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Robust Model Predictive Controller Design ⋆

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Abstract: This paper addresses to the problem of designing, modelling and practical realization of robust model predictive control which ensures a parameter dependent quadratic stability and guaranteed cost for linear polytopic systems. The design procedure based on BMI and LMI is reduced to off-line output feedback gain calculation. A numerical example and an application to a real process is given to illustrate the effectiveness of the proposed method.

Keywords: MPC, Robust stability, Polytopic model, LMI, Output feedback

1. INTRODUCTION

Model predictive control (MPC) is a modern controller which is widely used in industrial applications. The main idea of MPC is to predict the future behaviour of the system over a finite receding horizon and to find the optimal value of the system input with respect to predictions and given constraints. There exist several different realizations of MPC. Extensive overview of MPC algorithms can be found in Maciejowski [2002], Rossiter [2003].

The drawbacks of the most common MPC algorithms are the on-line computational complexity and the feasibility problem which in the past limited the use of MPC to plants with a slow time response because it solves optimization problem in each sample time. Another drawback when the plant model is uncertain or unstable is the lack of guaranteed stability and robustness of the closed-loop system that can cause a poor performance of the system. Numerous design procedures were developed to guarantee robust stability. They are based on the infinite time horizon with the on-line LMI optimization [Kothare et al., 1996], the min-max optimization with a terminal constraint in an invariant set [Scokaert and Mayne, 1998, Lofberg, 2003], the off-line LMI optimization to calculate a sequence of output feedback laws and the on-line selection of the appropriate law [Ding et al., 2008] or the robust tube-based control design approach [Mayne et al., 2005]. The problems of the computational complexity and the robust stability are solved in the robust explicit MPC [Kerrigan and Maciejowski, 2004].An excellent survey of robust MPC design can be found in [Bemporad and Morari, 1999, Mayne et al., 2000, Rakovic, 2012]. Alternatively, in many implementations the MPC is applied without input and output constraints [Camacho and Bordons, 1999]. Then the algorithm reduces to linear feedback controller. But the design procedure still does not guarantee the stability or the robust stability and the closed-loop stability can be checked only by simulation. Recently the new MPC algorithm which guarantees the robustness of the closed-loop system have been presented in Nguyen et al. [2013], Veselý and Rosinová [2010], Veselý et al. [2010].

In this paper we pursue the ideas of Veselý et al. [2010] where the robust MPC algorithm has been designed on the base of the polytopic system quadratic stability and the input constraints, but the implementation problem in the real plants has been left out. However, the quadratic stability generally provides quite conservative results. To find a less conservative approach in this paper the parameter dependent quadratic stability(PDQS) [de Oliveira et al., 1999, Peaucelle et al., 2000] is used. The controller feedback gains calculation is formulated in the form of bilinear matrix inequalities (BMI). Because currently available BMI solvers do not allow to work with high order systems the problem of calculating an MPC output feedback gain needs to be transformed from BMI to LMI with a linearisation approach. The designed controller solves the main drawbacks of the standard MPC because it guarantees the robust stability and requires only a simple on-line calculation based on the output feedback and the state estimation for a prediction model. The problem of MPC control algorithm design using PDQS and its practical implementation is the main result of the paper.

The main principle of the controller design is described in the section 2. In the section 3 an MPC design procedure without constraints is derived. The last section contains a numerical example and a practical implementation to a real unstable magnetic levitation system with results from experiments. The following notational conventions will be adopted: The inequality $P > 0$ ($P < 0$) denotes the positive (negative) definiteness of the matrix, I denotes the identity matrix with a corresponding dimension and the notation $y(k+h|k)$ is for simplicity reasons replaced by $y(k+h)$ and denotes a h step ahead prediction of y calculated in the sample time k.

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2. PRELIMINARIES AND PROBLEM FORMULATION

Let the polytopic linear discrete time system be described by

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\xi)\tilde{x}(k) + \tilde{B}(\xi)u(k) \\ \tilde{y}(k) &= \tilde{C}x(k) \end{aligned} \tag{1}$$

where $\tilde{x}(k) \in R^n$, $u(k) \in R^m$, $\tilde{y}(k) \in R^l$ are state, control input and output variables of the system respectively. The matrices $\tilde{A}(\xi)$ and $\tilde{B}(\xi)$ belong to the convex set S, with M vertices $S_1, \ldots, S_M$ that can be formally defined as:

$$\begin{aligned} S :&\Big\{ \tilde{A}(\xi), \tilde{B}(\xi) : \left(\tilde{A}(\xi), \tilde{B}(\xi)\right) = \sum_{i=1}^{M} \xi_i \left(\tilde{A}_i, \tilde{B}_i\right), \\ &\sum_{i=1}^{M} \xi_i = 1,\ \xi_i \geq 0 \Big\} \end{aligned} \tag{2}$$

Matrices $\tilde{A}_i$, $\tilde{B}_i$ and $\tilde{C}$ are known with constant coefficients of corresponding dimensions. Let consider the following integrator to force disturbance rejection and to achieve set-point tracking

$$z(k+1) = z(k) - \tilde{C}\tilde{x}(k) + w(k) \tag{3}$$

where $w(k)$ is a desired set-point value. Adding the integrator (3) to (1) one obtains:

$$\begin{aligned} x(k+1) &= A(\xi)x(k) + B(\xi)u(k) + B_w w(k) \\ y(k) &= Cx(k) \end{aligned} \tag{4}$$

where

$$\begin{aligned} &x(k) = \begin{bmatrix} \tilde{x}(k) \\ z(k) \end{bmatrix}, \quad A(\xi) = \begin{bmatrix} \tilde{A}(\xi) & 0 \\ -\tilde{C} & I \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ I \end{bmatrix}, \\ &C = \begin{bmatrix} \tilde{C} & 0 \\ 0 & I \end{bmatrix}, \quad B(\xi) = \begin{bmatrix} \tilde{B}(\xi) \\ 0 \end{bmatrix}, \quad y(k) = \begin{bmatrix} \tilde{y}(k) \\ z(k) \end{bmatrix} \end{aligned} \tag{5}$$

$\tilde{y}(k)$ is a plant output for the proportional part of the controller and $z(k)$ is a plant output for the integral part of the controller. Optionally, the derivative part of the controller can be added in the form of the first difference:

$$y_d(k) = \tilde{y}(k-1) - \tilde{y}(k) = \tilde{C}\tilde{x}(k-1) - \tilde{C}\tilde{x}(k) \tag{6}$$

Then the system (4) is augmented as follows:

$$\begin{aligned} &x(k) = \begin{bmatrix} \tilde{x}(k) \\ z(k) \\ z_d(k) \end{bmatrix}, \ A(\xi) = \begin{bmatrix} \tilde{A}(\xi) & 0 & 0 \\ -\tilde{C} & I & 0 \\ I & 0 & 0 \end{bmatrix}, \ B_w = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, \\ &C = \begin{bmatrix} \tilde{C} & 0 & 0 \\ 0 & I & 0 \\ -\tilde{C} & 0 & \tilde{C} \end{bmatrix}, \ B(\xi) = \begin{bmatrix} \tilde{B}(\xi) \\ 0 \\ 0 \end{bmatrix}, \ y(k) = \begin{bmatrix} \tilde{y}(k) \\ z(k) \\ y_d(k) \end{bmatrix} \end{aligned} \tag{7}$$

The selection of the system (4) is based on (5) or (7) depending on the desired controller structure. Simultaneously with (4) we consider the nominal plant of the system in the form:

$$\begin{aligned} x(k+1) &= A_0 x(k) + B_0 u(k) + B_w w(k) \\ y(k) &= Cx(k) \end{aligned} \tag{8}$$

where $A_0 = \sum_{i=1}^{M} A_i \xi_{i0}$ and $B_0 = \sum_{i=1}^{M} B_i \xi_{i0}$ for any values of ξ_{i0} which satisfy $\sum_{i=1}^{M} \xi_{i0} = 1$, $\xi_{i0} \geq 0$.

Remarks:

(1) If the MPC design is based on the quadratic stability, the rate of change of coefficients ξ_i may reach any value which satisfies (2).

(2) If the MPC design procedure is based on the parameter dependent quadratic stability, ξ_i are constant and unknown.

(3) In the references [Veselý et al., 2010, Nguyen et al., 2013, Veselý and Rosinová, 2010], for the model predictive control design procedure there are two main ideas presented. In the first [Veselý et al., 2010] ξ_{i0}, $i = 1, \ldots, M$ are known and constant. In the last two references coefficients ξ_{i0} (for the realization of the prediction model) are known but may be changed.

(4) For the next we assume that the system model (4) and the nominal model (8) satisfy to remark 2.

The nominal model (8) will be used for the construction of the prediction model and (4) is considered as a real plant description providing the plant output. Therefore, we assume that for time k, output $y(k)$ is obtained from the uncertain plant model (4), and predicted outputs for time $k+1, \ldots, k+N_y$ will be obtained from the model prediction, when the nominal model (8) is used. The prediction is carried out over a finite output horizon N_y and a control horizon N_u ($N_u \leq N_y$). Note that we consider $F_{ij} = 0$ for $i = N_u+1, \ldots, N_y$. Therefore for the next development we consider $N_u = N_y$.

Consider the predictive control algorithm with output feedback as follows:

$$\begin{aligned} u(k+i) &= \sum_{j=0}^{N_y} \bar{F}_{ij} \left(\tilde{y}(k+j) - w(k+j)\right) + \\ &+ E_{ij} z(k+j) = \sum_{j=0}^{N_y} F_{ij} y(k+j) - \bar{F}_{ij} w(k+j) \\ i &= 0, 1, \ldots, N_y \end{aligned} \tag{9}$$

where $F_{ij} = \left[\bar{F}_{ij}\ E_{ij}\right]$, $\bar{F}_{ij} \in R^{m \times l}$, $E_{ij} \in R^{m \times l}$ are proportional and integral controller gain matrices.

States and outputs of the system for time instant k are obtained from real system measurements:

$$x(k+1) = A(\xi)x(k) + B(\xi)u(k) + B_w w(k) \tag{10}$$

$$y(k) = Cx(k) \tag{11}$$

for time instants $k+h$, $h = 1, \ldots, N_y$ the nominal model is used:

- $h = 1$
$$\begin{aligned} x(k+2) &= A_0 x(k+1) + B_0 u(k+1) \\ &\quad + B_w w(k+1) \\ &= A_0 A(\xi)x(k) + A_0 B(\xi)u(k) + B_0 u(k+1) \\ &\quad + A_0 B_w w(k) + B_w w(k+1) \\ y(k+1) &= Cx(k+1) \end{aligned}$$
- $\vdots$
- h
$$\begin{aligned} x(k+h+1) &= A_0^h A(\xi)x(k) + \sum_{i=1}^{h} A_0^{h-i} B_0 u(k+i) + \\ &+ A_0^h B(\xi)u(k) + \sum_{i=0}^{h} A_0^{h-i} B_w w(k+i) \\ y(k+h) &= Cx(k+h) \end{aligned}$$

For $h = 1, \ldots, N_y$ in the matrix form one obtains:

$$\begin{aligned} x_f(k+1) &= A_f x(k) + B_f v(k) + B_{wf} w_f(k) \\ y_f(k) &= C_f x_f(k) \end{aligned} \tag{12}$$

where

$$x_f(k) = \begin{bmatrix} x(k) \\ \vdots \\ x(k+N_y) \end{bmatrix}, \quad w_f(k) = \begin{bmatrix} w(k) \\ \vdots \\ w(k+N_y) \end{bmatrix}, \quad v(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k+N_y) \end{bmatrix}, \quad y_f(k) = \begin{bmatrix} y(k) \\ \vdots \\ y(k+N_y) \end{bmatrix} \tag{13}$$

$$A_f = \begin{bmatrix} A(\xi) \\ A_0 A(\xi) \\ \vdots \\ A_0^{N_y} A(\xi) \end{bmatrix}, \ C_f = \begin{bmatrix} C & 0 & \dots & 0 \\ 0 & C & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C \end{bmatrix},$$
$$B_f = \begin{bmatrix} B(\xi) & 0 & \dots & 0 \\ A_0 B(\xi) & B_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_0^{N_y} B(\xi) & A_0^{N_y-1} B_0 & \dots & B_0 \end{bmatrix}, \tag{14}$$
$$B_{wf} = \begin{bmatrix} B_w & 0 & \dots & 0 \\ A_0 B_w & B_w & \dots & 0 \\ A_0^{N_y} B_w & A_0^{N_y-1} B_w & \dots & B_w \end{bmatrix}$$

Matrices (14) in the system with prediction (12) are used only for calculation of robust controller gains (matrix A_f is augmented with zeros to square matrix). In the practical implementation with respect to remark 4 matrices $A(\xi)$ and $B(\xi)$ are replaced with A_0 and B_0 when ξ_{i0} is constant. Then for the model prediction we obtain matrices A_{fp} and B_{fp} in the form:

$$A_{fp} = \begin{bmatrix} A(\xi_0) \\ A_0^2 \\ \vdots \\ A_0^{N_y+1} \end{bmatrix} B_{fp} = \begin{bmatrix} B(\xi_0) & 0 & \dots & 0 \\ A_0 B_0 & B_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_0^{N_y} B_0 & A_0^{N_y-1} B_0 & \dots & B_0 \end{bmatrix}$$

If ξ_{i0} is known and can change value then all matrices A_0 and B_0 in (8) should be recalculated in every sample time.

Control algorithm (9) can be rewritten to matrix form:

$$v(k) = F y_f(k) - \bar{F} w_f(k) \tag{15}$$

$$F = \begin{bmatrix} F_{00} & F_{01} & \dots & F_{0N_y} \\ F_{10} & F_{11} & \dots & F_{1N_y} \\ \vdots & \vdots & \dots & \vdots \\ F_{N_y0} & F_{N_y1} & \dots & F_{N_yN_y} \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} \bar{F}_{00} & \bar{F}_{01} & \dots & \bar{F}_{0N_y} \\ \bar{F}_{10} & \bar{F}_{11} & \dots & \bar{F}_{1N_y} \\ \vdots & \vdots & \dots & \vdots \\ \bar{F}_{N_y0} & \bar{F}_{N_y1} & \dots & \bar{F}_{N_yN_y} \end{bmatrix} \tag{16}$$

Matrices F_{ij}, $i, j = 0, 1, \dots, N_y$ are output feedback gains with constant entries to be determined by minimizing the cost function as follows:

$$J = \sum_{k=0}^{\infty} \tilde{J}(k) \tag{17}$$

$$\begin{aligned} \tilde{J}(k) &= \sum_{j=0}^{N_y} x^T(k+j) q_j x(k+j) + \sum_{j=0}^{N_u} u^T(k+j) r_j u(k+j) \\ &= x_f^T(k) Q x_f(k) + v^T(k) R v(k) \end{aligned} \tag{18}$$

$q_j \in R^{n\times n}$, $r_j \in R^{m\times m}$, Q and R are positive definite matrices.

The following lemmas will be used in the next development.

Lemma 1. Suppose there exists a Lyapunov function $V(x_f(k), \xi)$ such that for the first difference of a Lyapunov function $\Delta V(x_f(k), \xi) = V(x_f(k+1), \xi) - V(x_f(k), \xi)$ the solution of (12) with control algorithm (9) the following inequality holds

$$\Delta V(x_f(k), \xi) < 0 \tag{19}$$

then the closed-loop system is robustly stable.

Lemma 2. Consider the system (4), if there exists a control law (9) and a Lyapunov function $V(x_f(k), \xi) = x_f^T(k) P(\xi) x_f(k)$ such that the closed-loop system is robustly stable and for the first difference of a Lyapunov function on the closed-loop system solution holds:

$$\Delta V(x_f(k), \xi) + \tilde{J}(k) \le 0 \tag{20}$$

then the control law (9) is said to be the guaranteed cost control law with $J \le J^*$.

Proof: Summing (20) from $k = k_0$ to $k = \infty$ the upper bound on J is obtained

$$J \le V(x_f(k_0), \xi) = J^* \tag{21}$$

The design procedure is given as follows:

$$\min_F J \tag{22}$$

with constraints:

(1) System model (12) with given N_y and N_u.
(2) Robust stability constraint and performance (20).

Note that in the above design procedure the optimisation results are the gain matrices F_{ij}, $i, j = 0, 1, \dots, N_y$ which can be calculated off-line using the above design procedure which reduces to the standard robust controller design procedure. From Peaucelle et al. [2000] the following lemma is obtained:

Lemma 3. Assume that the uncertain system with control algorithm is given by (12) and (15). The closed-loop system is parameter dependent quadratic stable if there exists a parameter dependent Lyapunov matrix $P(\xi) = \sum_{i=1}^{M} P_i \xi_i > 0$, auxiliary matrices H and G such that the following inequality holds:

$$\begin{aligned} &\begin{bmatrix} U_1 & -H + A_{ci}^T G \\ -H^T + G^T A_{ci} & P_i - (G + G^T) \end{bmatrix} < 0 \\ &U_1 = -P_i + A_{ci}^T H^T + H A_{ci} + Q + C_f^T F^T R F C_f \\ &A_{ci} = A_{fi} + B_{fi} F C_f, \quad i = 1, 2, \dots, M \end{aligned} \tag{23}$$

Because of BMI the following approach transforms above inequality to LMI. On the base of Veselý et al. [2011] the two step LMI desing procedure is obtained:

(1) Solve the following matrix equations:

$$\begin{aligned} & A_{ci}^T P_i A_{ci} - P_i + Q + C_f^T F^T R F C_f < 0 \\ & - P_i + A_{ci}^T H^T + H A_{ci} + Q + C_f^T F^T R F C_f < 0 \\ & A_{ci} = A_{fi} + B_{fi} F C_f, \quad i = 1, 2, \dots, M \end{aligned} \tag{24}$$

where Q and R are weighting matrices in the cost function (17). Positive definite matrices P_i and matrices F, H represent unknown variables. Equations (24) are BMI. In order to transform (24) to linear matrix inequalities at first a Shur complement is used:

$$\begin{bmatrix} -P_i + Q & C_f^T F^T & (A_{fi} + B_{fi} F C_f)^T \\ F C_f & -R^{-1} & 0 \\ A_{fi} + B_{fi} F C_f & 0 & -P_i^{-1} \end{bmatrix} < 0$$

$$\begin{bmatrix} U_2 & C_f^T F^T & (H^T + B_{fi} F C_f)^T \\ F C_f & -R^{-1} & 0 \\ H^T + B_{fi} F C_f & 0 & -I \end{bmatrix} < 0$$

$$\begin{aligned} U_2 = & -P_i + A_{fi}^T H^T + H A_{fi} - H H^T - \\ & - C_f^T F^T B_{fi}^T B_{fi} F C_f + Q \end{aligned}$$

Then a linearisation approach described in [Han and Skelton, 2003] transforms nonlinear diagonal terms to LMI by finding its upper bounds. Where for a given matrix W the linearisation is:

$$lin(X^{-1}, X_k) = X_k^{-1} - X_k^{-1}(X - X_k)X_k^{-1} \tag{25}$$

$$\begin{aligned} lin(XWX, X_k) = & X_k W X_k - X W X_k - \\ & - X_k W X \end{aligned} \tag{26}$$

This linearisation approach requires several iterations to find a solution. At first an initial value for X_k is selected. In the next iteration X_k is equal to the calculated value of X from the previous iteration.

(2) Check the robust stability. The LMI condition of parameter dependent quadratic stability [Peaucelle et al., 2000] is (23).

The quality of the above controller design method was experimentally proved in [Vozák and Veselý, 2013].

In order to find F which is suitable for the application on real processes with respect to noise suppression and maximal values of manipulated variables it is possible to use additional constraints. One of the possibilities is to constraint elements of $F = \{f_{ij}\}$:

(1) $c_{1min} \leq f_{ij} \leq c_{1max}$ for $j = 1, 4, 7, \dots$, $\forall i$ (proportional part of the controller)
(2) $c_{2min} \leq f_{ij} \leq c_{2max}$ for $j = 2, 5, 8, \dots$, $\forall i$ (integral part of the controller)
(3) $c_{3min} \leq f_{ij} \leq c_{3max}$ for $j = 3, 6, 9, \dots$, $\forall i$ (derivative part of the controller)

For the next development assume that the gain matrices F_{ij} are calculated off-line. The main problem is modelling and practical implementation of control algorithm (9) on the base of plant measurable output vectors $y(k)$ and states $x(k)$. Note that designed gain matrices F_{ij} guarantee closed-loop stability, robustness and ensure guaranteed cost.

3. PRACTICAL IMPLEMENTATION

In this section we construct for a given N_y and N_u the final control algorithm for the case of robustly stable predictive control without input and output constraints.

Multiplying (10) from the left side by C_f and substituting (15) to (10)

$$\begin{aligned} y_f(k) = & C_f A_f x(k) + C_f B_f F y_f(k) - C_f B_f \bar{F} w_f(k) \\ & + C_f B_{wf} w_f(k) \end{aligned} \tag{27}$$

Let matrix $C_f B_f F$ be partitioned as follows:

$$C_f B_f F = \begin{bmatrix} M_1 & \overline{M}_2 \end{bmatrix} \tag{28}$$

where M_1 is the first block column of matrix $C_f B_f F$.

$$M_1 = \begin{bmatrix} C B_0 F_{00} \\ C A_0 B_0 F_{00} \\ \vdots \\ C A_0^{N_y} B_0 F_{00} \end{bmatrix} \tag{29}$$

and $\overline{M}_2$ is the other part of matrix $C_f B_f F$. From (28) and (27) one obtains:

$$\begin{aligned} C_f B_f F y_f(k) = & M_1 y(k) + \begin{bmatrix} \overline{M}_2 & 0 \end{bmatrix} y_f(k+1) \\ = & M_1 y(k) + M_2 y_f(k+1) \end{aligned} \tag{30}$$

Using (30) and (27):

$$\begin{aligned} [I - M_2] y_f(k+1) = & C_f A_f x(k) + M_1 y_f(k) \\ & - C_f \left(B_f \bar{F} - B_{wf} \right) w_f(k) \end{aligned} \tag{31}$$

From (9) rewrite the control algorithm:

$$\begin{aligned} u(k) = & F_{00} y(k) + \begin{bmatrix} F_{01} & \dots & F_{0N_y} & 0 \end{bmatrix} y_f(k+1) - \\ & - \begin{bmatrix} \bar{F}_{00} & \dots & \bar{F}_{0N_y} \end{bmatrix} w_f(k) \end{aligned} \tag{32}$$

Substituting (31) to (32) and after some manipulation:

$$u(k) = D_1 x(k) + (F_{00} + D_2)\, y(k) - \left(D_3 + \bar{F}_0 \right) w_f(k) \tag{33}$$

$$D_1 = \begin{bmatrix} F_{01} & \dots & F_{N_y} & 0 \end{bmatrix} [I - M_2]^{-1} C_f A_f \tag{34}$$

$$D_2 = \begin{bmatrix} F_{01} & \dots & F_{N_y} & 0 \end{bmatrix} [I - M_2]^{-1} M_1 \tag{35}$$

$$D_3 = \begin{bmatrix} F_{01} & \dots & F_{N_y} & 0 \end{bmatrix} [I - M_2]^{-1} C_f (B_f \bar{F} - B_{wf}) \tag{36}$$

$$\bar{F}_0 = \begin{bmatrix} \bar{F}_{00} & \dots & \bar{F}_{0N_y} \end{bmatrix} \tag{37}$$

Matrices D_1, D_2, D_3 for the case of $\xi_{i0} = const.$, $i = 1, 2, \dots, M$ are constant too and calculated off-line but for the case when ξ_{i0} changes, matrices D_1, D_2, D_3 should be recalculated in every sample time. Because gain matrices F and $\bar{F}$ are constant and guarantee the closed-loop robust stability the calculation of D matrices is straightforward.

4. EXPERIMENTAL RESULTS

In order to prove applicability of the MPC algorithm two examples are presented. The first is a simulation of randomly generated system. In the second example the algorithm was implemented to control an unstable magnetic levitation system.

4.1 Random system

The system model is generated as a random discrete time model in the affine form:

$$\begin{aligned} \tilde{A}(q) = \tilde{A}_{q0} + q_1 \tilde{A}_{q1} + q_2 \tilde{A}_{q2} \\ \tilde{B}(q) = \tilde{B}_{q0} + q_1 \tilde{B}_{q1} + q_2 \tilde{B}_{q2} \end{aligned} \tag{38}$$

where $q_1, q_2 \in \langle -1, 1 \rangle$.

$$\begin{aligned} & \tilde{A}_{q0} = \begin{bmatrix} 0.7060 & 0.2769 \\ 0.0318 & 0.0462 \end{bmatrix}, \quad \tilde{A}_{q1} = \begin{bmatrix} 0.1265 & 0.0557 \\ 0.0195 & 0.1094 \end{bmatrix} \\ & \tilde{A}_{q2} = \begin{bmatrix} 0.0656 & 0.0849 \\ 0.0036 & 0.0934 \end{bmatrix} \end{aligned} \tag{39}$$

$$\tilde{B}_{q0} = \begin{bmatrix} 0.6787 \\ 0.7577 \end{bmatrix} \tilde{B}_{q1} = \begin{bmatrix} 0.7431 \\ 0.3922 \end{bmatrix} \tilde{B}_{q2} = \begin{bmatrix} 0.6555 \\ 0.1712 \end{bmatrix} \quad (40)$$

From (38) four vertices of polytopic model (1) are obtained:

$$\begin{aligned} \tilde{A}_1 &= \tilde{A}_{q0} + \tilde{A}_{q1} + \tilde{A}_{q2}, & \tilde{B}_1 &= \tilde{B}_{q0} + \tilde{B}_{q1} + \tilde{B}_{q2} \\ \tilde{A}_2 &= \tilde{A}_{q0} - \tilde{A}_{q1} + \tilde{A}_{q2}, & \tilde{B}_2 &= \tilde{B}_{q0} - \tilde{B}_{q1} + \tilde{B}_{q2} \\ \tilde{A}_3 &= \tilde{A}_{q0} + \tilde{A}_{q1} - \tilde{A}_{q2}, & \tilde{B}_3 &= \tilde{B}_{q0} + \tilde{B}_{q1} - \tilde{B}_{q2} \\ \tilde{A}_4 &= \tilde{A}_{q0} - \tilde{A}_{q1} - \tilde{A}_{q2}, & \tilde{B}_4 &= \tilde{B}_{q0} - \tilde{B}_{q1} - \tilde{B}_{q2} \end{aligned} \quad (41)$$

Calculated output feedback matrix F with $Q = I$, $R = I$, $N_y = N_u = 7$ and system model structure (5) is:

$$F = \left[\begin{matrix} -0.5525 & 0.3945 & -0.0141 & 0.0120 & -0.0057 \\ -0.6459 & 0.7151 & 0.0147 & 0.0070 & 0.0064 \\ -0.2396 & 0.2647 & 0.0024 & -0.0074 & 0.0011 \\ -0.1947 & 0.2148 & 0.0006 & -0.0104 & 0.0004 \\ -0.1964 & 0.2164 & 0.0004 & -0.0106 & 0.0003 \\ -0.2069 & 0.2279 & 0.0006 & -0.0104 & 0.0004 \\ -0.2076 & 0.2279 & 0.0005 & -0.0102 & 0.0003 \\ -0.0580 & 0.0589 & -0.0016 & -0.0051 & -0.0006 \end{matrix}\right.$$

$$\begin{matrix} 0.0272 & -0.0028 & 0.0342 & -0.0009 & 0.0384 & 0.0007 \\ -0.0079 & 0.0045 & -0.0145 & 0.0038 & -0.0192 & 0.0032 \\ -0.0101 & 0.0005 & -0.0117 & -0.0000 & -0.0127 & -0.0005 \\ -0.0114 & -0.0001 & -0.0123 & -0.0006 & -0.0128 & -0.0010 \\ -0.0115 & -0.0001 & -0.0123 & -0.0006 & -0.0128 & -0.0011 \\ -0.0114 & -0.0001 & -0.0123 & -0.0006 & -0.0128 & -0.0010 \\ -0.0111 & -0.0001 & -0.0119 & -0.0006 & -0.0124 & -0.0010 \\ -0.0034 & -0.0004 & -0.0029 & -0.0003 & -0.0025 & -0.0003 \end{matrix}$$

$$\left.\begin{matrix} 0.0410 & 0.0025 & 0.0426 & 0.0070 & 0.0431 \\ -0.0234 & 0.0023 & -0.0273 & -0.0019 & -0.0309 \\ -0.0133 & -0.0009 & -0.0137 & -0.0018 & -0.0137 \\ -0.0129 & -0.0015 & -0.0127 & -0.0019 & -0.0122 \\ -0.0129 & -0.0015 & -0.0126 & -0.0019 & -0.0121 \\ -0.0130 & -0.0014 & -0.0128 & -0.0019 & -0.0123 \\ -0.0125 & -0.0014 & -0.0123 & -0.0017 & -0.0119 \\ -0.0022 & -0.0003 & -0.0020 & 0.0002 & -0.0017 \end{matrix}\right]$$

Matrix $\bar{F}$ is a proportional gain of the controller defined in (9). Then matrices (34)-(37) are:

$$\begin{aligned} D_1 &= [-0.0181 \;\; -0.2541 \;\; 0.2384] \\ D_2 &= [0.2942 \;\; -0.2815] \\ D_3 &= [0.0558 \;\; -0.2270 \;\; -0.2003 \;\; -0.1666 \\ &\quad -0.1285 \;\; -0.0874 \;\; -0.0444 \;\; -0.0005] \\ F_{00} &= [-0.5525 \;\; 0.3945] \\ \bar{F}_0 &= [-0.5525 \;\; -0.0141 \;\; -0.0057 \;\; -0.0028 \\ &\quad -0.0009 \;\; 0.0007 \;\; 0.0025 \;\; 0.0070] \end{aligned} \quad (42)$$

Simulated system output and input is in fig. 1.

4.2 Magnetic levitation

The main principle of magnetic levitation model CE152 made by HUMUSOFT [HUMUSOFT, 1996] is to control the position of a steel ball between a coil and an induction sensor. The process was identified in three working points (position $y = 0.3,\ 0.5,\ 0.75[mu]$) and a discrete-time polytopic model in the state-space with the sampling period $T_s = 0.001$ was created:

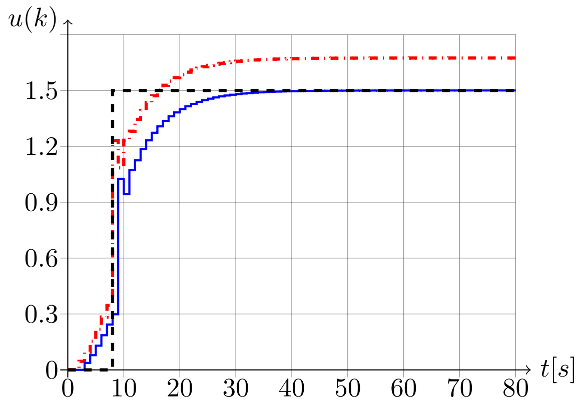

Fig. 1. Simulated system input (dash-dotted line), output (solid line) and set-point (dashed line).

$$\begin{aligned} \tilde{A}_1 &= \begin{bmatrix} 2.006 & 1 \\ -1.003 & 0 \end{bmatrix} \tilde{B}_1 = \begin{bmatrix} 0.00341 \\ 0.00341 \end{bmatrix} \\ \tilde{A}_2 &= \begin{bmatrix} 2.009 & 1 \\ -1.005 & 0 \end{bmatrix} \tilde{B}_2 = \begin{bmatrix} 0.0052 \\ 0.00521 \end{bmatrix} \\ \tilde{A}_3 &= \begin{bmatrix} 2.0 & 1 \\ -0.995 & 0 \end{bmatrix} \tilde{B}_3 = \begin{bmatrix} 0.00508 \\ 0.00508 \end{bmatrix} \\ \tilde{C} &= [1 \;\; 0] \end{aligned} \quad (43)$$

The matrix F calculated with $Q = 1$, $R = 85000$, $N_y = N_u = 3$, $c_{1max} = -c_{1min} = 5$, $c_{2max} = -c_{2min} = 5$ and $c_{3max} = -c_{3min} = 0.03$ is:

$$F = \left[\begin{matrix} -2.2557 & 4.5427 & 0.002 & 3.5318 & -2.3153 & 0.0009 \\ -0.2055 & -0.1212 & -0.000 & -0.0411 & 0.0496 & -0.000 \\ -0.0779 & -0.0788 & -0.000 & -0.0277 & 0.0265 & -0.000 \\ 0.0017 & -0.0240 & -0.000 & -0.0088 & 0.0054 & -0.000 \end{matrix}\right.$$

$$\left.\begin{matrix} -4.1428 & -3.9659 & 0.0008 & -4.6655 & 1.7631 & 0.0008 \\ 0.1021 & 0.0849 & -0.0000 & 0.2200 & -0.0213 & -0.0000 \\ 0.0545 & 0.0520 & -0.0000 & 0.1268 & -0.0040 & -0.0000 \\ 0.0111 & 0.0144 & -0.0000 & 0.0313 & 0.0034 & -0.0000 \end{matrix}\right]$$

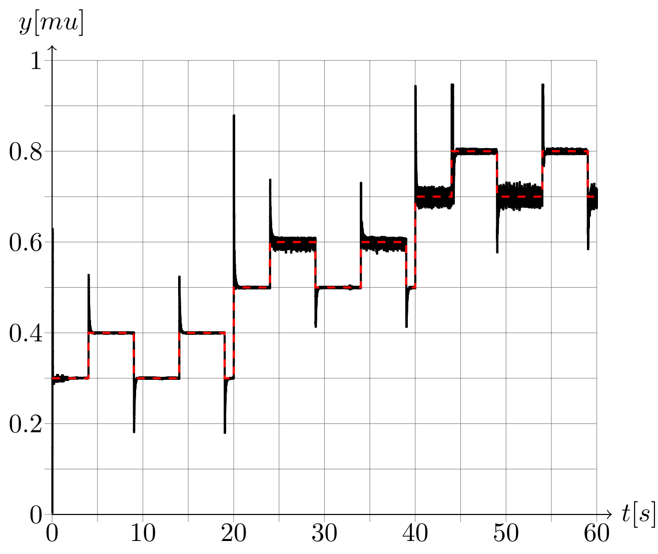

Fig. 2. Measured system output (solid line) and set-point (dashed line)

The equation of the observer is written as:

$$\hat{x}(k+1) = \tilde{A}_0\hat{x}(k) + \tilde{B}_0 u(k) + L\tilde{C}\big(x(k) - \hat{x}(k)\big) \quad (44)$$

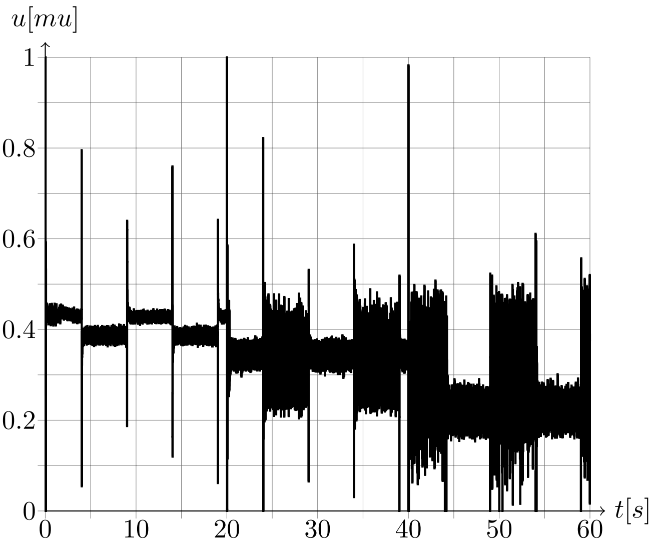

Fig. 3. Measured system input

Observer gain matrix L was computed by solving the dual problem to the discrete-time LQ with weighting matrices $Q_L = 10I$ and $R_L = I$ of appropriate dimensions.

$$L = \begin{bmatrix} 1.8542 \\ -0.9635 \end{bmatrix} \tag{45}$$

System output and input measured on real process experiments are in figures 2 and 3. The figures show that

- the system is robustly stable in all working points,
- the output has zero steady state error.

5. CONCLUSION

A new predictive control algorithm guarantees the robust stability of the closed-loop and the guaranteed cost control. Due to off-line calculation of gain matrices its practical implementation is straightforward and with only simple on-line calculations required. In the case without input and output constraints it obviously reduces to linear feedback controller with constant gain matrices. For this approach a control algorithm was derived and practically implemented to control unstable magnetic levitation system. Obtained results prove good performance of the controller.

REFERENCES

A. Bemporad and M. Morari. Robust Model Predictive Control: A Survey. *Robustness in Identification and Control*, 245:207–226, 1999.

E.F. Camacho and C.A. Bordons. *Model Predictive Control*. Springer-Verlag GmbH, 1999. ISBN 3540762418.

M.C. de Oliveira, J. Bernussou, and J.C. Geromel. A new discrete-time robust stability condition. *Systems & Control Letters*, 37(4):261 – 265, 1999. ISSN 0167-6911.

Baocang Ding, Yugeng Xi, Marcin T. Cychowski, and Thomas OMahony. A synthesis approach for output feedback robust constrained model predictive control. *Automatica*, 44(1):258 – 264, 2008. ISSN 0005-1098.

Jeongheon Han and R.E. Skelton. An lmi optimization approach for structured linear controllers. In *Decision and Control, 2003. Proceedings. 42nd IEEE Conference*, volume 5, pages 5143 – 5148 Vol.5, dec. 2003.

HUMUSOFT. Ce 152 magnetic levitation model, user's manual. 1996.

Eric C. Kerrigan and Jan M. Maciejowski. Feedback min-max model predictive control using a single linear program: robust stability and the explicit solution. *International Journal of Robust and Nonlinear Control*, 14 (4):395–413, 2004. ISSN 1099-1239.

Mayuresh V. Kothare, Venkataramanan Balakrishnan, and Manfred Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361 – 1379, 1996. ISSN 0005-1098.

J. Lofberg. Approximations of closed-loop minimax mpc. In *Decision and Control, 2003. Proceedings. 42nd IEEE Conference on*, volume 2, pages 1438–1442 Vol.2, 2003.

J. M. Maciejowski. *Predictive control with constraints*. Prentice Hall, 2002.

D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789 – 814, 2000.

D.Q. Mayne, M.M. Seron, and S.V. Rakovi. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2):219 – 224, 2005. ISSN 0005-1098.

Quang Thuan Nguyen, Vojtech Veselý, and Danica Rosinová. Design of robust model predictive control with input constraints. *International Journal of Systems Science*, 44(5):896–907, 2013.

D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou. A new robust D-stability condition for real convex polytopic uncertainty. *Systems & Control Letters*, 40: 21–30, 2000.

Sasa V. Rakovic. Invention of prediction structures and categorization of robust mpc syntheses. In *4th IFAC Nonlinear Model Predictive Control Conference*, volume 4, pages 245–273, August 2012.

J.A. Rossiter. *Model-Based Predictive Control: A Practical Approach*. CRC Press Control Series. CRC Press, 2003. ISBN 9780849312915.

P. O M Scokaert and D.Q. Mayne. Min-max feedback model predictive control for constrained linear systems. *Automatic Control, IEEE Transactions on*, 43(8):1136–1142, 1998. ISSN 0018-9286.

Vojtech Veselý and Danica Rosinová. Robust output feedback model predictive control design. *Archives of Control Sciences*, 20(3):253–266, 2010.

Vojtech Veselý, Danica Rosinová, and Martin Foltin. Robust model predictive control design with input constraints. *Isa Transactions*, 49:114–120, 2010.

Vojtech Veselý, Danica Rosinová, and Vladimír Kučera. Robust static output feedback controller LMI based design via elimination. *Journal of the Franklin Institute*, 348(9):2468 – 2479, 2011. ISSN 0016-0032.

Daniel Vozák and Vojtech Veselý. Robust output feedback controllers for mimo systems, bmi and lmi approach. In *Process Control (PC), 2013 International Conference on*, pages 142–146, 2013.