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Multiperiod Mean-Variance Portfolio Optimization with General Correlated Returns ⋆

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Abstract: While rich empirical evidence always shows certain degree of dependency among different time periods for the returns of risky assets, the current literature on dynamic portfolio selection has been dominated by the results under an independency assumption, although in various forms. We consider in this paper a multiperiod mean-variance (MV) portfolio selection problem for a market with multiple risky assets whose returns are statistically correlated among time periods. Instead of assuming some particular stochastic processes to model the correlation, we adopt a formulation with a general form of correlation, which enables us to better matching our model with real markets. Recognizing the fact that, under this general setting, parameters in the portfolio policy become path-dependent adaptive processes themselves, we solve the problem analytically and derive an explicit form for the optimal portfolio policy, which remains as a linear affine function of the current wealth.

Keywords: Multi-period portfolio selection, Multi-period mean-variance formulation, Stochastic control, Dynamic programming

1. INTRODUCTION

The mean-variance formulation proposed by Markowitz (1952) more than half century ago laid the foundation for modern financial analysis. After the static MV portfolio selection theory was extended to multi-period MV portfolio selection by Li and Ng (2000) and to continuous-time MV portfolio selection by Zhou and Li (2000), both in 2000, the past decade has witnessed significant advancement of both theory and methodologies for dynamic MV portfolio selection by leaps and bounds, see for example, Li et al. (2001), Lim and Zhou (2002), Zhu et al. (2004), Bielecki et al. (2005), Cui et al. (2012), and Cui et al. (2013). While almost all the studies in continuous-time assume that the stock prices follow a vector-valued Geometric Brownian motion, most studies in discrete-time assume the time independency of the return vector, with a few exceptions.

Abundant empirical evidence shows that the returns of the risky assets always exhibit certain degree of dependency among time periods, e.g., see Campbell and Viceira (2002) and reference therein. Thus, it is necessary and meaningful to investigate MV portfolio optimization models with correlated returns. The last decade has witnessed a few attempts to incorporate a dependency structure of the returns into the MV portfolio model. Çakmak and Özekici (2006) adopt a Morkov chain to model the switching of the market conditions. Costa and Araujo (2008) study such a Markovian switching model with no bankruptcy restriction on the wealth. Following the same line, Costa and Oliveira (2012) extend such a method to a more general mean-variance control problem. While the majority of the current literature assumes particular stochastic processes to model the correlation of the asset returns, investigation on dynamic mean-variance models with a general correlation form of returns does emerge recently. A general correlation form is assumed in Xu and Li (2008) for returns at different time periods for a market with only one risky asset and one risk free asset. General forms of correlation structure are also assumed for returns in the portfolio selection formulations of Dokuchaev (2007) and Dokuchaev (2012). In their work on time cardinality constrained dynamic MV portfolio selection, Gao et al. (2013) investigate a dynamic MV formulation with general correlation for a market with multiple risky assets and one riskless asset. In this work, we consider first a market with multiple risky assets only and general form of correlation, and derive the analytical portfolio policy of the dynamic MV portfolio optimization model. We discuss next the case where a risk free asset is also included in the market.

The remaining of this paper is organized as follows. We first formulate the dynamic mean-variance portfolio selection problem with and without risk free asset in Section 2.

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We then derive in Section 3 the optimal portfolio policies for these problems. We give in Section 4 an example to illustrate our solution procedure. Finally, we conclude the paper in Section 5. We use $\pi(\cdot)$ and $v(\cdot)$ to denote the optimal control (policy) and the optimal value of problem $(\cdot)$. The notations $\mathbf{1}$, $\mathbf{0}$ and $\mathbf{I}$ stand, respectively, for the vector with all elements being 1, zero matrix, and the identity matrix with a proper dimension.

2. PROBLEM FORMULATION

We assume that the capital market consists of n risky assets, all of which evolve within a time horizon of T periods, $t = 0, 1, \ldots, T-1$. An investor with initial endowment x_0 enters the market in period 0 and allocates his wealth among these n assets at the beginning of each of the T periods from $t = 0$ to $t = T-1$. We denote the return vector of the n risky assets in period t as

$$e_t \triangleq \left(e_t^1, e_t^2, \cdots, e_t^n\right)'$$

for $t = 0, \cdots, T-1$, which is a square integrable random vector. In our study, we allow return vectors in different time periods, $\{e_t,\}_{t=0}^{T-1}$, to be statistically correlated. All the underlying uncertainties are modeled by a complete probability space $(\Omega, \mathbb{P}, \mathcal{F})$ with the structure of the filtration satisfying $\mathcal{F}_t \subset \mathcal{F}_{t+1}$, $t = 0, \cdots, T-1$ and $\mathcal{F}_T = \mathcal{F}$. The filtration $\mathcal{F}_t$ represents the information available at stage t.[1] We use the notations $\mathrm{E}_t[\cdot]$, $\mathrm{Cov}_t[\cdot]$ and $\mathrm{Var}_t[\cdot]$ to denote the conditional expectation $\mathrm{E}[\cdot|\mathcal{F}_t]$, the conditional covariance matrix $\mathrm{Cov}[\cdot|\mathcal{F}_t]$ and the conditional variance $\mathrm{Var}[\cdot|\mathcal{F}_t]$, respectively. The following assumption is reasonable in a financial market.

Assumption 1. We assume that the conditional covariance matrices, $\mathrm{Cov}_t[e_t] := \mathrm{E}_t[e_t e_t'] - \mathrm{E}_t[e_t]\mathrm{E}_t[e_t]'$, are positive definite for all $t = 0, \ldots, T-1$.

Let u_t^i be the dollar amount invested in the i-th risky asset at time t, $i = 1, \cdots, n$, and x_t be the wealth level at time t. Then, under the self-financing constraint, the wealth process evolves according to

$$x_t = \mathbf{1}'u_t, \ t = 0, \cdots, T-1, \tag{1}$$

$$x_{t+1} = e_t'u_t, \ t = 0, \cdots, T-1, \tag{2}$$

where $u_t \triangleq \left(u_t^1, u_t^2, ..., u_t^n\right)'$. The dynamics in (1) and (2) can be understood in the following way. At stage t, the current wealth is x_t and the wealth of next period, x_{t+1}, is a function of random return e_t and the portfolio allocation u_t, which are subject to budget constraint $\mathbf{1}'u_t = x_t$. Since x_t is the realized wealth at time t, the budget constraint $\mathbf{1}'u_t = x_t$ is a deterministic constraint at time t. The investor is seeking a best mean-variance investment strategy, u_t, for $t = 0, \cdots, T-1$, such as to minimize the variance of terminal wealth under a constraint that the expected return is not less than a given aspiration level $\epsilon > 0$,

$$(\mathcal{P}(\epsilon)): \quad \min_{u_t} \ \mathrm{Var}[x_T]$$

$$\text{Subject to: } \begin{cases} \mathrm{E}[x_T] \geq \epsilon, \\ \{u_t, x_t\} \text{ satisfies (1) and (2).} \end{cases}$$

In this paper, we also consider a formulation in which the risk free asset is also included in the market. Let the returns of the risk free asset be r_t, $t = 0, \cdots, T-1$, which is assumed to be deterministic in this paper. When the risk free asset is included in the portfolio, we can represent allocation in the risk-free asset by $x_t - \mathbf{1}'u_t$, which gives rise to the wealth in the next time period as, $x_{t+1} = r_t(x_t - \mathbf{1}'u_t) + e_t'u_t$, for $t = 1, \cdots, T-1$. The wealth process can be further written as the following compact form,

$$x_{t+1} = r_t x_t + P_t' u_t, \quad \text{for } t = 0, \cdots, T-1, \tag{3}$$

where $P_t \triangleq \left(P_t^1, P_t^2, \cdots, P_t^n\right)'$ is the excess return vector, which is defined by $P_t = e_t - r_t\mathbf{1}$. In a market consisting of multiple risky assets and a risk free asset, the mean-variance portfolio optimization problem becomes

$$(\mathcal{P}_f(\epsilon)): \quad \min_{u_t} \ \mathrm{Var}[x_T]$$

$$\text{Subject to: } \begin{cases} \mathrm{E}[x_T] \geq \epsilon, \\ \{x_t, \, u_t\} \text{ satisfies (3).} \end{cases}$$

3. OPTIMAL PORTFOLIO POLICY

3.1 Solution of problem $(\mathcal{P}(\epsilon))$

We first solve problem $(\mathcal{P}(\epsilon))$. Solving problem $(\mathcal{P}(\epsilon))$ is equivalent to minimizing a weighted sum of the mean-variance pair for some $\omega > 0$,

$$(\bar{\mathcal{P}}(\omega)) \quad \min_{u_t} \ \omega \mathrm{Var}[x_T] - \mathrm{E}[x_T]$$

$$\text{Subject to: } \{u_t, x_t\} \text{ satisfies (1) and (2).}$$

Before we give the solution of problem $(\bar{\mathcal{P}}(\omega))$, we introduce the following stochastic processes α_t, β_t, η_t, D_t and d_t for $t = T-1, \cdots, 0$, which play important roles in solving $(\bar{\mathcal{P}}(\omega))$,

$$\alpha_t := \mathbf{1}/\mathbf{1}'D_t^{-1}\mathbf{1}, \tag{4}$$

$$\beta_t := (\mathbf{1}'D_t^{-1}d_t)/(\mathbf{1}'D_t^{-1}\mathbf{1}), \tag{5}$$

$$\eta_t := \mathrm{E}_t[\eta_{t+1}] + d_t'D_t^{-1}d_t - (\mathbf{1}'D_t^{-1}d_t)^2/(\mathbf{1}'D_t^{-1}\mathbf{1}) \tag{6}$$

$$D_t := \mathrm{E}_t[\alpha_{t+1}e_t e_t'], \tag{7}$$

$$d_t := \mathrm{E}_t[\beta_{t+1}e_t'], \tag{8}$$

where $\alpha_T = 1$, $\beta_T = 1$ and $\eta_T = 0$.

Lemma 2. Given $a > 0$, $h \in \mathbb{R}^n$ and $H \in \mathbb{S}_{++}^n$. If $H - hh' \succ 0$, then we have

$$h'H^{-1}h = 1 - \frac{1}{h'(H - hh')^{-1}h}, \tag{9}$$

$$\begin{pmatrix} a^2 & ah' \\ ah & H \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \frac{-1}{a}h' \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \frac{1}{a^2} & \mathbf{0} \\ \mathbf{0} & (H - hh')^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & \mathbf{0} \\ \frac{-1}{a}h & \mathbf{I} \end{pmatrix}. \tag{10}$$

The proof of Lemma 2 can be found in Horn and Johnson (1990). Note that the processes α_t and η_t, $t = 0, \cdots, T-1$, possess the following properties.

Lemma 3. It holds true that $\alpha_t > 0$ and $0 < \eta_t < 1$ almost surly for $t = 0, \cdots, T-1$.

Proof. (i) Under Assumption 1, since $\eta_T = 1$, $\mathrm{E}_{T-1}[e_t e_t'] \succ 0$ holds true at stage $t = T-1$, which implies $\alpha_{T-1} > 0$. Now assume that $\alpha_{k+1} > 0$. At stage k, we have $\mathrm{E}_k[\alpha_{k+1}e_k e_k'] \succ 0$, which also implies $\alpha_k > 0$.

[1] Mathematically, $\mathcal{F}_t$ can be defined as the smallest σ-algebra generated by $e_0, \cdots, e_{t-1}$.

(ii) Now we prove that $0 < \eta_t < 1$ for all $t = T-1, \cdots, 0$. Since $\alpha_t > 0$, under Assumption 1, we have $D_t \succ 0$, for all $t = 0, \cdots, T-1$. Thus, the following inequality holds for $t = T-1, \cdots, 0$,

$$\begin{aligned}(d_t' D_t^{-1} d_t)\big((d_t' D_t^{-1} d_t)(\mathbf{1}' D_t^{-1} \mathbf{1}) - (\mathbf{1}' D_t^{-1} d_t)^2\big) \\ = \Big((\mathbf{1}' D_t^{-1} d_t) d_t - (d_t' D_t^{-1} d_t)\mathbf{1}\Big) D_t^{-1} \\ \times \Big((\mathbf{1}' D_t^{-1} d_t) d_t - (d_t' D_t^{-1} d_t)\mathbf{1}\Big) > 0. \quad (11)\end{aligned}$$

At stage $t = T-1$, based on (11), we have

$$\eta_{T-1} = d_{T-1}' D_{T-1}^{-1} d_{T-1} - \beta_{T-1}^2/\alpha_{T-1} > 0. \quad (12)$$

Equality (9) implies

$$d_t' D_t^{-1} d_t = 1 - \frac{1}{\Big(1 + d_{T-1}'\Big(D_{T-1} - d_{T-1} d_{T-1}'\Big)^{-1} d_{T-1}\Big)}. \quad (13)$$

Substituting (13) to (12) yields $1 - \eta_{T-1} > (\beta_{T-1})^2/\alpha_{T-1}$, which further implies $\eta_{T-1} < 1$. Now we assume that

$$(1 - \eta_{k+1}) > (\beta_{k+1})^2/\alpha_{k+1} > 0. \quad (14)$$

Note that (14) implies that $0 < \eta_{k+1} < 1$. Together with (11), we have $\eta_k > 0$. To prove $\eta_k < 1$, we first show that the following equality is true,

$$(1 - \mathrm{E}_k[\eta_{k+1}]) D_k - d_k d_k' \succ 0. \quad (15)$$

Given any $x \in \mathbb{R}^n$, due to the induction assumption in (14) and $\alpha_{k+1} > 0$, we have

$$\alpha_{k+1} x'(e_k e_k') x + 2\beta_{k+1}(x' e_k) + (1 - \mathrm{E}_k[\eta_{k+1}]) \quad (16)$$

for any $x \in \mathbb{R}^n$. Note that the inequality in (16) holds for any realization of α_{k+1}, β_{k+1}, η_{k+1}, and e_k. Taking conditional expectation of (16) with respect to $\mathcal{F}_k$ gives rise to

$$x' D_k x + 2 d_k' x + (1 - \eta_{k+1}) > 0, \ \forall x \in \mathbb{R}^n, \quad (17)$$

which further implies,

$$\begin{bmatrix} D_k & d_k \\ d_k' & 1 - \mathrm{E}_k[\eta_{k+1}] \end{bmatrix} \succ 0. \quad (18)$$

Since $1 - \mathrm{E}_k[\eta_{k+1}] > 0$, applying the Schur's complement theory (Horn and Johnson (1990)) to (18) yields

$$D_k - d_k' d_k/(1 - \mathrm{E}_k[\eta_{k+1}]) \succ 0,$$

which completes the proof for (15). We also have

$$\begin{aligned} & 1 - \eta_k \\ &= 1 - \mathrm{E}_k[\eta_{k+1}] - d_k' D_k^{-1} d_k + \beta_k^2/\alpha_k \\ &= (1 - \mathrm{E}_k[\eta_{k+1}])\Big(1 - d_k' \left((1 - \mathrm{E}_k[\eta_{k+1}]) D_k\right)^{-1} d_k\Big) + \frac{\beta_k^2}{\alpha_k} \\ &= \Big(\frac{1 - \mathrm{E}_k[\eta_{k+1}]}{1 + d_k' \left((1 - \mathrm{E}_k[\eta_{k+1}]) D_k - d_k' d_k\right)^{-1} d_k}\Big) + \frac{\beta_k^2}{\alpha_k}. \quad (19)\end{aligned}$$

Combining (19) and (15) yields $1 - \eta_k > \beta_k^2/\alpha_k > 0$, which further implies $\eta_k < 1$. □

Due to the nonseparability of the variance term in problem $(\bar{\mathcal{P}}(\omega))$ in the sense of dynamic programming, we adopt the same idea as in Li and Ng (2000) by constructing the following auxiliary problem $\mathcal{A}(\lambda, \omega)$,

$$\mathcal{A}(\omega, \lambda): \ \min \ \mathrm{E}[\omega x_T^2 - \lambda x_T],$$
$$\text{Subject to: } \{u_t, x_t\} \text{ satisfies (1) and (2).}$$

The auxiliary problem $\mathcal{A}(\omega, \lambda)$ can be solved explicitly.

Proposition 4. The optimal policy for problem $\mathcal{A}(\omega, \lambda)$ is

$$\begin{aligned} u_t^*(\omega, \lambda) = \frac{\lambda}{2\omega} \mathrm{E}_t[\alpha_{t+1} e_t e_t']^{-1} \mathrm{E}_t[\beta_{t+1} e_t] + (\alpha_t x_t - \frac{\lambda \beta_t}{2\omega}) \\ \times \mathrm{E}_t[\alpha_{t+1} e_t e_t']^{-1} \mathbf{1}, \quad (20)\end{aligned}$$

where α_t and β_t are defined in (4) and (5), respectively. Furthermore, under the optimal policy $u_t^*(\omega, \lambda)$, we have

$$\mathrm{E}[x_T(\omega, \lambda)] = \beta_0 x_0 + \frac{\lambda \eta_0}{2\omega}, \quad (21)$$

$$\mathrm{E}[x_T^2(\omega, \lambda)] = \alpha_0 x_0^2 + \frac{\lambda^2 \eta_0}{4\omega^2}, \quad (22)$$

$$\begin{aligned}\mathrm{Var}[x_T(\omega, \lambda)] = (\alpha_0 - \beta_0^2) x_0^2 - \frac{\lambda^2}{4\omega^2}(\eta_0 - \eta_0^2) \\ - \frac{\lambda \eta_0 \beta_0 x_0}{\omega}. \quad (23)\end{aligned}$$

Proof. Define the value function of problem $\mathcal{A}(\omega, \lambda)$ as

$$J_t(x_t) \triangleq \min_{u_\tau, \tau \geq t} \mathrm{E}_t\big[\omega x_T^2 - \lambda x_T\big]. \quad (24)$$

Applying dynamic programming recursion and the smooth property of the conditional expectation to (24) yields $J_t(x_t) = \max_{u_t} \mathrm{E}_t\Big[J_{t+1}(x_{t+1})\Big]$. We claim that the value function (24) is of a quadratic form,

$$J_t(x_t) = \omega \alpha_t x_t^2 - \lambda \beta_t x_t - \frac{\lambda^2}{4\omega} \eta_t, \quad (25)$$

where α_t, β_t, η_t are defined in (4), (5) and (6), respectively. Such a claim can be proved by induction method. At stage T, since $\alpha_T = 1$, $\beta_T = 1$ and $\eta_T = 0$, we have $J_T(x_T) = \omega x_T^2 - \lambda x_T$. Assume that the claim in (25) is true at stage $t = k+1$. At stage $t = k$, the value function (24) can be written as follows by using (2),

$$\begin{aligned} J_k(x_k) &= \min_{x_k = \mathbf{1}' u_k} \mathrm{E}_t\big[\omega \alpha_{k+1} x_{k+1}^2 - \lambda \beta_{k+1} x_{k+1} - \frac{\lambda^2}{4\omega} \eta_{k+1}\big] \\ &= \min_{x_k = \mathbf{1}' u_k} \Big(\omega u_k' D_k u_k - \lambda d_k' u_k - \frac{\lambda^2}{4\omega} \mathrm{E}_k[\eta_{k+1}]\Big). \quad (26)\end{aligned}$$

To solve (26), we consider the following Lagrangian function by introducing multiplier ρ,

$$L(\rho) = \omega u_k' D_k u_k - \lambda d_k' u_k + \rho(x_k - \mathbf{1}' u_k).$$

Applying the optimality condition of $L(\rho)$ gives rise to

$$\nabla_{u_k} L(\rho) = 2\omega D_k u_k - \lambda d_k - \rho^* \mathbf{1} = 0,$$

which implies that

$$u_k^* = \frac{1}{2\omega} D_k^{-1}(\lambda d_k + \rho^* \mathbf{1}), \ \rho^* = \frac{2\omega x_k - \lambda \mathbf{1}' D_k^{-1} d_k}{\mathbf{1}' D_k^{-1} \mathbf{1}}.$$

Since $\alpha_{k+1} > 0$ by Lemma 3, $J_k(x_k)$ is a convex function with respect to u_k. We can conclude that the optimal policy is indeed given by (20). Substituting u_k^* into (26) yields

$$\begin{aligned} J_k(x_k) =& \frac{(\rho^*)^2}{4\omega} \mathbf{1}' D_k^{-1} \mathbf{1} - \frac{\lambda^2}{4\omega}(d_k' D_k^{-1} d_k + \mathrm{E}_k[\eta_{k+1}]) \\ =& \omega \alpha_k x_k^2 - \lambda \beta_k x_k - \frac{\lambda^2}{4\omega} \eta_k, \end{aligned}$$

where $\alpha_k = 1/\mathbf{1}' D_k^{-1} \mathbf{1}$, $\beta_k = (\mathbf{1}' D_k^{-1} d_k)/(\mathbf{1}' D_k^{-1} \mathbf{1})$ and $\eta_k = \mathrm{E}_k[\eta_{k+1}] + d_k' D_k^{-1} d_k - (\mathbf{1}' D_k^{-1} d_k)^2/(\mathbf{1}' D_k^{-1} \mathbf{1})$, which completes the proof the optimality of policy (20). Under policy (20), we claim that the first and second order conditional moments of x_T preserve the following form,

$$\mathrm{E}_t[x_T(\omega,\lambda)] = \beta_t x_t + \frac{\lambda\eta_t}{2\omega}, \tag{27}$$

$$\mathrm{E}_t[x_T^2(\omega,\lambda)] = \alpha_t x_t^2 + \frac{\lambda^2\eta_t}{4\omega^2}. \tag{28}$$

It is obvious that the claims in (27) and (28) hold true at stage T. We assume that such a claim is also true at time $k+1$. At time k, we have

$$\begin{aligned}\mathrm{E}_k[x_T] &= \mathrm{E}\big[\mathrm{E}[x_T|\mathcal{F}_{k+1}]|\mathcal{F}_k\big] = \mathrm{E}_k[\beta_{k+1}e_k' u_k^* + \frac{\lambda\eta_{k+1}}{2\omega}] \\ &= \mathrm{E}_k\Big[\beta_{k+1}e_k'\big(\frac{\lambda}{2\omega}D_k^{-1}d_k + (\alpha_t x_t - \frac{\lambda\beta_k}{2\omega})D_k^{-1}\mathbf{1}\big) \\ &\quad + \frac{\lambda}{2\omega}\eta_{k+1}\Big] \\ &= \beta_k x_k + \frac{\lambda\eta_k}{2\omega}.\end{aligned}$$

Similarly, the second order moment can be computed as

$$\begin{aligned}\mathrm{E}_k[x_T^2] &= \mathrm{E}\big[\mathrm{E}[x_T^2|\mathcal{F}_{k+1}]|\mathcal{F}_k\big] \\ &= \mathrm{E}_k[\alpha_{k+1}u_k^{*\prime}e_k e_k' u_k^* + \frac{\lambda^2\eta_{k+1}}{4\omega^2}] \\ &= \frac{x_k^2}{\mathbf{1}'D_k^{-1}\mathbf{1}} + \frac{\lambda^2}{4\omega}\Big(d_k'D_k^{-1}d_k - \frac{(\mathbf{1}'D_k^{-1}d_k)^2}{\mathbf{1}'D_k^{-1}\mathbf{1}} \\ &\quad + \mathrm{E}_k[\eta_{k+1}]\Big) \\ &= \alpha_k x_k^2 + \frac{\lambda^2\eta_k}{4\omega^2},\end{aligned}$$

which completes the proof. □

Proposition 5. The following policy solves problem $(\bar{\mathcal{P}}(\omega))$,

$$\begin{aligned}u_t^*(\omega) =& \frac{1+2\omega\beta_0x_0}{2\omega(1-\eta_0)}\mathrm{E}_t[\alpha_{t+1}e_te_t']^{-1}(\mathrm{E}_t[\beta_{t+1}e_t] - \beta_t\mathbf{1}) \\ &+ \alpha_t x_t \mathrm{E}_t[\alpha_{t+1}e_te_t']^{-1}\mathbf{1},\end{aligned} \tag{29}$$

with the optimal mean-variance pair of the terminal wealth x_T given by

$$\mathrm{E}[x_T] = \frac{\beta_0x_0}{1-\eta_0} + \frac{\eta_0}{2\omega(1-\eta_0)}, \tag{30}$$

$$\mathrm{Var}[x_T] = (\alpha_0 - \frac{\beta_0^2}{1-\eta_0})x_0^2 + \frac{\eta_0}{4\omega^2(1-\eta_0)}. \tag{31}$$

Furthermore, the efficient frontier can be expressed as

$$\begin{aligned}&(\mathrm{E}[x_T] - \frac{\beta_0x_0}{1-\eta_0})^2 \\ &= \frac{\eta_0}{1-\eta_0}(\mathrm{Var}[x_T] - \alpha_0x_0^2 + \frac{\beta_0^2x_0^2}{1-\eta_0}).\end{aligned} \tag{32}$$

Proof. Implied by Theorem 2 in Li and Ng (2000), the optimal policy of problem $(\bar{\mathcal{P}}(\omega))$ takes the same form as (20). We now identify λ^* such that $\pi(\mathcal{A}(\lambda^*,\omega))$ also solves problem $(\bar{\mathcal{P}}(\omega))$. From Theorem 2 in Li and Ng (2000), the optimal λ^* solves the following equation,

$$\lambda^* = 1 + 2\omega\mathrm{E}[x_T]|_{\pi^*(\mathcal{A}(\omega,\lambda^*))}. \tag{33}$$

Combining (33) and (21) yields

$$\lambda^* = 1 + 2\omega(\beta_0x_0 + \frac{\lambda^*\eta_0}{2\omega}).$$

Solving the above equation gives rise to

$$\lambda^* = \frac{2\omega\beta_0x_0+1}{1-\eta_0}. \tag{34}$$

Substituting λ^* into the optimal policy $\pi(\mathcal{A}(\omega,\lambda))$ and the expressions in (21) and (23) gives rise to the optimal policy (29) of problem $(\bar{\mathcal{P}}(\omega))$ and the mean-variance efficient pair of the terminal wealth given in (30) and (31). The efficient frontier (32) is achieved by eliminating ω from (30) and (31). □

Now we reach the final stage to solve problem $(\mathcal{P}(\epsilon))$ by utilizing the optimal policy of problem $(\bar{\mathcal{P}}(\omega))$.

Proposition 6. The optimal policy (29) of problem $\bar{\mathcal{P}}(\omega)$ solves problem $(\mathcal{P}(\epsilon))$ with

$$\omega \triangleq \frac{\eta_0}{2\big(\beta_0x_0 - (1-\eta_0)\epsilon\big)}. \tag{35}$$

Proof. We introduce Lagrangian multiplier $\omega \geq 0$ for problem $(\mathcal{P}(\epsilon))$,

$$\mathcal{L}(\epsilon,\omega): \max_{u_t} \ \omega\mathrm{Var}[x_T] + (\epsilon - \mathrm{E}[x_T])$$

$$\text{Subject to}: \{x_t,u_t\} \text{ satisfies (1) and (2).}$$

By weak duality, it is clear that $v(\mathcal{L}(\epsilon,\omega)) \geq v(\mathcal{P}(\epsilon))$. On the other hand, note that solving problem $(\mathcal{L}(\epsilon,\omega))$ is equivalent to solving problem $(\bar{\mathcal{P}}(\omega))$. Thus, policy $\pi(\bar{\mathcal{P}}(\omega))$ also solves problem $(\mathcal{L}(\epsilon,\omega))$. Under the optimal policy $\pi(\bar{\mathcal{P}}(\omega))$, the expected value and the variance, $\mathrm{E}[x_T(\omega)]$ and $\mathrm{Var}[x_T(\omega)]$, are given in (30) and (31), respectively. The strong duality, $v(\mathcal{L}(\sigma,\omega)) = v(\mathcal{P}(\epsilon))$, holds once the feasible condition $\mathrm{E}[x_T(\omega)] = \epsilon$ is satisfied. Solving

$$\frac{\beta_0x_0}{1-\eta_0} + \frac{\eta_0}{2\omega(1-\eta_0)} = \epsilon$$

gives rise to the result in (35). □

If all the returns e_t are independent among different periods, the conditional expectation degenerates to the unconditional expectation. In particular, we have $\mathrm{E}_t[\alpha_te_te_t'] = \alpha_t\mathrm{E}[e_te_t']$ and $\mathrm{E}_t[\beta_te_t] = \beta_t\mathrm{E}[e_t]$. Furthermore, the processes (4), (5) and (6) all become deterministic, for $t = T-1,\cdots,0$,

$$\begin{aligned}\alpha_t &= \frac{1}{\prod_{\tau=t}^{T-1}\mathbf{1}'\mathrm{E}[e_te_t']^{-1}\mathbf{1}}, \\ \beta_t &= \frac{\prod_{\tau=t}^{T-1}\mathbf{1}'\mathrm{E}[e_te_t']^{-1}\mathrm{E}[e_t]}{\prod_{\tau=t}^{T-1}\mathbf{1}'\mathrm{E}[e_te_t']^{-1}\mathbf{1}}, \\ \eta_t &= \eta_{t+1} + \frac{(\beta_{t+1})^2}{\alpha_{t+1}}\mathrm{E}[e_t]'\mathrm{E}[e_te_t']^{-1}\mathrm{E}[e_t] - \frac{(\beta_t)^2}{\alpha_t}.\end{aligned}$$

Substituting α_t, β_t and η_t into (29) and (32) yields the same optimal policy and efficient frontier given in Li and Ng (2000).

3.2 Solution of problem $(\mathcal{P}_f(\epsilon))$

Now we turn to consider problem $(\mathcal{P}_f(\epsilon))$, in which the risk free asset is also included in the portfolio. We first define the discount factor as $\gamma_t = \prod_{\tau=t}^{T-1} r_\tau$, for $t = 0,\cdots,T$.

In problem $(\mathcal{P}_f(\epsilon))$, we can assume that $\epsilon > x_0\gamma_0$; Otherwise, the optimal investment policy is to invest all the initial wealth in the risk-free account. To solve problem $(\mathcal{P}_f(\epsilon))$, we then define the following process recursively, for $t = T-1,\cdots,0$,

$$\rho_t = \mathrm{E}_t[\rho_{t+1}] - \mathrm{E}_t[\rho_{t+1}\mathcal{P}_t'](\mathrm{E}_t[\rho_{t+1}\mathcal{P}_t\mathcal{P}_t'])^{-1}\mathrm{E}_t[\rho_{t+1}\mathcal{P}_t], \tag{36}$$

with boundary condition $\rho_T = 1$. The process ρ_t possesses the following property,

Lemma 7. For all $t = T-1, \cdots, 0$, $0 < \rho_t < 1$ holds true almost surely.

Lemma 7 can be proved by a method similar to the one for Lemma 3. As the same as the way in solving $(\mathcal{P}(\epsilon))$, We now also consider the following problem of a weighting sum of the mean-variance pair for $(\mathcal{P}_f(\epsilon))$,

$$(\bar{\mathcal{P}}_f(\omega)): \quad \min_{u_t} \ \omega \mathrm{Var}[x_T] - \mathrm{E}[x_T]$$
$$\text{Subject to: } \{x_t,\ u_t\} \text{ satisfies (3).}$$

Proposition 8. The following optimal portfolio policy $\pi(\bar{\mathcal{P}}_f(\omega))$ solves problem $(\bar{\mathcal{P}}_f(\omega))$,

$$u_t^*(\omega) = -(r_t x_t - \prod_{\tau=0}^{t} r_\tau x_0 - \frac{1}{2\omega\gamma_{t+1}\rho_0})\mathrm{E}_t[\rho_{t+1}P_tP_t']^{-1} \times \mathrm{E}_t[\rho_{t+1}P_t], \quad (37)$$

for $t = T-1, \cdots, 0$, where ρ_t is given in (36). Furthermore, the expected value and variance of the optimal terminal wealth are expressed as

$$\mathrm{E}[x_T(\omega)] = \frac{1-\rho_0}{2\omega\rho_0} + x_0\gamma_0, \quad (38)$$

$$\mathrm{Var}[x_T(\omega)] = \frac{1-\rho_0}{4\omega^2\rho_0}, \quad (39)$$

respectively, and the efficient frontier is expressed as

$$\mathrm{Var}[x_T(\omega)] = \frac{\rho_0}{1-\rho_0}(\mathrm{E}[x_T(\omega)] - x_0\gamma_0)^2 \quad (40)$$

for $\mathrm{E}[x_T(\omega)] > x_0\gamma_0$.

Proof. Proposition 8 can be proved in the same way as the proof for Proposition 5. However, we provide an alternative proof here. We treat problem $(\bar{\mathcal{P}}_f(\omega))$ as a special case of problem $(\bar{\mathcal{P}}(\omega))$ by regarding the risk free asset as a special risky asset. Let $\hat{e}_t \in \mathbb{R}^{n+1}$ be the augmented return vector, i.e., $\hat{e}_t = (r_t, e_t')'$, for $t = 0, \cdots, T-1$. Due to the composition of $\hat{e}_t$, the processes α_t, β_t and η_t defined in (4), (5) and (6) can be expressed as

$$\alpha_t = \gamma_t^2\rho_t, \quad \beta_t = \gamma_t\rho_t, \quad \eta_t = 1-\rho_t, \quad (41)$$

where ρ_t is defined in (36) for $t = T, \cdots, 0$. Clearly, when $t = T$, the claim (41) is true. Now we assume that claim (41) holds true at time $t = k+1$. At time $t = k$, we have

$$\begin{aligned}
&\mathrm{E}_k[\alpha_{k+1}\hat{e}_k\hat{e}_k']^{-1} \\
&= \frac{1}{\gamma_{k+1}}\begin{pmatrix} r_k^2 b_k & r_k h_k' \\ r_k h_k & H_k \end{pmatrix}^{-1} \\
&= \frac{1}{\gamma_{k+1}}\begin{pmatrix} 1 & \frac{-1}{r_k b_k}h_k' \\ \mathbf{0} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \frac{1}{r_k^2 b_k} & \mathbf{0} \\ \mathbf{0} & b_k(b_k\mathbf{H}_k - h_k h_k')^{-1} \end{pmatrix} \\
&\times \begin{pmatrix} 1 & \mathbf{0} \\ \frac{-1}{r_k b_k}h_k & \mathbf{I} \end{pmatrix}, \quad (42)
\end{aligned}$$

where $b_k \triangleq \mathrm{E}_k[\rho_{k+1}]$ and $h_k \triangleq \mathrm{E}_k[\rho_{k+1}e_k]$, $H_k \triangleq \mathrm{E}_k[\rho_{k+1}e_k e_k]$. Using expression (42) leads to the following,

$$\begin{aligned}
&\begin{pmatrix} 1 & \mathbf{1}' \end{pmatrix}\mathrm{E}_k[\rho_{k+1}\hat{e}_k\hat{e}_k']^{-1}\begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} \\
&= \frac{1}{\gamma_k^2 b_k} + \frac{1}{\gamma_k^2 b_k}(h_k - b_k r_k\mathbf{1})'(b_k H_k - h_k h_k')^{-1} \\
&\quad \times (h_k - b_k r_k \mathbf{1}). \quad (43)
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{E}_k[\rho_{k+1}P_k] &= h_k - b_k r_k \mathbf{1}, \quad (44) \\
b_k H_k - h_k h_k' &= b_k \mathrm{E}_k[\rho_{k+1}P_kP_k'] \\
&\quad - \mathrm{E}_k[\rho_{k+1}P_k]\mathrm{E}_k[\rho_{k+1}P_k']. \quad (45)
\end{aligned}$$

Combining (43) with (45) and (44) and using Lemma 2 give rise to

$$\begin{aligned}
\alpha_k &= 1/\Big(\begin{pmatrix} 1 & \mathbf{1}' \end{pmatrix}\mathrm{E}_k[\rho_{k+1}\hat{e}_k\hat{e}_k']^{-1}\begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix}\Big) \\
&= \gamma_k^2(b_k - \mathrm{E}_k[\rho_{k+1}P_k']\mathrm{E}_k[\rho_{k+1}P_kP_k']^{-1}\mathrm{E}_k[\rho_{k+1}P_k]) \\
&= \gamma_k^2\rho_k.
\end{aligned}$$

By using the similar method, we can prove that $\beta_k = \gamma_k\rho_k$ and $\eta_k = 1-\rho_k$. Then substituting (41) to the optimal policy (29) and optimal mean-variance pair (30) and (31) for the augmented return $\hat{e}_t$ yields the optimal policy in (37) and the mean-variance pair in (38) and (39). □

Proposition 9. Problem $(\mathcal{P}_f(\epsilon))$ can be solved by policy $\pi(\bar{\mathcal{P}}_f(\omega^*))$ given in (37) with

$$\omega^* \triangleq \frac{1-\rho_0}{2\rho_0(\epsilon - \gamma_0 x_0)}. \quad (46)$$

Proof. Using the argument similar to the proof of Proposition 6, $(\mathcal{P}_f(\epsilon))$ is solved when $\mathrm{E}[x_T(\omega^*)] = \epsilon$ holds, i.e.,

$$\frac{1-\rho_0}{2\omega^*\rho_0} + x_0\gamma_0 = \epsilon, \quad (47)$$

based on (38). The solution to (47) is exactly the optimal ω^* given in (46). □

4. ILLUSTRATIVE EXAMPLE

We use one simple example to illustrate the computational procedure provided in Section 3 for problem $(\bar{\mathcal{P}}(\omega))$. We consider a case with two risky assets whose return process follows an AR(1) stochastic process model, i.e., $e_{t+1} = c + Ae_t + \xi$, where

$$A = \begin{pmatrix} 0.01 & -0.002 \\ -0.002 & 0.012 \end{pmatrix}, \quad c = \begin{pmatrix} 1.05 \\ 1.05 \end{pmatrix},$$

with initial value $e_{-1} = (1.07, 105)'$, and ξ being a disturbance term that takes only two values of $(0.055, -0.045)'$ and $(0.02, 0.06)'$ with probability 0.3 and 0.7, respectively. We also assume that the investment horizon is $T = 8$ and $\omega = 2$. Figure 1 depicts the tree-structure of the realization of the returns e_t for $t = 0, 1, \cdots, T$ (Only first 4 periods are plotted). The second column of Table 1 gives the details of the returns in each of these scenarios. The corresponding process α_t, η_t and η_t, for $t = T, T-1, \cdots, 0$, can be computed by (41). Note that the conditional expectation is computed according to each individual scenario (node) in the scenario tree (Figure 1). We give the details of α_t, β_t and η_t for $t = 0, 1, 2$ in the last three columns in Table 1.

From Proposition 5, by using the parameters listed in Table 1, we can compute the correspondent optimal portfolio policy for each of these scenario as follows.

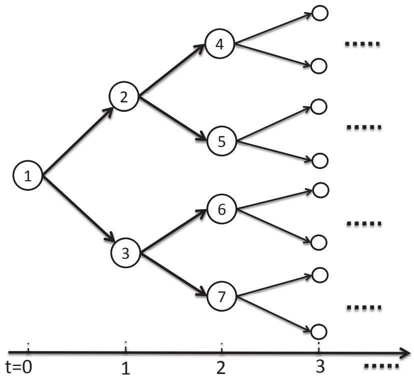

Fig. 1. The scenario tree of returns

Node	t	e_t	α_t	β_t	η_t
1	0	$(1.070, 1.050)'$	1.302	0.742	0.577
2	1	$(1.113, 1.015)'$	1.269	0.776	0.526
3	1	$(1.039, 1.120)'$	1.247	0.763	0.534
4	2	$(1.114, 1.015)'$	1.228	0.805	0.472
5	2	$(1.039, 1.120)'$	1.207	0.792	0.481
6	2	$(1.113, 1.016)'$	1.228	0.805	0.472
7	2	$(1.038, 1.121)'$	1.207	0.791	0.481

Table 1. The information of each scenario

$$\begin{aligned}
\text{node1} &: u_0(x_0) = (4.428, -3.428)'x_0 + (-5.140, 5.140)',\\
\text{node2} &: u_1(x_1) = (4.312, -3.428)'x_1 + (-5.140, 5.140)',\\
\text{node3} &: u_1(x_1) = (4.581, -3.581)'x_1 + (-5.732, 5.732)',\\
\text{node4} &: u_2(x_2) = (4.311, -3.311)'x_2 + (-5.734, 5.734)',\\
\text{node5} &: u_2(x_2) = (4.580, -3.580)'x_2 + (-6.148, 6.148)',\\
\text{node6} &: u_2(x_2) = (4.315, -3.315)'x_2 + (-5.740, 5.740)',\\
\text{node7} &: u_2(x_2) = (4.583, -3.583)'x_2 + (-6.153, 6.153)'.
\end{aligned}$$

The correspondent efficient frontier of $\{\sqrt{\text{Var}[x_T]}, \text{E}[x_T]\}$ is given as

$$(\text{E}[x_T] - 1.754)^2 = 1.364(\text{Var}[x_T] + 1.312 \times 10^{-6}).$$

5. CONCLUSION

For a market with the most general structure of correlation for returns of risky assets, we have derived analytically the optimal portfolio policy for the multiperiod mean-variance formulation. When the returns are correlated, the return history does contain valuable information for us to predict the future return. Thus, parameters in the portfolio policy become path-dependent adaptive processes themselves. It will be interesting to investigate further the prediction power hidden in our complicated optimal portfolio policy. By calibrating real market data into our model formulation, we could verify the validity and gain more insights from applying our derived policy.

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