = I$. In addition, from (31) and (32) and easy manipulation, we can confirm that when $\Delta(s)=0$, transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y have finite numbers of poles.

We have thus proved Theorem 1.

4. CONCLUSION

In this paper, we proposed the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output time-delay plants such that the controller works as a robust stabilizing multi-period repetitive controller for time-delay plants and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles. Since the robust stabilizing simple multi-period repetitive control system has merit such as the stability of control system wity uncertainty is guaranteed and the robust stabilizing simple multi-period repetitive control system can be easily designed, the practical application of the robust stabilizing simple repetitive control is expected.

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