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Robust Two Degree-Of-Freedom Control for MIMO System with Both Model and Signal Uncertainties

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Abstract: In the study, a model inversion based two degree-of-freedom (DOF) controller is proposed to achieve optimal robust performance for multi-input-multi-output (MIMO) system with existence of both model and signal uncertainties. The original plant dynamics are modified by a pre-compensator and a post-compensator to obtain a diagonally dominant matrix, where the selected dynamics of the plant are contained within the diagonal entries. The inversion of the diagonal matrix, as a result, reflects the inversion of the plant dynamics within a specified frequency range. The feedback controller is designed using a standard H_2 mixed sensitivity approach. The feedforward controller consists of plant inversion and a feedforward filter, which is optimized to balance the trade-off between nominal performance and robust performance under model uncertainty and signal uncertainty. A numerical example is used to demonstrate the improvement of the robust performance when using the optimal feedforward control.

1. INTRODUCTION

An inversion-based feedforward approach addresses the exogenous signals that are pre-known or pre-measured. It generates an input that compensates for the exogenous signals to achieve better tracking or disturbance rejection performance than using only a feedback controller (Clayton, Tien, Leang, Zou, and Devasia 2009), (Clayton, Tien, Fleming, Moheimani, and Devasia 2008). The input from the feedforward controller is usually injected into the existing feedback loop to form a 2 DOF control framework as shown in Fig. 1 .

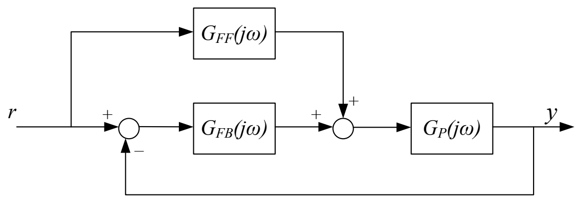

Fig. 1. Two DOF control schematic

Fig. 1 shows a standard structure for 2 DOF control, where $G_p(j\omega)$ is an n-input, n-output continuous MIMO system, $G_p(j\omega)=\left[g_p(j\omega)\right]_{m\times m}$ is an $m\times m$ matrix of rational proper transfer functions. $G_{FB}(j\omega)$ and $G_{FF}(j\omega)$ are the feedback controller and feedforward controller, respectively. The reference vector for the output to track is $r=\left[r_1(j\omega)\quad r_2(j\omega)\quad \cdots \quad r_m(j\omega)\right]^T$ and the output is $y=\left[y_1(j\omega)\quad y_2(j\omega)\quad \cdots \quad y_m(j\omega)\right]^T$. The sensitivity function from the reference r to the tracking error e is shown in Eq. (1), from which it can be concluded that the essence of feedforward control is to make $G_{FF}(j\omega)$ the inversion of the plant dynamics $G_p(j\omega)$ (Skogestad and Postlethwaite 2005).Such types of feedforward controllers are often non-causal, and use previewed exogenous signals $r(\omega)$ to generate the compensating input signals (Zou and Devasia 1999). However, perfect compensation by use of this type of feedforward input is usually not applicable for real systems mainly due to two factors: (i) the mismatch between the model and the real plant dynamics and (ii) the inaccuracy in the signal information. We define these two factors as model uncertainty and signal uncertainty, respectively.

$$e(\omega)=S(j\omega)\left(I-G_P(j\omega)G_{FF}(j\omega)\right)r(\omega) \tag{1}$$

Model uncertainty, and its effect on inversion based feedforward control, have been studied widely. The conditions required of uncertain SISO system are discussed in (Devasia 2002). The influence of various model uncertainties on the error signal is presented in (Faanes and Skogestad 2004). Research on robust control of SISO uncertain system can be found in (Lee and Salapaka 2009),(Adam and Marchetti 2004) and (Vilanova, Arrieta, Ibeas, Balaguer, and Pedret 2008). Due to the difficulty of obtaining the explicit expression of a model inversion for MIMO systems, the robust feedforward control for MIMO systems usually uses a diagonal transfer function matrix to approximate the plant dynamics. Therefore, it is important to find some diagonally dominant matrix that represents the plant dynamics. Relevant work can be found in (Garcia-Sanz, Eguinoa, and Bennani 2009),(Karimi-Ghartemani and Mobed 2008) and (Peng, Xu, Zou, and Zhang 2012). In this paper, the authors propose a pre-compensator and post-compensator framework to make the diagonal entries contain the dominant poles. This method specifies the design by including the chosen poles in the diagonal matrix, which is intuitively straightforward to implement.

Signal uncertainties have received far less attention, compared with model uncertainty, in robust feedforward controller design. However, inaccuracy of the signal information $r(\omega)$ can make the compensating input deviate from its desired value significantly, which causes extra error in the output. The authors have studied feedforward controller designs dealing with signal uncertainty for SISO systems in (Xie and Alleyne 2014). The work is expanded to MIMO systems in this paper.

The rest of this paper is organized as follows. Section 2 shows the framework for the proposed 2 DOF control and discusses the design procedure by examining the norm of the error. Section 3 shows the method to design the pre-compensator and post-compensator. Section 4 introduces a robust feedback controller. Optimal feedforward controller design is presented in Section 5. A numerical example is then used to demonstrate the effectiveness of the proposed 2 DOF control, followed by a conclusion.

2. PROBLEM FORMULATION

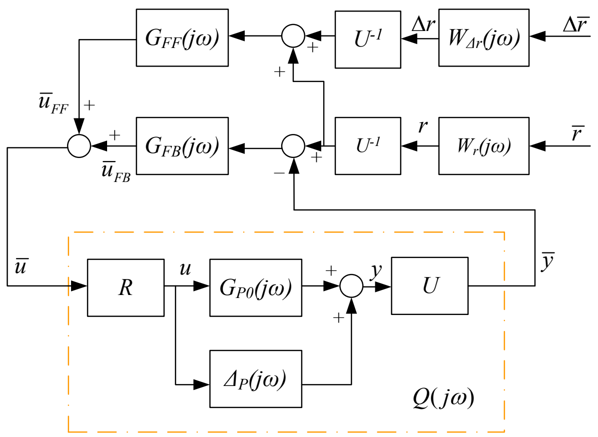

Fig. 2. Transformation for two DOF control system

When considering both the model and signal uncertainty, the system in Fig. 1 can be transformed to Fig. 2 for convenience of analysis in the following content of this paper. $G_{P0}(j\omega)$ is the nominal model, and $\Delta_P(j\omega)$ is the additive uncertainty of $G_{P0}(j\omega)$. The true plant dynamics can be represented as Eq. (2). $R \in \mathbb{R}^{m\times m}$ and $U \in \mathbb{R}^{m\times m}$ are two invertible matrices working as compensators to transform the transfer function matrix $G_{P0}(j\omega)$ to a diagonally dominant matrix $Q_0(j\omega)$ with their relationship shown in Eq. (3). The resulting transformed plant $Q(j\omega)$ is shown in Eq. (4). $\bar{r}$ is a normalized reference signal weighted by $W_r(j\omega)$ as shown in Eq. (5), and $\Delta\bar{r}$ is the normalized reference signal uncertainty weighted by $W_{\Delta r}(j\omega)$ as shown in Eq. (6). Therefore, both $\bar{r}$ and $\Delta\bar{r}$ satisfy the relationships describes in Eq. (7) and (8). u, y are the control input and the plant output, respectively, and $\bar{u}$, $\bar{y}$ are the corresponding transformed input and output signals defined in Eqs. (9) and (10).

$$G_P(j\omega) = G_{P0}(j\omega) + \Delta_P(j\omega) \tag{2}$$

$$Q_0(j\omega) = RG_{P0}(j\omega)U \tag{3}$$

$$Q(j\omega) = RG_{P0}(j\omega)U + R\Delta_P(j\omega)U \tag{4}$$

$$\bar{r}(\omega) = W_r(j\omega)r(\omega) \tag{5}$$

$$\Delta\bar{r}(\omega) = W_{\Delta r}(j\omega)\Delta r(\omega) \tag{6}$$

$$|\bar{r}(\omega)|_2 = 1 \text{ for } \forall\omega \tag{7}$$

$$|\Delta\bar{r}(\omega)|_2 = 1 \text{ for } \forall\omega \tag{8}$$

$$u(\omega) = R\bar{u}(\omega) \tag{9}$$

$$\bar{y}(\omega) = Uy(\omega) \tag{10}$$

With the setup in Fig. 2, the transfer function of the two DOF control system from $\bar{r}$ and $\Delta\bar{r}$ to $\bar{y}$ is given by Eq. (11).

$$\bar{y}(\omega) = S(j\omega)Q(j\omega)\begin{pmatrix} G_{FF}(j\omega)\begin{pmatrix} U^{-1}W_r(j\omega)\bar{r}(\omega) \\ +U^{-1}W_{\Delta r}(j\omega)\Delta\bar{r}(\omega)\end{pmatrix} \\ +G_{FB}(j\omega)U^{-1}W_r(j\omega)\bar{r}(\omega)\end{pmatrix} \tag{11}$$

where $S(j\omega) \triangleq \left(I + Q(j\omega)G_{FB}(j\omega)\right)^{-1}$.

The transformed error $\bar{e}(\omega)$, defined in Eq. (12), can be expressed in Eq. (13).

$$\bar{e}(\omega) = U^{-1}r(\omega) - \bar{y}(\omega) \tag{12}$$

$$\bar{e}(\omega) = S(j\omega)\begin{pmatrix} U^{-1}(j\omega)W_r(j\omega)\bar{r}(\omega) \\ -Q(j\omega)G_{FF}(j\omega)U^{-1}(j\omega)W_r(j\omega)\bar{r}(\omega) \\ -Q(j\omega)G_{FF}(j\omega)U^{-1}(j\omega)W_{\Delta r}(j\omega)\Delta\bar{r}(\omega)\end{pmatrix}$$

Lemma 1: the norm of the transformed tracking error, $\bar{e}(\omega)$, is less than or equal to the multiplication of the feedforward tracking error function $\varepsilon_{FF}(\omega)$ and the closed loop sensitivity function norm $\|S(j\omega)\|_F$, where $\|\cdot\|_F$ is the Frobenius-norm. That is

$$\|\bar{e}(\omega)\|_2 \le \|S(j\omega)\|_F\,\varepsilon_{FF}(\omega) \tag{14}$$

where

$$\begin{aligned}\varepsilon_{FF}(\omega) \triangleq & \left\|\left(I - Q(j\omega)G_{FF}(j\omega)\right)U^{-1}W_r(j\omega)\right\|_F \\ & + \left\|Q(j\omega)G_{FF}(j\omega)U^{-1}W_{\Delta r}(j\omega)\right\|_F\end{aligned} \tag{15}$$

Proof:

$$\left\|\bar{e}(\omega)\right\|_2 = \left\| S(j\omega) \begin{pmatrix} U^{-1}W_r(j\omega)\bar{r}(\omega) \\ -Q(j\omega)G_{FF}(j\omega)U^{-1}W_r(j\omega)\bar{r}(\omega) \\ -Q(j\omega)G_{FF}(j\omega)U^{-1}W_{\Delta r}(j\omega)\Delta\bar{r}(\omega) \end{pmatrix} \right\|_2$$

$$\leq \left\|S(j\omega)\right\|_F \begin{pmatrix} \left\|(I-Q(j\omega)G_{FF}(j\omega))U^{-1}W_r(j\omega)\right\|_F \left\|\bar{r}(\omega)\right\|_2 \\ +\left\|Q(j\omega)G_{FF}(j\omega)U^{-1}W_{\Delta r}(j\omega)\right\|_F \left\|\Delta\bar{r}(\omega)\right\|_2 \end{pmatrix}$$

$$= \left\|S(j\omega)\right\|_F \varepsilon_{FF}(\omega)$$

Q.E.D.

Lemma 2: the norm of the original error $\left\|e(\omega)\right\|_2$ satisfies the relationship with $\left\|\bar{e}(\omega)\right\|_2$ as

$$\left\|e(\omega)\right\|_2 \leq \delta \left\|\bar{e}(\omega)\right\|_2 \text{ for } \forall\omega \tag{16}$$

where δ is a positive real number.

Proof:

With the definition for $\bar{e}(\omega)$ in Eq. (12) and $\bar{y}(\omega)$ in Eq. (10), $\bar{e}(\omega)$ and $e(\omega)$ satisfy the relationship as shown in Eq. (17).

$$e(\omega) = U\bar{e}(\omega) \tag{17}$$

Therefore, the following relationship is derived:

$$\left\|e(\omega)\right\|_2 = \left\|U\bar{e}(\omega)\right\|_2 \leq \left\|U\right\|_2 \left\|\bar{e}(\omega)\right\|_2 = \delta \left\|\bar{e}(\omega)\right\|_2$$

where δ is chosen as the maximum singular value of U.
Q.E.D.

With the conclusions given by lemma 1 and lemma 2, the relationship in (18) is true. Therefore, we can separately design the feedback controller to minimize $\left\|S(j\omega)\right\|_F$ and the feedforward controller to minimize $\varepsilon_{FF}(\omega)$, the combined effect of which will decrease the upper bound for the error.

$$\left\|e(\omega)\right\|_2 \leq \delta \left\|S(j\omega)\right\|_F \varepsilon_{FF}(\omega) \text{ for } \forall\omega \tag{18}$$

3. COMPENSATOR DESIGN

$R \in \mathbb{R}^{m\times m}$ and $U \in \mathbb{R}^{m\times m}$ are two compensators to transform the transfer function matrix $Q_0(j\omega)$ to be the sum of a diagonal transfer function matrix $Q_d(j\omega)$ and a model uncertainty matrix $\Delta_{FF}(j\omega)$. With R and U properly designed, $Q_d(j\omega)$ is able to capture the frequency domain behavior of $Q_0(j\omega)$ within a given frequency range.

A minimal state space realization of $G_{P0}(j\omega)$ is written as Eq. (19). Define $x = P\bar{x}$ with P structured in Eq. (20). With input and output signal transformation defined in Eqs. (9) and (10) , a new state space system for $Q_0(j\omega)$ can be written as Eq. (21) with the coefficient matrix defined in Eqs. (22) - (24).

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{19}$$

$$P = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \tag{20}$$

where A , B and C have dimension as $n\times n$, $n\times m$ and $m\times n$, respectively. r_i , $i = 1,2\cdots n$, is the *i*-th eigenvector of A .

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}\bar{u} \\ \bar{y} &= \bar{C}\bar{x} \end{aligned} \tag{21}$$

$$\bar{A} = P^{-1}AP = diag\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix} \tag{22}$$

where λ_i , $i = 1,2\cdots n$ is the *i*-th eigenvalue of A .

$$\bar{B} = P^{-1}BR \tag{23}$$

$$\bar{C} = UCP \tag{24}$$

Lemma 3: for system in Eq. (21), there always exist invertible matrices R and U satisfying Eqs. (25) and (26).

$$P^{-1}BR = \begin{bmatrix} e_1 + \bar{b}_{1,m+1}e_{m+1} + \cdots + \bar{b}_{1,n}e_n \\ e_2 + \bar{b}_{2,m+1}e_{m+1} + \cdots + \bar{b}_{2,n}e_n \\ \cdots \\ e_m + \bar{b}_{m,m+1}e_{m+1} + \cdots + \bar{b}_{m,n}e_n \end{bmatrix}^T \tag{25}$$

$$UCP = \begin{bmatrix} e_1 + \bar{a}_{1,m+1}e_{m+1} + \cdots + \bar{a}_{1,n}e_n \\ e_2 + \bar{a}_{2,m+1}e_{m+1} + \cdots + \bar{a}_{2,n}e_n \\ \cdots \\ e_m + \bar{a}_{m,m+1}e_{m+1} + \cdots + \bar{a}_{m,n}e_n \end{bmatrix} \tag{26}$$

where e_i are the standard basis for $\mathbb{R}^{n\times 1}$, and $\bar{a}_{i,j}$, $\bar{b}_{i,j}$ are real number coefficients with $i = 1,2\cdots,m$ and $j = m+1, m+2,\cdots,n$.

Proof:

Because Eq. (19) is minimal realization for $G_{P0}(j\omega)$, its controllability matrix C_m in Eq. (27) has rank as n . Define a new matrix $\bar{\mathrm{C}}_m$ in Eq. (28). It can be written as Eq. (29). The rank of the constructed block diagnal matrix R_m is $n\times m$. According to the Sylvester Inequality, the inequality (30) is true. Since $\bar{\mathrm{C}}_m$ has only n rows, the rank of $\bar{\mathrm{C}}_m$ is n .

$$\mathrm{C}_m = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{27}$$

$$\bar{\mathrm{C}}_m = \begin{bmatrix} P^{-1}BR & P^{-1}ABR & \cdots & P^{-1}A^{n-1}BR \end{bmatrix} \tag{28}$$

$$\bar{\mathrm{C}}_m = P^{-1}\,\mathrm{C}_m\,R_m \tag{29}$$

where $R_m = \begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & R \end{bmatrix}$.

$$rank(\bar{C}_m) \geq rank\left(P^{-1}\right) + rank\left(C_m\right) + rank\left(R_m\right) - n \times m - n = n \tag{30}$$

Rewrite matrix $\bar{C}_m$ in the form of Eq. (31). It is equivalent to the controllability matrix for the system in Eq. (21); therefore, the system of Eq. (21) is controllable. With similar derivation, it can be shown that the system of Eq. (21) is also observable. The controllability and observability prove that Eq. (21) is a minimal realization of $Q_0(j\omega)$. Its transfer function can be written as Eq. (32).

$$\bar{C}_m = \left[P^{-1}BR \quad P^{-1}AP^{-1}PBR \quad \cdots \quad P^{-1}A^{n-1}P^{-1}PBR\right] \tag{31}$$

$$\begin{aligned} Q_0(j\omega) &= UCP\left(j\omega I - \bar{A}\right)^{-1} P^{-1}BR \\ &= UCPdiag\begin{pmatrix} \frac{1}{j\omega - \lambda_1} & \frac{1}{j\omega - \lambda_2} & \cdots & \frac{1}{j\omega - \lambda_n} \end{pmatrix} P^{-1}BR \end{aligned} \tag{32}$$

There are m row vectors of $P^{-1}B$ that are independent with respect to each other, which can be shown as follows. Assuming the number of independent row vectors is less than m, there must exist certain elementary matrix operations such that some r^{th} row of $P^{-1}BR$ is trivial, where $R \in \mathbb{R}^{m\times m}$ is chosen as a full rank operational matrix. Assuming the i^{th} row vector of $P^{-1}BR$, denoted as $\bar{b}_i$, is a zero vector. Then the term $\frac{1}{j\omega - \lambda_i}$ will not appear in the matrix product $diag\begin{pmatrix} \frac{1}{j\omega - \lambda_1} & \frac{1}{j\omega - \lambda_2} & \cdots & \frac{1}{j\omega - \lambda_n} \end{pmatrix} P^{-1}BR$; it thereby will not show up in the transfer function of $Q_0(j\omega)$. This implies that the system in Eq. (21) is not a minimal realization, which is not ture according to the analysis above.

Since there are m independent row vectors of $P^{-1}B$, there exist certain column operation such that Eq. (33) is satisfied when a proper sequence of signal values in Eq (22) is selected. The result in Eq. (33) is equivalent to the expression in Eq. (25).

$$P^{-1}BR = \begin{bmatrix} I_{m\times m} \\ \Phi_{(n-m)\times m} \end{bmatrix} \tag{33}$$

where $\Phi_{(n-m)\times m} \in R^{(n-m)\times m}$ is a matrix with any m row vectors independent if $(n-m) > m$.

Similiary, it can be proven that there always exists an invertible matrix U such that Eq. (26) is satisfied. Q.E.D.

Theorem 1: choosing R and U satisfying Eqs. (25) and (26), the system $Q_0(j\omega)$ in Eq. (21) can be decomposed into two transfer function matrices $Q_d(j\omega)$ and $\Delta_{Q\_d}(j\omega)$ shown in Eqs. (35) and (36) .

$$Q_0(j\omega) = Q_d(j\omega) + \Delta_{Q\_d}(j\omega) \tag{34}$$

$$Q_d(j\omega) = diag\begin{pmatrix} \frac{1}{s-\lambda_1} & \frac{1}{s-\lambda_2} & \cdots & \frac{1}{s-\lambda_m} \end{pmatrix} \tag{35}$$

$$\Delta_{Q\_d}(j\omega)_{i,j} = \sum_{k=m-1}^{n} \bar{a}_{i,k}\bar{b}_{j,k} \frac{1}{j\omega - \lambda_k} \tag{36}$$

Proof:
Substitute Eqs. (25) and (26) into Eq. (32) to obtain:

$$\begin{aligned} Q_0(j\omega) &= \\ &= \begin{bmatrix} \frac{1}{j\omega-\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{j\omega-\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{j\omega-\lambda_m} \end{bmatrix} \\ &+ \begin{bmatrix} \sum_{k=m+1}^{n} \bar{a}_{1,k}\bar{b}_{1,k}\frac{1}{j\omega-\lambda_k} & \cdots & \sum_{k=m+1}^{n} \bar{a}_{1,k}\bar{b}_{n,k}\frac{1}{j\omega-\lambda_k} \\ \vdots & \ddots & \vdots \\ \sum_{k=m+1}^{n} \bar{a}_{n,k}\bar{b}_{1,k}\frac{1}{j\omega-\lambda_k} & \cdots & \sum_{k=m+1}^{n} \bar{a}_{n,k}\bar{b}_{n,k}\frac{1}{j\omega-\lambda_k} \end{bmatrix} \\ &= Q_d(j\omega) + \Delta_{Q\_d}(j\omega) \end{aligned} \tag{37}$$

Q.E.D.

Remark 1. Theorem 1 provides a way to decompose a MIMO system $Q_0(j\omega)$ to be a diagonal system $Q_d(j\omega)$ and another system $\Delta_{Q\_d}(j\omega)$. The advantage of such a decomposition is that the diagonal matrix $Q_d(j\omega)$ preserves the dynamics of the original system at the selected poles λ_i, $i = 1,\cdots,m$, and leaves the dynamics associated with other poles λ_i, $i = m+1,\cdots,n$, in the matrix $\Delta_{Q\_d}(j\omega)$. The user is able to choose which poles are to be preserved by arranging the eigenvectors in Eq. (20) when constructing P. In many applications, poles at lower frequencies play a more important role in influencing the dynamical behavior of the system. Therefore λ_i, $i = 1,\cdots,m$, can be specified such that they include the m poles corresponding to the eigenvalues with smallest magnitudes.

Remark 2. Theorem 1 together with the system transformation in Fig. 2 shows a way to design model inversion based feedforward controller $G_{FF}(j\omega)$. $Q_d(j\omega)$ is treated as the system model, and $\Delta_{Q\_d}(j\omega)$ is lumped together with the original model uncertainty as the new model uncertainty for feedforward control. The inversion of $Q_d(j\omega)$ is easily obtained and it is non-casual in many cases. This is feasible for non-casual feedforward control when the reference, or disturbance, signal is pre-known. Although omitting $\Delta_{Q\_d}(j\omega)$ increases the feedforward control error, its impact can be limited by choosing less important poles in $\Delta_{Q\_d}(j\omega)$ as discussed in Remark 1. The design method for $G_{FF}(j\omega)$ and the influence of $\Delta_{Q\_d}(j\omega)$ will be further discussed in Section 5.

Remark 3. This method to obtain an approximately diagonal system can also be used for decentralized feedback control of square systems, which have equal number of the inputs and outputs.

Remark 4. Since the diagonal system considers dominant poles, there might be a significant static gain included in $\Delta_{Q\_d}(j\omega)$. For many systems, the accuracy of the static gain is important, which needs to be addressed in the controller design. Details about this will be further discussed in Section 5.

4. FEEDBACK CONTROLLER DESIGN

The main objective of feedback controller design is to achieve robust stability and maintain robust performance with model uncertainty $\Delta_P(j\omega)$. When designing the feedback controller, the feedforward loop in Fig. 2 is negelected. In this paper, a H_2 mixed sensitivity function is used to design the robust feedback controller. The three terms in the mixed sensitivity function, shown in Eq. (38), represents the three design requirements for the closed loop system respectively: tracking performance, robust stability, and control gain attentuation. $W_p(j\omega)$, $W_t(j\omega)$ and $W_u(j\omega)$ are the three weighting transfer functions designed to specify the empahsis on each objective.

$$\min_{G_{FB}(j\omega)} \left\| \begin{matrix} W_p(j\omega)\dfrac{1}{1+Q_0(j\omega)G_{FB}(j\omega)} \\ W_t(j\omega)\dfrac{Q_0(j\omega)G_{FB}(j\omega)}{1+Q_0(j\omega)G_{FB}(j\omega)} \\ W_u(j\omega)\dfrac{G_{FB}(j\omega)}{1+Q_0(j\omega)G_{FB}(j\omega)} \end{matrix} \right\|_2 \quad (38)$$

The synthesis for the mixed sensitivity problem in Eq. (38) follows the standard procedure with the system formulated as in Fig. 3. The solution of the optimal state feedback controller is a stablizing controller G_{FB} for the augmented system $P_{aug}(j\omega)$ such that the H_2 norm of $P_{aug}(j\omega)$ is minimized. Here, $P_{aug}(j\omega)$ is defined as the system shown in Eq. (39),. The robust stability can be tested by examining whether the inequality (40) is true for the designed G_{FB}.

$$P_{aug}(j\omega) = \begin{bmatrix} W_p(j\omega)I & -W_p(j\omega)Q_0(j\omega) \\ 0 & W_t(j\omega)Q_0(j\omega) \\ 0 & W_u(j\omega) \\ I & -Q_0(j\omega) \end{bmatrix} \quad (39)$$

$$\left\|\Delta_P(j\omega)\right\|_2 \left\|P_{cl}(j\omega)\right\|_2 \le 1 \quad (40)$$

In (40), $P_{cl}(j\omega)$ is the transfer function from $y_{\Delta p}$ to $u_{\Delta p}$ in the closed loop system.

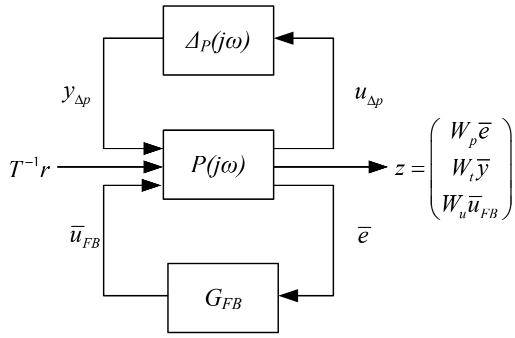

Fig. 3. Block representation for the robust feedback controller design

5. MODEL INVERSION BASED FEEDFORWARD CONTROL

The feedforward control for the transformed system in Fig. 2 is designed based on the model inversion of the plant. $Q_d^{-1}(j\omega)$ is simply the inversion of the diagonal matrix $Q_d(j\omega)$ and is shown in Eq. (41). Due to the existence of $\Delta_{Q\_d}(j\omega)$, $Q_d^{-1}(j\omega)$ deviates from the inversion of the nominal plant $Q_0^{-1}(j\omega)$. As discussed at the end of Section 3, the static error between $Q_d^{-1}(j\omega)$ and $Q_0^{-1}(j\omega)$ needs to be compensated in some cases. Define ω_{st_eff} to be a frequency below which the static compensation takes effect. The compensated model inversion becomes Eq. (42), and the compensating term has a first order effect in the proposed method. The feedforward controller takes the form of Eq. (43) consisting of the compensated matrix and a feedforward filter $G_{FF\_Q}(j\omega)$. $G_{FF\_Q}(j\omega)$ is a transfer function matrix to be designed to minimize $\varepsilon_{FF}(\omega)$ with consideration of both the signal and model uncertainties.

$$Q_d^{-1}(j\omega) = diag\begin{pmatrix} j\omega-\lambda_1 & j\omega-\lambda_2 & \cdots & j\omega-\lambda_m \end{pmatrix} \quad (41)$$

$$\bar{Q}_{d\_inv}(j\omega) = Q_d^{-1}(j\omega) + \frac{\omega_{st\_eff}}{s+\omega_{st\_eff}}\left(Q^{-1}(0) - Q_d^{-1}(0)\right) \quad (42)$$

$$G_{FF}(j\omega) = \bar{Q}_{d\_inv}(j\omega)G_{FF\_Q}(j\omega) \quad (43)$$

With the model inversion based feedforward controller defined in Eq. (43), $\varepsilon_{FF}(\omega)$ can be rewritten as Eq. (44).Denoting the inversion of $\bar{Q}_{d\_inv}(j\omega)$ as $Q_{d\_comp}(j\omega)$, $\Delta Q_{FF}(j\omega)$ in Eq. (45) is defined to be the difference between the true system $Q(j\omega)$ and $Q_{d\_comp}(j\omega)$ in Eq. (45). Therefore, $\Delta Q_{FF}(j\omega)$ consists of two parts: (i) the uncertainty caused by uncertain information when modeling the system and, (ii) the transfer function matrix truncated during the model inversion as shown in Eq. (46).

$$\varepsilon_{FF}(\omega)=\left\|\begin{matrix}\left(I-\left(I+\Delta Q_{FF}(j\omega)\bar{Q}_{d\_inv}(j\omega)\right)G_{FF\_Q}(j\omega)\right)\\ \times U^{-1}W_r(j\omega)\end{matrix}\right\|_F \quad (44)$$
$$+\left\|\left(I+\Delta Q_{FF}(j\omega)\bar{Q}_{d\_inv}(j\omega)\right)G_{FF\_Q}(j\omega)U^{-1}W_{\Delta r}(j\omega)\right\|_F$$

$$\Delta Q_{FF}(j\omega)=Q(j\omega)-Q_{d\_comp}(j\omega)=R\Delta_P(j\omega)U+\Delta_{FF}(j\omega) \quad (45)$$

$$\Delta_{FF}(j\omega)=Q_0(j\omega)-Q_{d\_comp}(j\omega) \quad (46)$$

$$\min_{G_{FF\_Q}(j\omega)}\left(\sup_{\Delta Q_{FF}(j\omega)}\left(\left\|\begin{matrix}\left(\begin{matrix}I-\left(I+\bar{Q}_{d\_inv}(j\omega)\Delta Q_{FF}(j\omega)\right)G_{FF\_Q}(j\omega)\\ \times U^{-1}W_r(j\omega)\end{matrix}\right)\\ \left(\begin{matrix}\left(I+\bar{Q}_{d\_inv}(j\omega)\Delta Q_{FF}(j\omega)G_{FF\_Q}(j\omega)\right)\\ \times U^{-1}W_{\Delta r}(j\omega)\end{matrix}\right)\end{matrix}\right\|_\infty\right)\right) \quad (47)$$

The feedforward control problem can be specified as the minimization problem in (47), which is solved by the following two design steps.

Step 1: Find the balance between the nominal tracking performance and robustness to signal uncertainty

By observing Eq. (47), the norm of the upper expression represents the feedforward tracking error for the reference $\bar{r}$. The norm of the lower expression represnts the performance robustness to the signal uncertainty $\Delta\bar{r}$. With both norms desired to be small, there is a conflict of objectives when designing $G_{FF\_Q}(j\omega)$. To simplify the analysis and focus on balance the two competing objectives, the problem of Eq. (47) is first written as Eq. (48) with $\bar{G}_{eq}(j\omega)$ defined in Eq. (49).

$$\min_{G_{FF\_Q}(s)}\left\|\begin{matrix}\left(I-\bar{G}_{eq}(s)\right)\times U^{-1}W_r(s)\\ \bar{G}_{eq}(s)U^{-1}W_{\Delta r}(s)\end{matrix}\right\|_\infty \quad (48)$$

$$\bar{G}_{eq}(j\omega)=\left(I+\Delta Q_{FF}(j\omega)\bar{Q}_{d\_inv}(j\omega)\right)G_{FF\_Q}(j\omega) \quad (49)$$

The problem in Eq. (48) can be reformulated as an H_2 optimal control problem for an auxiliary system in Fig. 4. $G_{P\_a1}(j\omega)$ is an auxiliary plant chosen by the designer. Typically it is chosen as a low order system to decrease the order of the resulting controller. K_{a1} is the state feedback controller to be designed. By defining the relationship in Eq. (50), the problem in Eq. (48) is equivalent to a mixed sensitity problem for the auxiliary system and can be solved by standard synthesis method. The optimal $\bar{G}_{eq}(j\omega)$, denoted as $\bar{G}^*_{eq}(j\omega)$, is the sensitivity function obtained by the optimization of the mixed sensitivity problem.

$$S_{a1}(j\omega)=\bar{G}_{eq}(j\omega) \quad (50)$$

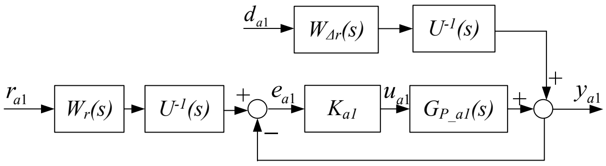

Fig. 4. Auxiliary system for designing $\bar{G}_{eq}(j\omega)$

Step 2: Find the balance between the nominal tracking performance and robustness to model uncertainty

With the result obtained in Step 1, the design problem is transformed into finding a $G_{FF\_Q}(j\omega)$ so that $\bar{G}_{eq}(j\omega)$ is close to $\bar{G}^*_{eq}(j\omega)$ in the presence of $\Delta Q_{FF}(j\omega)$. Let $G_{FF\_Q}(j\omega)$ to be designed with the form in Eq. (51). The difference between $\bar{G}^*_{eq}(j\omega)$ and $\bar{G}_{eq}(j\omega)$ is defined as $\Delta\bar{G}_{eq}(j\omega)$ shown in Eq. (52). The design task is therefore to minimize the norm of $\Delta\bar{G}_{eq}(j\omega)$ under the worst case model uncertainty condition for $\Delta Q_{FF}(j\omega)$.

$$G_{FF\_Q}(j\omega)=\bar{G}_{FF\_Q}(j\omega)\bar{G}^*_{eq}(j\omega) \quad (51)$$

$$\begin{aligned}\Delta\bar{G}_{eq}(j\omega)&=\bar{G}^*_{eq}(j\omega)-\bar{G}_{eq}(j\omega)\\ &=\left(I-\bar{G}_{FF\_Q}(j\omega)\right)\bar{G}^*_{eq}(j\omega)\\ &\quad-\bar{G}_{FF\_Q}(j\omega)\bar{Q}_{d\_inv}(j\omega)\Delta Q_{FF}(j\omega)\bar{G}^*_{eq}(j\omega)\end{aligned} \quad (52)$$

Lemma 4: *an upper bound of* $\left\|\Delta\bar{G}_{eq}(j\omega)\right\|_\infty$ is given by

$$\left\|\Delta\bar{G}_{eq}(j\omega)\right\|_\infty\le\left\|\bar{G}^*_{eq}(j\omega)\right\|_\infty\left(\begin{matrix}\left\|\left(I-\bar{G}_{FF\_Q}(j\omega)\right)\right\|_\infty+\\ \left\|\bar{G}_{FF\_Q}(j\omega)\bar{Q}_{d\_inv}(j\omega)\Delta\bar{Q}_{FF}(j\omega)\right\|_\infty\end{matrix}\right) \quad (53)$$

where $\Delta\bar{Q}_{FF}(j\omega)$ is a transfer function s.t.

$$\Delta\bar{Q}^*_{FF}(j\omega)\Delta\bar{Q}_{FF}(j\omega)\ge\bar{\delta}(\omega) \quad (54)$$

$\bar{\delta}(\omega)$ is defined as the max singular value of $\Delta Q_{FF}(j\omega)$.

Proof:

$$\begin{aligned}&\left\|\Delta\bar{G}_{eq}(j\omega)\right\|_\infty\\ &=\left\|\begin{matrix}\left(I-\bar{G}_{FF\_Q}(j\omega)\right)\bar{G}^*_{eq}(j\omega)\\ -\bar{G}_{FF\_Q}(j\omega)\bar{Q}_{d\_inv}(j\omega)\Delta Q_{FF}(j\omega)\bar{G}^*_{eq}(j\omega)\end{matrix}\right\|_\infty\\ &\le\left\|\bar{G}^*_{eq}(j\omega)\right\|_\infty\left(\begin{matrix}\left\|\left(I-\bar{G}_{FF\_Q}(j\omega)\right)\right\|_\infty+\\ \left\|\bar{G}_{FF\_Q}(j\omega)\bar{Q}_{d\_inv}(j\omega)\Delta\bar{Q}_{FF}(j\omega)\right\|_\infty\end{matrix}\right)\end{aligned}$$

$\left\|\left(I-\bar{G}_{FF\_Q}(j\omega)\right)\right\|_\infty$ and $\left\|\bar{G}_{FF\_Q}(j\omega)Q_d^{-1}(j\omega)\Delta\bar{Q}_{FF}(j\omega)\right\|_\infty$ in (53) show, respectively, the two competing objectives: to achieve small nominal tracking error and to decrease the error induced by model uncertianty. To minimize the upper bound in the right side of (53), a technique similar to Step 1 can be employed. Due to space limitations, the details for this design procedure is omitted. The optimal $\bar{G}_{FF\_Q}(j\omega)$, denoted as $\bar{G}^*_{FF\_Q}(j\omega)$, is the sensitivity function obtained by the optimization of the mixed sensitivity problem for the auxiliary system.

With the two design steps described above, the design for the feedforward controller is finalized in Eq. (55). $\bar{Q}_{d\_inv}(j\omega)$ is the inversion of the approximated model, which captures the system behavior within a chosen frequency range. $\bar{G}^*_{eq}(j\omega)$ and $\bar{G}^*_{FF\_Q}(j\omega)$ are designed seperately to balance the trade-offs between the nominal tracking performance and the robustness with respect to signal and model uncertainties.

$$G_{FF}(j\omega) = \bar{Q}_{d\_inv}(j\omega)\bar{G}^*_{FF\_Q}(j\omega)\bar{G}^*_{eq}(j\omega) \tag{55}$$

6. NUMERICAL EXAMPLE

Consider the linear MIMO system defined in Eq. (56), one minimal realization of which is shown in Eq. (57). With the process described in Section 3, the pre-compensator and post-compensator are calculated as Eqs. (58) and (59) respectively. This results in the $Q_0(j\omega)$ in Eq. (60) according to Theorem 1. The comparison between $Q_0(j\omega)$ and the diagonal transfer function matrix $Q_d(j\omega)$ is shown in Fig 5. The diagonal entries fit well and the off-diagonal entries have much smaller magnitudes compared to the diagonal entries. Therefore, $Q_d(j\omega)$ is able to capture the major dynamics of $Q_0(j\omega)$. For this particular case, static error compensation is not necessary, which means we let $\bar{Q}_{d\_inv}(j\omega) = Q_d^{-1}(j\omega)$.

$$y = \begin{bmatrix} \dfrac{1}{s(s+25)} & \dfrac{1}{s+25} \\ \dfrac{1}{s+3} & \dfrac{1}{(s+3)(s+30)} \end{bmatrix} u \tag{56}$$

$$\dot{x} = \begin{bmatrix} -25 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -30 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \tag{57}$$

$$R = \begin{bmatrix} -0.0371 & 1.0015 \\ 1.0015 & -0.0401 \end{bmatrix} \tag{58}$$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{59}$$

$$Q_0(j\omega) = \begin{bmatrix} \dfrac{1}{s} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix} + \begin{bmatrix} \dfrac{0.0015}{s+25} & \dfrac{-0.037}{s+25} \\ \dfrac{-0.04}{s+30} & \dfrac{0.0015}{s+30} \end{bmatrix} \tag{60}$$

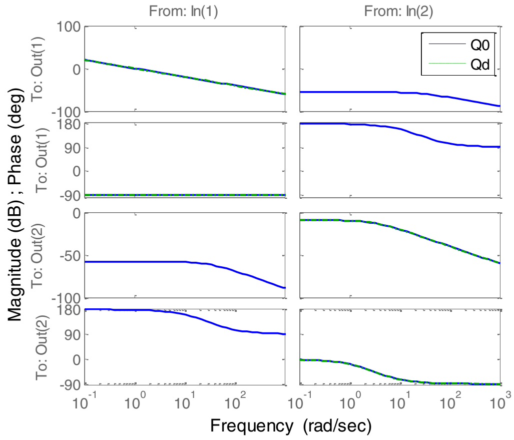

Fig. 5. Comparison between the frequency properties of $Q_0(j\omega)$ and $Q_d(j\omega)$.

For feedback controller design, the weighting functions are chosen as Eqs. (61)- (63). The Bode diagram of the resulting closed loop sensitivity function from $\bar{r}$ to $\bar{e}$ is shown in Fig 6. The sensitivity function of the diagonal input-output pairs gives a bandwidth of ~1.2rad/s.

$$W_p = \frac{s+10}{s+0.1} \tag{61}$$

$$W_t = \frac{50s+10}{s+10} \tag{62}$$

$$W_u = 0.1 \tag{63}$$

For the feedforward controller design, assume the upper bound of the model uncertainty is Eq. (64) and the weighting functions for the signal and signal uncertainty are Eq. (65) and Eq. (66), respectively. Follow the design procedure in Section 5 to obtain the feedforward controller. The corresponding sensitivity functions from the reference signal $\bar{r}$ and the signal uncertainty $\Delta\bar{r}$ to the error are given in Fig 7 and 8, respectively. Compared to Fig. 6, the magnitude of the sensitivity functions are much smaller in all frequency ranges and is always less than 0dB. The sensitivity functions when using direct model inversion feedforward are also shown in Figs. 7 and 8. The nominal tracking performance in Fig. 7 is sacrified in the optimal feedforward design to decrease the error caused by the signal uncertainty as shown in Fig. 8.

$$\Delta\bar{Q}_{FF}(j\omega) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{s}{(s+1.5)(s+100)} \tag{64}$$

$$W_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{s(s+1)} \tag{65}$$

$$W_{\Delta r} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{(s+2)(s+1)} \tag{66}$$

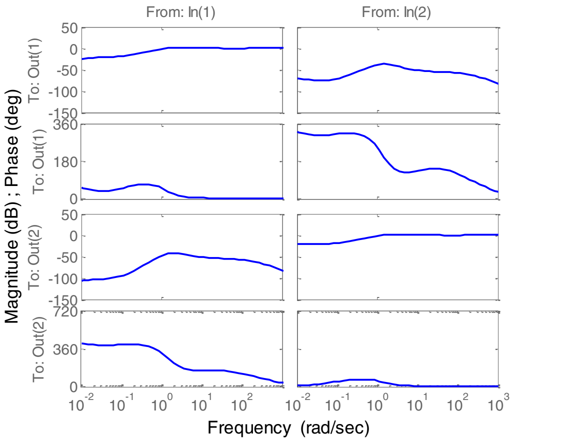

Fig. 6. Frequency properties of sensitivity functions from $\bar{r}$ to $\bar{e}$

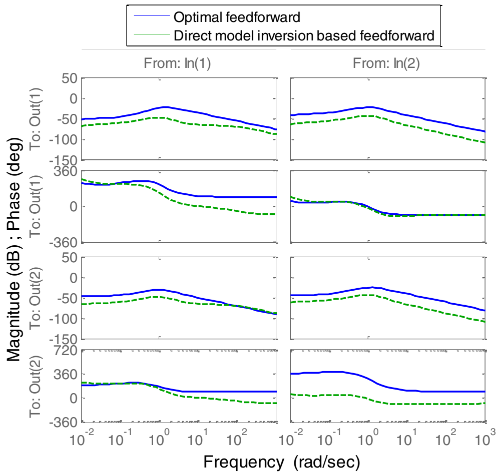

Fig. 7. Comparison between the frequency properties of transfer function from $\bar{r}$ to $\bar{e}$ with and without optimal filters.

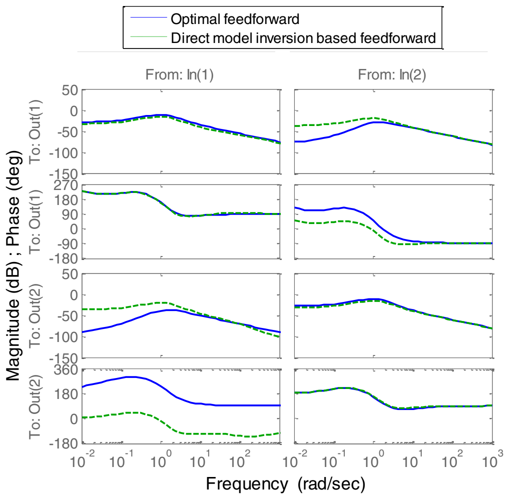

Fig. 8. Comparison between the frequency properties of transfer function from $\Delta\bar{r}$ to $\bar{e}$ with and without optimal filters.

7. CONCLUSIONS

This paper presents an optimal two DOF controller for MIMO systems. The feedback controller is designed following the standard robust controller synthesis to obtain closed loop robust stability and performance. The feedforward controller, on the other hand, is designed considering both the model uncertainty and signal uncertainty. In order to achieve robust performance, it contains two parts: (i) model inversion and (ii) an optimal filter. A method is proposed in this paper to construct a diagonally dominant matrix, where the dynamics of selected poles populate the diagonal entries. This simplifies the procedure of obtaining the inverse model. The optimal filter, on the other hand, is designed to consider the trade-off between nominal performance with the error caused by the model uncertainty and signal uncertainty. Future work will examine implementation of the design approach on an experimental system to illustrate its applicability on physical MIMO systems.

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