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On Stock Trading Over a Lattice via Linear Feedback

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Abstract: This paper considers the stochastic characterization of the returns garnered from the use of linear feedback to trade stocks. For simplicity of presentation, we begin with the simple case of a single stock and assume that the underlying stock price is governed by the classical binomial model of Cox, Ross and Rubinstein (CRR). From this starting point, we derive the resulting probability mass function (PMF) for the discrete-time trading gains and losses. Our use of the CRR model differs from typical applications in that the quantification of the effect of feedback on gains and losses is our focal point rather than option valuation. Our analysis is then generalized to the case of two correlated stock prices governed by a quadrinomial lattice model; i.e., the binomial lattice result can be obtained as a special case of the quadrinomial analysis using two "perfectly" correlated stocks. The technical novelty in this paper is the exploitation of a symmetry property of the trading gains and losses $g(N)$ at stage $k = N$. That is, using the fact that $g(N)$ is invariant under permutations of the random single-step returns $\rho_i(k)$ for each stock, instead of having 2^{2N} point masses describing its probability distribution, only $(N+1)^2$ are required. The theory is illustrated via numerical examples and the paper also describes how one might generalize the analysis to a portfolio with $n > 2$ stocks.

Keywords: Financial Systems, Stochastic Systems, Uncertain Dynamic Systems, Stock Trading

1. INTRODUCTION

This paper is part of a growing body of literature on "technical analysis" that addresses problems which arise in stock trading and portfolio balancing from a control-theoretic point of view. The literature on these topics can be subdivided into two categories: The first category, called *model-based approaches*, includes [1]-[16], and typically involves a parametric model for the financial process under consideration. The second category of papers, called *model-free approaches*, includes [17]-[22]. In these cases, the time-varying stock price $S(k)$ has no predictive model for its evolution. In this context, the objective is to construct a feedback control trading strategy that adjusts the amount invested $I(k)$ over time. Subsequently, as is the case in this paper, the performance is judged by studying the probability distribution associated with the trading gains or losses $g(k)$.

While the set of references above covers a broad array of issues, the focal point in this paper is stock trading via linear feedback control. That is, at stage k, the amount invested is given by

$$I(k) = I_0 + Kg(k)$$

with I_0 being the initial amount invested and K being the feedback gain. When I_0 and K are positive, the interpretation of the feedback law above is that the trader is "long" and benefits from increases in the stock price $S(k)$. Conversely, when I_0 and K are negative, the trader is "short" and profits are accrued if $S(k)$ decreases. Given this setting, the problem formulations in papers [11] and [19]-[22] are closest to the one given here.

The research described herein falls within the domain of model-based approaches mentioned above. More specifically, the take-off point for this work is the celebrated paper by Cox, Ross, and Rubinstein (CRR) which begins with a binomial lattice model for the evolution of a stock's price $S(k)$ over time; see [24]. With linear feedback as above in place, as $S(k)$ evolves over the lattice, we study the probability distribution for the gains $g(N)$ at stage $k = N$. This type of analysis involving trading based on feedback control is novel in that it differs from the classical use of the CRR model to price options.

Binomial Lattice Model: This model is described in terms of $u > 1$, $d < 1$ and a given probability p of "up." Then, going forward in time from some stock price S at stage k, it is assumed that the next price at stage $k+1$ is Su with probability p and Sd with probability $1-p$. Figure 1 illustrates the first two stages of a binomial lattice. Each node shows the two ways the stock price can change at each stage. Many papers

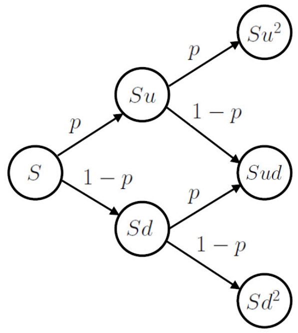

Fig. 1. Two Stages of a Binomial Lattice

have been written that expand on this model. For instance, in [25], a flexible binomial model is presented with a 'tilt' parameter λ that alters the shape and span of the binomial lattice. When $\lambda = 0$, the classical CRR model is recovered.

In Section 2, we provide a preliminary result. That is, in Theorem 2.1, using the CRR model to generate the stock price, we characterize the probability distribution for the trading gains $g(N)$ at stage $k = N$. This initial result, given in the binomial lattice framework, is a special case of the analysis in Section 3. It is provided for pedagogical purposes and provides a simple-to-understand exposition of the key ideas while avoiding the more detailed combinatorics required for the proof of the more general result in Section 3.

For the binomial lattice, the number of possible paths is 2^N and N can be quite large in practice. For example, in high-frequency trading over the course of an hour, one can easily see $N = 100$ investment updates. With this issue in mind, the technical novelty in this paper is the exploitation of a symmetry property of the trading gains and losses $g(N)$ at stage $k = N$. That is, using the fact that $g(N)$ is invariant under permutations of the random *single-step returns*

$$\rho(k) \doteq \frac{S(k+1) - S(k)}{S(k)}; \quad k = 0, 1, 2, ..., N-1,$$

instead of having 2^N point masses describing its probability distribution, only $N + 1$ are required.

Quadrinomial Lattice Model: In Section 3, we consider a generalization of the binomial case that involves two correlated stocks. To this end, we work with a model very similar to the one considered in [26] in option valuation; see also [27]-[28] for other examples of option valuation via lattices and [29] for an introduction to lattice models in finance. For the two-stock case under consideration, we have $u_1 > 1$ and $d_1 < 1$ "up-down parameters" for the first stock price $S_1(k)$ and $u_2 > 1$ and $d_2 < 1$ for the second stock price $S_2(k)$. In this more general case, beginning with price pair (S_1, S_2) at stage k in the lattice, there are four possible branches and associated probabilities for the next price pair at stage $k+1$. Namely, we transition to (S_1d_1, S_2d_2) with probability p_{00}, (S_1d_1, S_2u_2) with probability p_{01}, (S_1u_1, S_2d_2) with probability p_{10} and (S_1u_1, S_2u_2) with probability p_{11}. These transitions are depicted in Figure 2 and the main result, a description of the probability mass function (PMF) for $g(N)$, is given in Theorem 3.1.

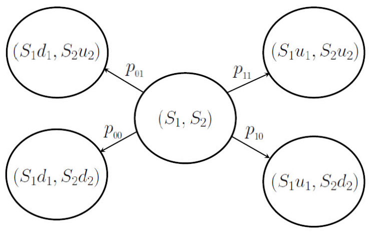

Fig. 2. Single Transition Possibilities in Two-Stock Case

For the quadrinomial case, again exploiting symmetry with respect to daily returns $\rho_i(k)$ for each stock, the 2^{2N} sample paths collapse down to $(N+1)^2$ point masses in the PMF. Finally, we note that the binomial lattice results are a special case of the quadrinomial lattice formulation with $p_{10} = p_{01} = 0$, $d_1 = d_2$ and $u_1 = u_2$.

Additional Considerations: In Section 4, we provide numerical examples demonstrating the application of the two theorems established in the preceding sections. Finally, in Section 5, we provide conclusions and directions for further research.

2. TRADING THE BINOMIAL LATTICE

In this section, we give the first result in this paper: a formula for the probability mass function which results from trading a single stock via the linear feedback control $I(k) = I_0 + Kg(k)$ over a binomial lattice described by the triple (u, d, p). As a preliminary for the proof, we first derive sample path solutions for $g(k)$ which result from a realization of the (u, d) sequences.

Indeed, suppose that $\rho(0), \rho(1), ..., \rho(N-1), \rho(N)$ is a sample path of stock returns generated from some underlying stochastic process. Then, when we arrive at stage k with the cumulative trading gains $g(k)$, to update to stage $k+1$, the increment to the trading gains $\Delta g(k)$ is obtained by multiplying the percentage change in the stock price $\rho(k)$ by the amount being invested $I(k)$. That is,

$$\Delta g(k) \doteq g(k+1) - g(k) = \rho(k)I(k).$$

Now substituting the linear feedback

$$I(k) = I_0 + Kg(k)$$

above and simplifying, we obtain update dynamics

$$g(k+1) = (1 + K\rho(k))g(k) + \rho(k)I_0.$$

By viewing the recursion above as a linear time-varying system with input $\rho(k)I_0$, a formula for $g(N)$ is readily obtained via classical state-space methods. Indeed, via a lengthy but straightforward calculation, the solution is

$$g(N) = \frac{I_0}{K}\left[\prod_{i=0}^{N-1}(1 + K\rho(i)) - 1\right].$$

We are now prepared to provide a characterization of the probability mass function for the random variable $g(N)$ above.

2.1 Theorem: *Given the linear feedback control stock trading strategy $I(k) = I_0 + Kg(k)$ and binomial lattice triple (u, d, p), let*

$$x_i \doteq \frac{I_0}{K}\left[(1 + K(u-1))^i (1 + K(d-1))^{N-i} - 1\right]$$

for $i = 0, 1, 2, ..., N$. Then, the probability mass function for the trading gain or loss $g(N)$ is the sum of Dirac Delta functions given by

$$f_N(x) = \sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} \delta(x - x_i).$$

Proof: To find $f_N(x)$, for the random variable $g(N)$, we first note that there are 2^N possible paths for the lattice associated with the price $S(k)$. However, since $g(N)$ is a symmetric function of the $\rho(k)$, it is invariant to any permutation of these returns. Hence, $g(N)$ can only take on $N+1$ possible values given by

$$x_i \doteq \frac{I_0}{K}\left[(1 + K(u-1))^i (1 + K(d-1))^{N-i} - 1\right]$$

where $i = 0, 1, \ldots, N$. These values span from the case where the stock goes down for N consecutive periods to the case where the stock goes up for N consecutive periods. Now for these $N+1$ points comprising the sample space for $g(N)$, we find the probability mass for each of them. Indeed, for x_i, there are

$$N_i = \binom{N}{i}$$

possible paths with each such path having probability

$$p_i = p^i(1-p)^{N-i}.$$

Now summing over these $N+1$ possibilities leads to

$$f_N(x) = \sum_{i=0}^{N} N_i p_i \delta(x - x_i);$$
$$= \sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} \delta(x - x_i). \square$$

3. TRADING THE QUADRINOMIAL LATTICE

In this section, we provide the main result, a generalization of Theorem 2.1. That is, we provide a formula for the probability mass function of the trading gain $g(N)$ which results from trading two correlated stocks with the two linear feedback controls $I_1(k) = I_{0,1} + K_1 g_1(k)$ and $I_2(k) = I_{0,2} + K_2 g_2(k)$ over a quadrinomial lattice. For $i = 1, 2$, note that $I_{0,i}$ is the initial amount invested in stock i. and K_i is the feedback gain. Per Section 1, the $I_{0,i}$ and K_i can be positive or negative and this price model is described by "up-down" parameters (u_1, d_1, u_2, d_2) and transition probabilities $(p_{00}, p_{01}, p_{10}, p_{11})$. Recalling the analysis in Section 2, the individual trading gains or losses are

$$g_i(N) = \frac{I_{0,i}}{K_i} \left[\prod_{j=0}^{N-1} (1 + K_i \rho_i(j)) - 1 \right]$$

where

$$\rho_i(k) \doteq \frac{S_i(k+1) - S_i(k)}{S_i(k)}$$

for $i = 1, 2$ and $k = 0, 1, 2, \ldots, N-1$.

3.1 Theorem: *Given the linear feedback stock trading strategies $I_1(k) = I_{0,1} + K_1 g_1(k)$ and $I_2(k) = I_{0,2} + K_2 g_2(k)$, quadrinomial lattice with parameters (u_1, d_1, u_2, d_2) and transition probabilities $(p_{00}, p_{01}, p_{10}, p_{11})$, let*

$$x_{1,i} \doteq \frac{I_{0,1}}{K_1} \left[(1 + K_1(u_1 - 1))^i (1 + K_1(d_1 - 1))^{N-i} - 1 \right];$$
$$x_{2,i} \doteq \frac{I_{0,2}}{K_2} \left[(1 + K_2(u_2 - 1))^i (1 + K_2(d_2 - 1))^{N-i} - 1 \right]$$

for $i = 0, 1, 2, ..., N$. Then, the probability mass function $f_N(x)$ for the overall trading gain or loss

$$g(N) \doteq g_1(N) + g_2(N)$$

is the sum of Dirac Delta functions given by

$$f_N(x) = \sum_{i=0}^{N} \sum_{k=0}^{i} \sum_{j=k}^{N-i+k} \binom{N}{i} \binom{i}{k} \binom{N-i}{j-k} \times p_{00}^{N-i-j+k} p_{01}^{j-k} p_{10}^{i-k} p_{11}^{k} \delta(x - x_{1,i} - x_{2,j}).$$

Proof: To find $f_N(x)$ for the random variable $g(N)$, we first note that there are 2^{2N} possible paths for the lattice associated with the price pair $(S_1(k), S_2(k))$. Now, arguing as in the proof of Theorem 2.1, both $g_1(N)$ and $g_2(N)$ are symmetric functions of their corresponding returns $\rho_i(k)$ for $i = 1, 2$. Hence, each of these gains can only take on $N+1$ possible values at most. In other words $g_i(N)$ is invariant to any permutation of $(\rho_i(0), \rho_i(1), \ldots, \rho_i(N-1))$. Now, for $i = 1, 2$, the $N+1$ possible values for the $g_i(N)$ are given by

$$x_{i,j} \doteq \frac{I_{0,i}}{K_i} \left[(1 + K_i(u_i - 1))^j (1 + K_i(d_i - 1))^{N-j} - 1 \right]$$

for $j = 0, 1, \ldots, N$. These values are obtained by spanning all the "up-down" possibilities $x_{i,j}$ for the two stocks. Consequently, there are at most $(N+1)^2$ possible values for $g(N)$ obtained from sums of the form $x_{1,i} + x_{2,j}$. These sums range from $x_{1,0} + x_{2,0}$ to $x_{1,N} + x_{2,N}$. Now for these $(N+1)^2$ points comprising the sample space for $g(N)$, we need to find the probability mass for each of them.

To achieve this, we need to count up all the outcomes where after N periods of trading, i values of the single-step returns $(\rho_1(0), \rho_1(1) \ldots, \rho_1(N-1))$ in $g_1(N)$ are $u_1 - 1$ and j values of $(\rho_2(0), \rho_2(1) \ldots, \rho_2(N-1))$ in $g_2(N)$ are $u_2 - 1$ where i and j range from 0 to $N-1$. Now, for fixed i and j, let k represent the number of periods where both the first stock and the second stock go up on the same period.

For a particular outcome of $g_1(N)$ where there are i values of $\rho_i(k)$ set to $u_1 - 1$ and $N-i$ values set to $d_1 - 1$ and there are

$$N_{ijk} \doteq \binom{i}{k} \binom{N-i}{j-k}$$

possible sample paths for $g(N)$ for $k = 0, 1, \ldots, i$. Notice that $\binom{i}{k}$ is the number of ways for the first and second stock to have k out of i periods where both their prices go up on the same period. In addition, $\binom{N-i}{j-k}$ is the number of ways where the first stock price goes down and the second stock price either goes up or goes down over the remaining $N-i$ periods of trading. Since each trading period is independent, the probabilities associated with these sample paths are all the same and given by

$$p_{ijk} = p_{00}^{N-i-j+k} p_{01}^{j-k} p_{10}^{i-k} p_{11}^{k}$$

for $k = 0, 1, \ldots, i$. Now summing over all the $(N+1)^2$ possibilities leads to

$$f_N(x) = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{i} N_{ijk} p_{ijk} \delta(x - x_{1,i} - x_{2,j})$$
$$= \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{i} \binom{N}{i} \binom{i}{k} \binom{N-i}{j-k} \times p_{00}^{N-i-j+k} p_{01}^{j-k} p_{10}^{i-k} p_{11}^{k} \delta(x - x_{1,i} - x_{2,j})$$
$$= \sum_{i=0}^{N} \sum_{k=0}^{i} \sum_{j=k}^{N-i+k} \binom{N}{i} \binom{i}{k} \binom{N-i}{j-k} \times p_{00}^{N-i-j+k} p_{01}^{j-k} p_{10}^{i-k} p_{11}^{k} \delta(x - x_{1,i} - x_{2,j}).$$

In the summations above, we use the convention that $\binom{i}{k} = 0$ if $k > i$ and $\binom{N-i}{j-k} = 0$ if $j - k > N - i$ or $k > j$. $\square$

Remarks: As mentioned in Section 1, the PMF formula for $g(N)$ for the quadrinomial lattice in Theorem 3.1 is a generalization of the binomial lattice formula given in Theorem 2.1. That is, for the case of two "perfectly" correlated stocks where $p_{11} = p$, $p_{00} = 1-p$, $p_{10} = p_{01} = 0$, $K_1 = K_2 = K$, $I_{0,1} = I_{0,2} = I_0/2$, $u_1 = u_2$ and $d_1 = d_2$, then the PMF formula in Theorem 3 reduces to that given in Theorem 2.1.

A second special case of interest involves the so-called *Simultaneous Long-Short* (SLS) linear feedback control strategy. This feedback control strategy was first presented in [20] and subsequently pursued in [21] and [22]. In this strategy, the trader begins with position I_0 long and $-I_0$ short. Then as time evolves, the long investment $I_1(k)$ and the short investment $I_2(k)$ are modulated via linear feedback strategies

$$I_1(k) = I_0 + Kg_1(k);$$

$$I_2(k) = -I_0 - Kg_2(k)$$

where $g_1(k)$ and $g_2(k)$ are the trading gains or losses for the long and short positions respectively. The overall investment is given by

$$I(k) \doteq I_1(k) + I_2(k)$$

and the overall gain or loss is

$$g(k) \doteq g_1(k) + g_2(k).$$

In the context of the quadrinomial lattice framework, we recover the PMF of $g(N)$ for SLS by making the substitutions $p_{11} = p$, $p_{00} = 1-p$, $p_{01} = p_{10} = 0$, $u_1 = u_2$ and $d_1 = d_2$ in the quadrinomial lattice PMF formula for $f_N(x)$ in Theorem 3.1. This leads to the PMF given by

$$f_N(x) = \sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} \delta(x - x_i)$$

where

$$x_i \doteq \frac{I_0}{K} \Big[(1 + K(u-1))^i (1 + K(d-1))^{N-i} + (1 - K(u-1))^i (1 - K(d-1))^{N-i} - 2 \Big].$$

4. NUMERICAL EXAMPLES

In this section, we provide numerical examples illustrating the generation of the PMF for various trading scenarios for both the binomial and quadrinomial lattice models. In order to simulate real-world stock prices, we size the parameters u_i and d_i for both models to represent trading on the order of every minute. For all examples, when we find the PMF $f_{100}(x)$ representing $N = 100$ periods of trading, it roughly corresponds to 100 minutes of trading. We also record the maximum and minimum values of the gain represented by g_{min} and g_{max} respectively, the probability of loss, $p_{loss} = P(g(100) < 0)$ and the mean value of the gain represented by $E[g(100)]$.

4.1 Binomial Lattice Examples

For the single stock trading scenario, we assign up parameter $u = 1.001$, down parameter $d = 0.999$ and transition probability $p = 0.55$. For the long case, we set the initial investment to $I_0 = 1$ and the feedback gain to $K = 2$, while for the short case the initial investment is $I_0 = -1$ and the feedback gain is $K = -2$.

For the long trading example, in accordance with Theorem 2.1, the formula for the PMF is

$$f_{100}(x) = \sum_{i=0}^{100} \binom{100}{i} (0.55)^i (0.45)^{100-i} \times \delta\left(x - 0.5\left[(1.002)^i (0.998)^{100-i} - 1\right]\right).$$

Since the probability of the stock going up is higher than it is going down, the probability of winning, that is, having a positive value of $g(100)$, is greater than the probability of losing. Using the theorem, we obtain $g_{min} \approx -0.09072$, $g_{max} \approx 0.1106$, $E[g(100)] \approx 0.0101$ and $p_{loss} \approx 0.1827$.

For the short trading example, the formula for the PMF is

$$f_{100}(x) = \sum_{i=0}^{100} \binom{100}{i} (0.55)^i (0.45)^{100-i} \times \delta\left(x - 0.5\left[(0.998)^i (1.002)^{100-i} - 1\right]\right).$$

Since the probability of the stock going down is smaller than 0.5, with probability greater than 0.5, we obtain terminal gain $g(100) < 0$. Using the theorem, we obtain $g_{min} \approx -0.09072$, $g_{max} \approx 0.1106$, $E[g(100)] \approx -9.902 \times 10^{-3}$ and $p_{loss} \approx 0.8654$.

For comparison purposes, we show plots of the PMF for both the long and short cases in Figure 3. The long case is represented by the green curve while the short case is represented by the red curve. Note that for both PMF plots, we only show the envelope of the central portions of the distributions so that their overlapping regions can be clearly seen. The circles represent the actual impulses of the PMFs.

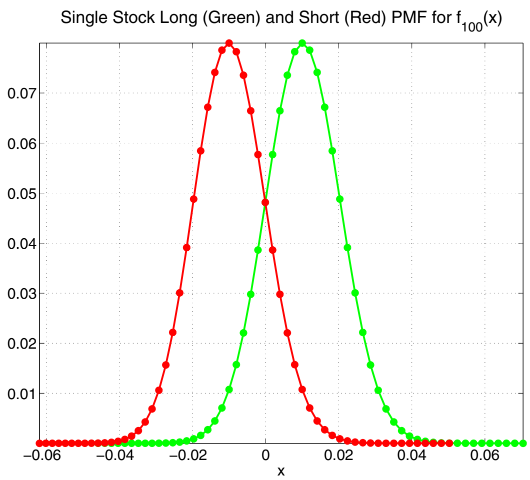

Fig. 3. Binomial Lattice Examples

4.2 Quadrinomial Lattice Examples

For the two-stock trading case, for our first example, we provide a plot of the trading gain PMF $f_{100}(x)$ where the first stock is a long trade while the second stock is a short trade. The quadrinomial lattice parameters are $u_1 = 1.001$, $u_2 = 1.002$, $d_1 = 0.999$, $d_2 = 0.998$, $p_{00} = 0.25$, $p_{01} = p_{10} = 0.125$ and $p_{11} = 0.5$. For the long trading stock, we take initial investment $I_{0,1} = 2$ and the feedback gain $K_1 = 2$, while for the short trading case, $I_{0,2} = -1$ and $K_2 = -2$.

For this example, using Theorem 3.1, we obtain PMF

$$f_{100}(x) = \sum_{i=0}^{100} \sum_{k=0}^{i} \sum_{j=k}^{100-i+k} \binom{100}{i} \binom{i}{k} \binom{100-i}{j-k} \times (0.25)^{100-i-j+k} (0.125)^{i+j-2k} (0.5)^k \times \delta(x - x_{1,i} - x_{2,j});$$

$$x_{1,i} = \left[(1.002)^i (0.998)^{100-i} - 1\right];$$

$$x_{2,j} = 0.5\left[(0.996)^j (1.004)^{100-j} - 1\right].$$

This trade produces a probability of winning that is noticeably bigger than 0.5 since $p_{loss} \approx 0.4145$. Our calculations also lead to $g_{min} \approx -0.3465$, $g_{max} \approx 0.4665$ and $E[g(100)] \approx 0.003635$. A plot of the PMF is shown in Figure 4. Note that we only show a smoothed envelope of the central portion of the distribution since the actual distribution is somewhat noisy and contains over $10,000$ point masses.

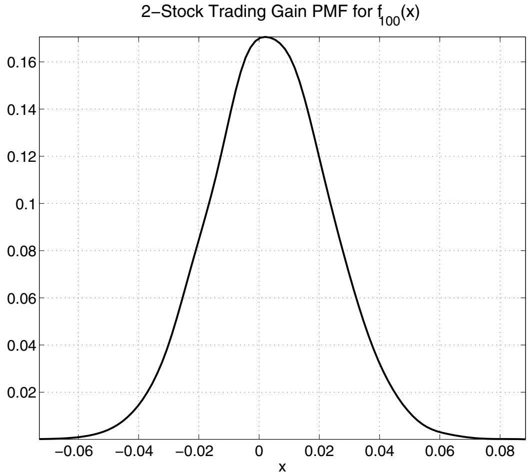

Fig. 4. Quadrinomial Lattice Example

Finally, we give an example of SLS trading with parameters $u_1 = u_2 = 1.001$, $d_1 = d_2 = 0.999$, $p_{00} = 0.45$, $p_{01} = p_{10} = 0$, $p_{11} = 0.55$, $I_{0,1} = 1$, $I_{0,2} = -1$, $K_1 = 2$ and $K_2 = -2$. From a "trading mechanics" point of view, this reduces to the same scenario for the binomial lattice example in Section 4.1. However, the PMF generated in this case is more "informative" because we obtain a PMF for the "combined" trading gain $g(N) = g_1(N) + g_2(N)$ rather than two individual PMFs, one for $g_1(N)$ and one for $g_2(N)$.

Figure 5 is a plot of the central portion of the PMF of $g(N)$ for this SLS case. Using the theorem we obtain $g_{min} \approx -2 \times 10^{-4}$, $g_{max} \approx 0.01986$, $E[g(100)] \approx 1.795 \times 10^{-4}$ and $p_{loss} \approx 0.6174$. Notice that the probability of losing for SLS is less than that of the binomial lattice short trade, but the probability of winning is smaller than that of the long stock trading case.

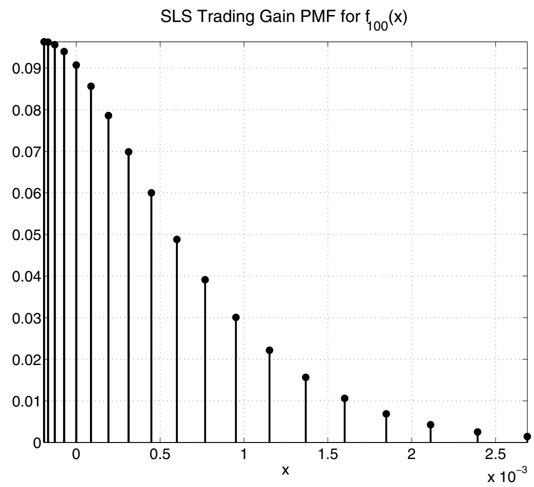

Fig. 5. SLS Quadrinomial Lattice Example

5. CONCLUSIONS AND FURTHER RESEARCH

In this paper, we provided formulae for the probability mass functions of the trading gains or losses $g(N)$ resulting from a linear feedback control strategy with prices generated via a lattice model. For simplicity of the exposition, we first obtained the PMF of $g(N)$ for the case of a binomial lattice with stock price $S(k)$. Subsequently, we generalized this result to the case of a quadrinomial lattice for a pair of correlated stock prices $(S_1(k), S_2(k))$. We believe that it should be possible to extend the ideas used for $n = 2$ stocks to $n = 3$ and beyond. To sketch the key ideas for an octonomial lattice obtained with $n = 3$ stocks, beginning with the transition probabilities $(p_{000}, p_{001}, p_{010}, p_{011}, p_{100}, p_{101}, p_{110}, p_{111})$, corresponding to combinations of up-down stock moves (u_i, d_i) for $i = 1, 2, 3$, the PMF for the trading gain or losses

$$g(N) \doteq g_1(N) + g_2(N) + g_3(N)$$

has the form

$$\begin{aligned} f_N(x) = \sum_{i_1=0}^{N} \sum_{i_2=0}^{N} \sum_{i_3=0}^{N} \sum_{k=0}^{i_1} & C(N, i_1, i_2, i_3, k) \\ & \times\ p_{000}^{N-i_1-i_2-i_3+k} \cdots p_{111}^{k} \\ & \times\ \delta(x - x_{1,i_1} - x_{2,i_2} - x_{3,i_3}); \end{aligned}$$

$$x_{i,j} \doteq \frac{I_{0,i}}{K_i}\left[(1 + K_i(u_i - 1))^j (1 + K_i(d_i - 1))^{N-j} - 1\right]$$

for $i = 1, 2, 3$ and $j = 0, 1, \ldots, N$. Note that the coefficient $C(N, i_1, i_2, i_3, k)$ is a product of appropriately constructed binomial coefficients and there are $(N+1)^3$ point masses comprising the PMF.

As a result of the symmetry of gains and losses $g(N)$ with respect to returns $\rho(k)$ entering into $g(N)$, it was seen that the computational complexity associated with PMF construction is dramatically reduced. For the quadrinomial lattice case, out of the possible 2^{2N} return sequences comprising the sample space, we only end up dealing with $(N+1)^2$ point masses. More generally, for the case of n stocks with an associated lattice of size 2^n, using our method and fully exploiting symmetry, the 2^{nN} possible return sequences collapse down to $(N+1)^n$ probability mass points. This number, while manageable for small (n, N) combinations, can easily become prohibitive for larger N and n. For example, with $N = 100$, in the examples given herein, the number of point masses in the two-stock case was about $10,000$ and this number increases to over $1,000,000$ for three stocks.

Given the growth rate described above, for $n > 2$ correlated stocks, in many cases development of a general formula for the PMF of $g(N)$ may be more of an academic exercise than of practical importance because the central portion of the PMF can readily be estimated via Monte Carlo simulation. To illustrate, for the case when $n = 3$ with $N = 100$, using $100,000$ sample paths we generated an approximation for the PMF of $g(100)$; see Figure 6. Notice that we only show a smoothed envelope of the central portion of the PMF since it consists of over a million point masses. In this example the parameters are $u_1 = 1.001$, $d_1 = 0.999$, $u_2 = 1.002$, $d_2 = 0.998$, $u_3 = 1.001$, $d_3 = 0.999$, $p_{000} = p_{001} = p_{010} = p_{011} = 1/8$, $p_{100} = p_{101} = 1/16$, $p_{110} = 1/4$, $p_{111} = 1/8$, $K_1 = K_2 = K_3 = 2$ and $I_{0,1} = I_{0,2} = I_{0,3} = 1$. Following the format of Section 4 we find that $g_{min} \approx -0.3465$, $g_{max} \approx 0.4665$, $E[g(100)] \approx 0.04022$ and $p_{loss} \approx 0.05004$.

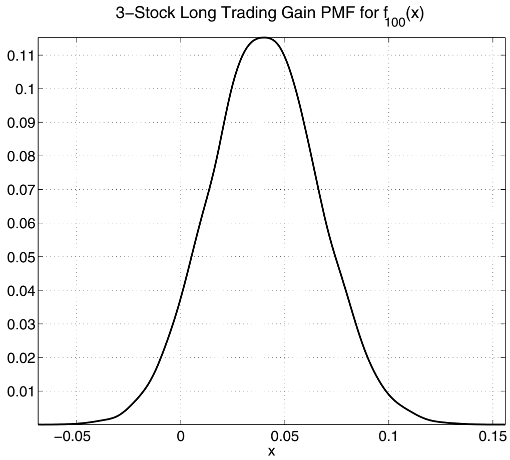

Fig. 6. Octonomial Lattice Example

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