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Discretization of Fractional-Order Differentiators and Integrators

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Abstract — *This paper introduces a closed form discretization method of fractional-order differentiators or integrators. Unlike the continued fraction expansion technique, or the infinite impulse response of second-order IIR-type filters, the proposed technique generalizes the Tustin operator to derive a stable and minimum 1st and 2nd-order discrete-time operators (DTO) that discretize continuous fractional-order differintegral operators. Such DTOs exploits the phase properties of the DTOs over a wide range of the frequency spectrum, which depend only on the order of the continuous operators. Moreover, the closed-form DTOs enable one to identify the stability regions of fractional-order discrete-time systems. The effectiveness of this work is demonstrated via several numerical examples.*

Index Terms—Discretization, Differintegrator, Fractional Calculus, Differentiation, Integration, Laplacian Operator

I. INTRODUCTION

Fractional-order systems represent a generalization of the integer-order ones. Most systems are best described by fractional-order dynamics of real or even complex orders. These systems enjoy hereditary effects and can be analyzed using the generalized fractional-order calculus. The hereditary effect implies that such systems are of infinite dimension, which must be approximated by finite-order models for realization purposes (Podlubny, 1994,1999; Oldham and Spanier, 1974; Krishna, 2011; Kilbas, et al., 2006, Samko, 1987).

Since embedded systems require fast computing, especially the ones that are described by fractional-order dynamics, there is a need to convert these systems into discrete-time ones. Hence, the discretization of the continuous differintegral Laplacian operator would pave the way to implement digital signal processors for such systems. It follows that a dynamic, stable, and a straightforward, discretization techniques are needed to achieve such goals.

There are two discretization techniques of differintergal continuous operators; a direct and an indirect one (Al-Alaoui, 1993, 2001, 2006). In the indirect discretization method one may first develop a rational continuous-time operator (CTO), which then discretized using any of the well know discretization techniques. The direct method, on the other hand, allows one to directly develop DTOs to discretize the fractional-order CTOs (Chen et al., 2003; Barbosa et al., 2005; Nei et al.; 2011).

The method of continued fraction expansion (CFE) is one of the early methods that is implemented in the indirect discretization approach to obtain rational approximations from irrational functions (Chen et al., 2003; Siami et al., 2001). One must be careful when using the CFE approach when discretizing closed-loop continuous fractional-order transfer functions; it may result in unstable models due to generating non-minimum phase DTOs. An alternative approach to the CFE was adopted by (Ortigueira and Serralheiro, 2007), where an IIR ARMA modeling are used to develop discrete-time fractional-order operators, which yields high order models.

Notice that some operators, such as the Al-Alaoui operator, are obtained by interpolating the trapezoidal and rectangular integration rules (Alaloui, 2001; Vinagre et al., 2000). The Interpolation and inversion processes may induce, in some cases, unstable fractional-order models.

This work introduces a direct discretization technique to discretize the continuous differintegral operators that depends only on its fractional order. The significance of the proposed method is achieved by identifying the stability region for discrete-time systems. Moreover, the proposed method yields a 2nd-order DTO with a frequency response that is almost identical to that of a 9th-order one introduced in (Al-Alaoui, 2001; Chen et al., 2002).

The paper is organized as follows: next section summarizes some preliminaries and background. Section III introduces the main results of this work. The stability regions of the new operators are discussed in section IV. Section V presents a comparative study between the proposed operators and those obtained using continued fraction expansion via numerical simulation. Finally, section VI highlights the concluding remarks.

II. PRELIMINARIES AND BACKGROUND

Fractional-order calculus is a generalization of the classical integer-order one (Oldham and Spanier, 1974). The fractional-order differintegrator operators are denoted by ${}_aD_t^{\pm\alpha}$. One may define the fractional-order integrator, ${}_aD_t^{-\alpha}$ by ${}_aI_t^{\alpha}$, where a and t represent the time limits and $\alpha \in \Re$ is the order of the operator. The digital implementation and synthesize of fractional-order controllers require proper discrete-time forms of the fractional-order differintegral operators (Ortigueira, and Serralheiro, 2007; Chen et al., 2003; Oustaloup et al., 2000); one must then look for a dynamic, accurate, and efficient discretization techniques to discretize the continuous fractional-order operators.

There are two discretization techniques; *direct and indirect discretization*. In the *indirect method*, the fractional-order operators are first approximated by a rational finite-order transfer functions; i.e., $s^{\pm\alpha} \approx \frac{N(s,\alpha)}{D(s,\alpha)}$ (Gupta, and Yadav, 2012; Oustaloup et al., 2000; Lubich, 1986), and then the resulting form is discretized using any existing technique such as Tustin method, Al-Alaoui operator, or a linear combination of other discrete-time operators. For example, the Al-Alaoui integral operator is simply a linear interpolation of the backward rectangular rule and the trapezoidal rule, i.e., $H(z) = aH_{Rect}(z) + (1-a)H_{Trap}(z)$, where $0 < a < 1$ (Al-Alaoui, 2006). A similar technique was also used to develop a dynamic hybrid digital integrator using a linear combination of Simpson's and the Trapezoidal integrators (Chen et al., 2003; Gupta et al., 2012). All these operators reduce the frequency deviation over limited range of the frequency bands, while they deteriorate their phase responses.

Figure 1 shows the frequency response of Tustin operator, $s \equiv H_T(z) = \frac{2(1-z^{-1})}{T(1+z^{-1})}$, the Al-Alaoui operators, $s \equiv H_a(z) = \frac{8(1-z^{-1})}{7T\left(1+\frac{z^{-1}}{7}\right)}$, and compared with Chen discrete-time operators (Chen and Moore, 2002). Another hybrid discrete-time differentiator that approximates an integer-order integrator was also introduced in (Chen et al., 2003) and given in equation (1) for completeness:

$$H(z) = \frac{6(z^2-1)}{T(3-a)(z+p_1)(z+p_2)};\ p_1 = \frac{3+a+2\sqrt{3a}}{3-a},\ p_2 = \frac{3+a-2\sqrt{3a}}{3-a} \quad (1)$$

Where T is the sampling time and $0 < a < 1$ is a scaling factor.
Equation (1) can be used to generate several quadratic forms that discretize $s^{\pm 1}$.

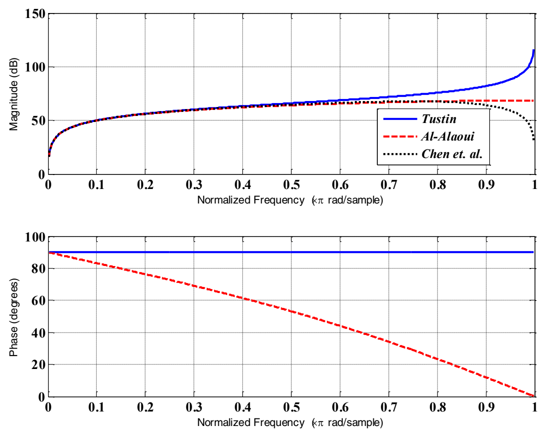

Figure 1: Frequency response of Tustin, Al-Alaoui, and the DTO of equation (1)

Figure 1 shows the frequency response of the three aforementioned operators, which approximate the differential operator, s^1 for $T = 0.001s$. The bilinear (Tustin) transformation exhibits a sever magnitude deviation at both ends of the frequency spectrum. The Al-Alaoui operator, on the other hand, reduces such deviation at high frequency and yields an almost linear phase due to the asymmetric pole-zero selection. Hence, any attempt to obtain an accurate discrete-time form for discrete-time systems using the CFE results in a high order rational z-transfer function, which is cumbersome and, in some cases, may yield unstable equivalent discrete-time systems.

Since the goal is to look for a closed-form discrete-time model for $s^{\pm\alpha}$, the *direct discretization* methods are exploited further to achieve such a goal. In all methods, one may replace the continuous frequency operator by a generating function, i.e., $s^{\pm\alpha} = \left(\omega(z^{-1})\right)^{\pm\alpha}$. For example, applying a *direct* discretization to the GL definition of the fractional-order integro-differential operator yields (Lubich, 1986; Vinagre et al., 2000; Podlubny, 1996, 1999),

$${}_aD_t^{\pm\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\mp\alpha}} \sum_{j=0}^{\infty} C_j^{\pm\alpha} f((t-j)h) \quad (2)$$

where $[\cdot]$ is the flooring operator, and where

$$C_j^{\pm\alpha} \equiv (-1)^j \binom{\pm\alpha}{j} = \left(1-\left(\frac{1\pm\alpha}{j}\right)\right) C_{j-1}^{\pm\alpha};\ C_0^{\pm\alpha} = 1;\ j = 1, \dots, n \quad (3)$$

Taking the $\mathcal{Z}$-transform of (2) and using the short memory principle (Vinagre et al., 2000), the following generating function may discretizes $s^{\pm\alpha}$, i.e.,

$$\left(\omega(z^{-1})\right)^{\pm\alpha} = T^{\mp\alpha}\left\{\sum_{j=0}^{[L/T]} C_j^{\pm\alpha}\, z^{-j}\right\} \quad (4)$$

where $T = h$ is the sampling time, and $L/T \equiv \left[\frac{nh-a}{h}\right]$ is an increasing memory size $L = nh - a$.

Equation (4) defines a transfer function of a FIR-type discrete-time model of $s^{\pm\alpha}$. The memory size, L, determines the accuracy of the generating function. A compromise between accuracy and size must be made to develop a realizable discrete-time operator.

The large memory size of the FIR discrete-time form of equation (4) does not yield an acceptable frequency response and would complicate the analysis and modeling of fractional-order systems. Therefore, one has to look for an IIR-type rational z-transfer function to minimize size of the discrete-time operators of $s^{\pm\alpha}$.

Most researchers perform a CFE to generate an IIR-discrete-time differentiators/integrators (Al-Alaoui, 2001; Chen et al., 2003; Lubich, 2006; Kilbas et al., 2006). The CFE does not always yield a stable minimum phase system, nor does it yield a flat-phase frequency response. Hence, one has to compromise between the size of the expansion and the type of the generating functions used. Several generating functions were used to discretize $s^{\pm\alpha}$, and listed below for completeness, to obtain a rational z-transfer functions of the fractional-order integrators/differentiators (DTFoI/DTFoD) (Chen et al., 2002, 2003; Vinagre et al., 2000, 2003; Nie et al., 2011):

a) Backward-Euler method alone by letting $\left(\omega(z^{-1})\right)^{\pm\alpha} = \left(\frac{1-z^{-1}}{T}\right)^{\pm\alpha}$.

b) Trapezoidal (Tustin) discretization rule alone to expand $\left(\omega(z^{-1})\right)^{\pm\alpha} = \left(\frac{2(1-z^{-1})}{T(1+z^{-1})}\right)^{\pm\alpha}$.

c) Al-Alaoui Operator alone to approximate $\left(\omega(z^{-1})\right)^{\pm\alpha} = \left(\frac{8}{7T}\left(\frac{1-z^{-1}}{1+\frac{z^{-1}}{7}}\right)\right)^{\pm\alpha}$.

d) A Hybrid interpolation of Simpson's and Trapezoidal discrete-time integrators of the form:

$$H(z) = aH_s(z) + (1-a)H_T(z);\ 0 < a < 1 \quad (5)$$

where $H_s(z) = \frac{T(1+4z^{-1}+z^{-2})}{3(1-z^{-2})}$, and $H_T(z) = \frac{T(1+z^{-1})}{2(1-z^{-1})}$.

The interpolation technique in (5) is a generalization of the first three methods. Since the magnitude of the frequency response of the integer-order integrator, s^{-1}, lies between that of the Simpson's and Trapezoidal discrete-time integrators (Al-Alaoui, 2001, 2006), then the linear combination in (5) for $0 < a < 1$ can be used to obtain a typical an IIR-type discrete-time operator as follows (Chen et al., 2003):

$$\left(\omega(z^{-1})\right)^{+\alpha} = k_o\left(\frac{1-z^{-2}}{(1+bz^{-1})^2}\right)^{\alpha} \quad (6)$$

where $\alpha \in [0,1]$, $k_o = \left(\frac{6z_2}{T(3-a)}\right)^{\alpha}$, and $b = z_2 = \frac{3+a-2\sqrt{3a}}{3-a}$.

Using the symbolic MATLAB Toolbox, one may obtain several forms of the CFE of (6) that approximates $\left(\omega(z^{-1})\right)^{\pm\alpha}$. For example, when $\alpha = 0.5$ and for $T = 0.001$ s, equation (6), yields several discrete-time transfer functions, $G_{(n,a)}(z)$, that discretize $s^{0.5}$, where n, and a represent the order and the weighting factor of the approximation, respectively (Chen et al., 2003, i.e.,

$$G_{(2,0.5)}(z^{-1}) = \frac{127+41.26\,z^{-1}-112.6\,z^{-2}}{4+2.98\,z^{-1}-z^{-2}} \quad (7a)$$

$$G_{(3,0.5)}(z^{-1}) = \frac{1501-503.6\,z^{-1}-1289\,z^{-2}+446.5z^{-3}}{47.26+4\,z^{-1}-23.63\,z^{-2}-z^{-3}} \quad (7b)$$

$$G_{(4,0.5)}(z^{-1}) = \frac{508.1-1501\,z^{-1}-4.478\,z^{-2}+1289z^{-3}-382.9z^{-4}}{16-40.54\,z^{-1}-12\,z^{-2}+20.27z^{-3}+z^{-4}} \quad (7c)$$

Figure 2 displays the frequency response of (7a,b,c) for $\omega \in (-\pi, \pi)$. The magnitude frequency response of the 2nd-order approximation yields a warping effect at high frequency, while the phase diagram of the three forms exhibit a decreasing phase value over most of the spectrum.

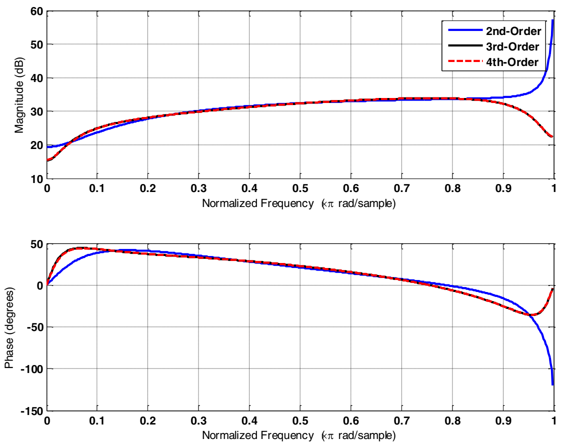

Figure 2: Frequency response of $s^{0.5}$ using (7a, b, c)

Remark 1: *The discrete-time approximation of* (7c), *reported in* (Chen et al., 2003), *represents an unstable non-minimum phase form since it has a pole and a zero outside the unit circle at* $p = 2.6298$, *and* $z = 2.6328$, *respectively. However, since* $p \approx z$, *then there is an almost pole-zero cancellation, which is evident from its frequency response shown in Figure 2. This would cause instability when implemented in larger systems. Furthermore, according to* (Chen, et al., 2003), *one must improve the phase performance of* $G_{(4,0.5)}(z)$ *by cascading a causal lead compensator* $z^{0.5} = z^{-0.5}/z^{-1}$, *which requires the implementation of a fractional sampler.*

Remark 2: *Since the CFE generates discrete-time forms of different orders, the stability region of such transformation is not straightforward and may not be guaranteed. Hence, one should look for a new form of discrete-time transformation that defines the stability region of the discrete-time fractional-order operators.*

III. EL-KHAZALI OPERATORS

As pointed out in section II, the CFE of any generating function could yield a higher order and an unstable non-minimum phase discrete-time approximation. The aim of this work is to avoid such subtleties by developing a dynamic closed-form solution to effectively discretize the fractional-order operators, $s^{\pm\alpha}$, which only depends on the order of the operator. Moreover, one would also be able to define the *z*-domain stability region of the DTO that approximates $s^{\pm\alpha}$.

A. First-Order Operators

The key point to develop a discrete-time operator that approximates $s^{\pm\alpha}$ is to generate a form that maintains an exact phase value over a wide range of frequency spectrum. The following first-order discrete-time operator is considered to generate a dynamic fractional-order *z*-transfer function of the form:

$$s^{\pm\alpha} \approx H(z) = \left(\frac{2}{T}\right)^{\pm\alpha}\left(\frac{(z\mp z_1(\alpha))}{(z\pm p_1(\alpha))}\right) \quad (8)$$

where $z_1(\alpha) \in \mathcal{R}$, and $p_1(\alpha) \in \mathcal{R}$ both depend on α.

Observe that (8) can describe both an integral/differential operators.

Theorem 1: Consider equation (8), where T is the sampling time. Let, $z_1(\alpha) = p_1(\alpha) = \frac{1}{\tan(2-\alpha)\pi/4}$; $0 < \alpha \leq 1$, then equation (8) discretizes the fractional-order operator $s^{\pm\alpha}$.

Proof: Consider first the fractional-order differentiator, $s^{+\alpha}$, which has a leading phase of $\alpha\pi/2$ and a unity gain over the entire spectrum. When $\alpha = 1$, $z_1(1) = p_1(1) = 1$, then equation (8) yields the well know Tustin transformation, i.e., $s = (2/T)(z-1)/(z+1)$, where $|s^{\alpha}| = |H(z)| = 2/T$. Now, for $0 < \alpha < 1$, $|z_1(\alpha)| = |p_1(\alpha)| < 1$, and the pole-zero maps of (8) are located inside the unit circle as shown in Figure 3.

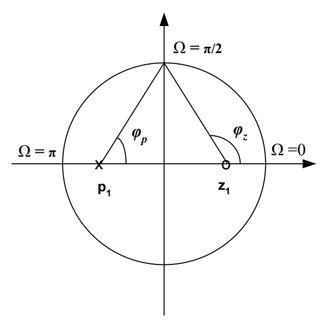

Figure 3: Pole-zero maps of (8)

Since equation (8) is expected to exhibit a flat phase response over the entire range of discrete frequencies,

$0 < \Omega < \pi$. Consider the test point when $\Omega = \pi/2$ (corresponding to half the sampling frequency), the phase angle of (8) must equal $\alpha\pi/2$, *i.e.*,

$$\varphi_z - \varphi_p = \alpha\pi/2 \tag{9}$$

Solving (9) and using the symmetry of the real pole and zero, i.e., $|z_1(\alpha)| = |p_1(\alpha)|$, then,

$$z_1(\alpha) = p_1(\alpha) = \frac{1}{\tan((2-\alpha)\pi/4)}; \quad 0 < \alpha < 1 \tag{10}$$

Finally, since $|z_1| = |p_1| \leq 1$ for $0 < \alpha \leq 1$, then equation (8) is a stable operator.

Figure 4 shows the frequency response of (8) for different values of α. The proposed first-order operator exhibits an acceptable response for $0.5 < \alpha \leq 1$. Obviously, the magnitude frequency response deviates off the exact values at both ends of the frequency spectrum since $|z_1(\alpha)| \to 1$, as $\alpha \to 1$.

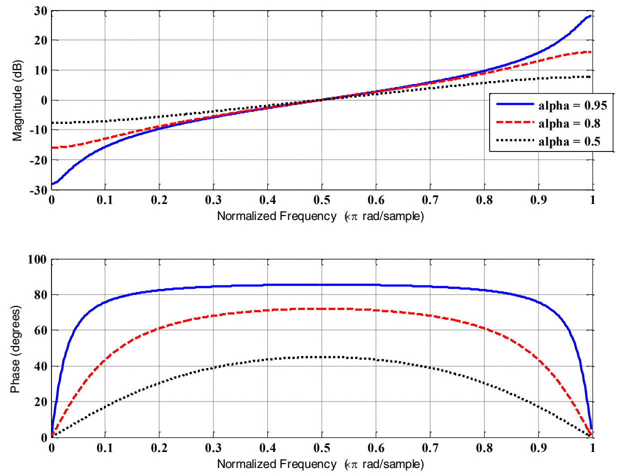

Figure 4: Frequency response of (8) for $T = 2$ s.

B. Second-Order Operator

The second-order discrete-time operator (DTO) has symmetric real poles and zeros. Similar to the continuous biquadratic approximation of the Laplacian operator, s^{α}, introduced in (El-Khazali, 2013), the following 2nd-order DTO may be considered to discretize $s^{+\alpha}$:

$$s^{\pm\alpha} \equiv H(z) = \left(\frac{2}{T}\right)^{\pm\alpha}\left(\frac{(z \mp z_1(\alpha))(z \mp z_2(\alpha))}{(z \pm p_1(\alpha))(z \pm p_2(\alpha))}\right) \tag{11}$$

where,

$$z_1(\alpha) = p_2(\alpha), \; z_2(\alpha) = p_1(\alpha), \; z_2(\alpha) = z_1(\alpha) - 1 \tag{12}$$

Substituting from (12) into (11) yields,

$$s^{\pm\alpha} = H(z(\alpha)) = \left(\frac{2}{T}\right)^{\pm\alpha}\frac{(z \mp z_1(\alpha))(z \mp (z_1(\alpha)-1))}{(z \pm (z_1(\alpha)-1))(z \pm z_1(\alpha))} \tag{13}$$

Theorem 2*:* Let T be a sampling time, $0 < \alpha \leq 1$, and consider (12), then the *z*-transfer function given by (13) discretizes the fractional-order operator, $s^{\pm\alpha}$, where

$$z_1(\alpha) = \frac{(\eta-2)+\sqrt{5\eta^2+4}}{2\eta}; \quad \eta = tan\left(\frac{\alpha\pi}{4}\right) \tag{14}$$

Proof: Using the same arguments of theorem 1, the phase contribution of the poles and zeros of (13) as depicted in Figure 5 is required to exhibit a flat phase over the entire spectrum, i.e.,

$$\varphi_{z_1} + \varphi_{z_2} - \varphi_{p_1} - \varphi_{p_2} = \alpha\pi/2 \tag{15}$$

Now, for symmetry, consider the midpoint of the normalized frequency spectrum at $\Omega = \pi/2$. Substituting from (12) into (13) and solving (15) yields,

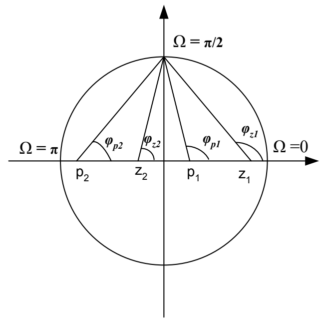

Figure 5: Pole-Zero maps of (13)

$$z_1(\alpha) = \frac{(\eta-2)+\sqrt{5\eta^2+4}}{2\eta} = p_2(\alpha), \; z_2(\alpha) = \frac{(\eta+2)-\sqrt{5\eta^2+4}}{2\eta} = p_1(\alpha) \tag{16}$$

Clearly, equation (16) gives a closed-form solution to the values of poles and zeros of (13). As $\alpha \to 1$, $z_1(1) = 1$, while $z_2(1) = 0$, and equation (13) reduces to the bilinear transformation $s^1 = \left(\frac{2}{T}\right)\left(\frac{z-1}{z+1}\right)$. Moreover, $|z(\alpha)| \leq 1$, for all $0 < \alpha \leq 1$.

Figure 6 displays the frequency response of (13) for $\alpha = 0.5, 0.7,$ and 0.9. Obviously, as α increases, the phase response of (13) exhibits a constant value over a wider range of frequency band. Figure 7, on the other hand, compares between the frequency response of the 1st and 2nd-order approximations for $s^{0.5}$, while Figure 8 displays a comparison between the frequency response of the two operators for $\alpha = 0.95$ and a sampling time $T = 0.001$ s. Notice that as $\alpha \to 1$, the frequency response of the 1st-order operator is almost identical to that of the 2nd-order one with less magnitude frequency deviation at both ends of the frequency spectrum. Hence, for large values of α, one may adopt the 1st-order operator instead of the 2nd-order one.

IV. Stability Region

The stability of continuous fractional-order systems was extensively investigated by many researchers (Matignon, 1996). It was verified that the stability region of such systems is defined off the cone that makes angles of $\pm\frac{\alpha\pi}{2}$ with the real axis (Matignon, 1996). In order to identify the stability region of the DTO of (8) or (13), the boundaries of the stable region of continuous fractional-order systems, defined by the vectors $s = Re^{\pm j\alpha\pi/2}$; $0 \leq R < \infty$, are mapped through the operators (8) or (13). Substituting $s = Re^{\pm j\alpha\pi/2}$ into (8) and solving for z yields the boundaries of the stability region of the 1st-order discrete-time operator given by (8), i.e.,

$$z = z_1(\alpha)\frac{(1+Re^{\pm j\alpha\pi/2})}{(1-Re^{\pm j\alpha\pi/2})} \tag{17}$$

Similarly, the boundaries of the stability region for the 2nd-order operator in (13) are given by:

$$z = \frac{-B+\sqrt{B^2-4AC}}{2A}, \tag{18}$$

where, $A = \left(1 - Re^{\pm\frac{j\alpha\pi}{2}}\right)$, $B = -(z_1(\alpha) + z_2(\alpha))\left(1 + Re^{\pm\frac{j\alpha\pi}{2}}\right)$, and $C = z_1(\alpha)z_2(\alpha)\left(1 - Re^{\pm\frac{j\alpha\pi}{2}}\right)$.

Figure 9 shows the stability region for both the 1st- and 2nd-order DTOs, which are contained inside the unit circle. It implies that both (8) and (13) yield a robust and stable representation to s^{α}. The low order of these operators yields a fast impulse response in the spatial domain; thus converging to its steady-state behavior faster than 9th-order DTOs.

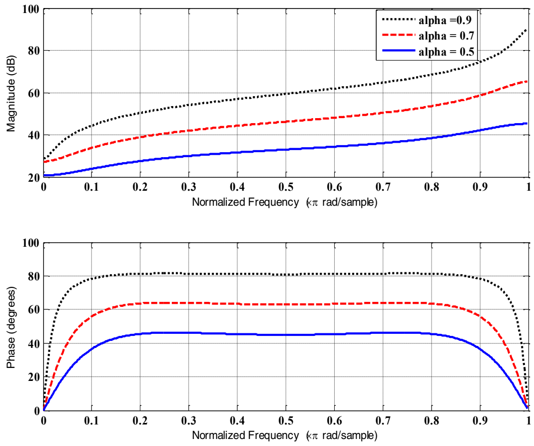

Figure 6: The frequency response of (13) for different values of α.

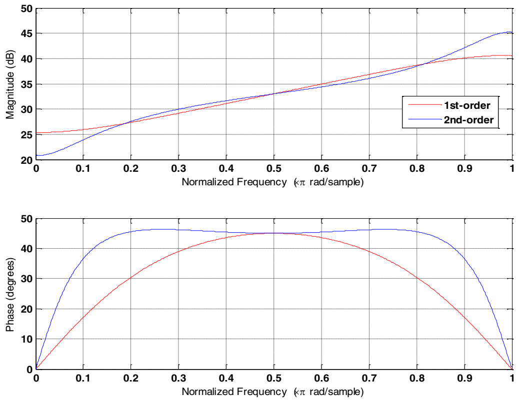

Figure 7: Frequency response of (8) and (13) for $s^{0.5}$.

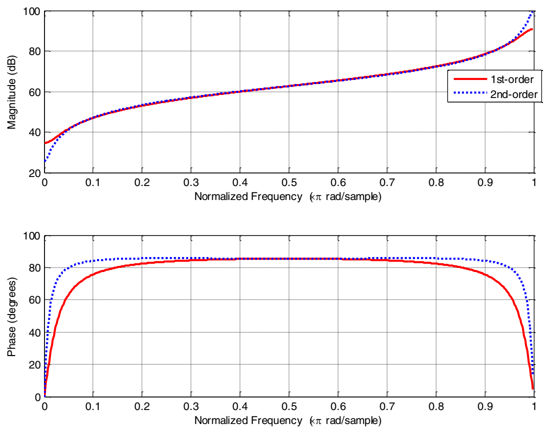

Figure 8: Frequency response of (8) and (13) for $s^{0.95}$

fast impulse response in the spatial domain; thus converging to its steady-state behavior faster than other forms of DTOs obtained by other methods. This point is left out due to space limitation.

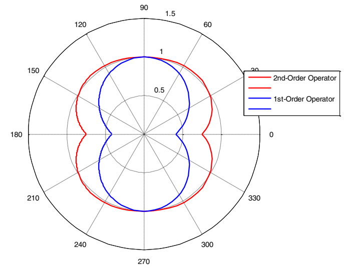

Figure 9: Stability regions of (8) and (13) for $\alpha = 0.5$.

Figure 10, on the other hand, shows the effect of the fractional-order, α, on the stability region of the 2nd-order operator (13). Clearly, as $\alpha \to 1$, the stability regions expand to coincides with the unit circle.

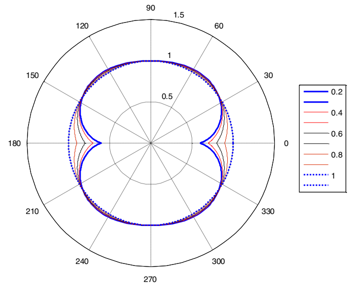

Figure 10: Stability region of (13) for different values of α

To obtain an accurate DTOs using the CFE technique, one may often requires a 7th-order or a 9th-order rational *z*-transfer function to discretize $s^{\pm\alpha}$ (Chen et al., 2002, 2003). The next section summarizes the effectiveness of the proposed operators of (8) and (13) against other forms of DTOs.

V. NUMERICAL SIMULATION

To appreciate the proposed DTOs of (8) and (13), their frequency response are compared with other forms of DTOs reported in (Al-Alaoui, 2001; Chen et al., 2003). The case when $\alpha = 0.5$ for $T = 0.001$ s is taken as a bench mark. Equation (13) yields,

$$s^{0.5} \cong \frac{44.7214 \ -22.0313z^{-1} \ -8.4670 \ \ z^{-2}}{1.0 + \ 0.4926z^{-1} \ -0.1893\ z^{-1}} \tag{19}$$

The following 9th-order DTO that discretizes $s^{0.5}$ using the CFE and reported in (Chen et al., 2003) is investigated against the one given by (19), i.e.,

$$G_9(z) = \frac{44.72\begin{pmatrix} z^9-0.5z^8-2z^7+0.875z^6+1.313z^5-0.4688z^4 \\ -0.3125z^3+0.07813z^2+0.01953z-0.001953 \end{pmatrix}}{\begin{pmatrix} z^9+0.5z^8-2z^7-0.875z^6+1.313z^5+0.4688z^4 \\ -0.3125z^3-0.07813z^2+0.01953z-0.001953 \end{pmatrix}} \tag{20}$$

Figure 11 displays the frequency response of (19) and (20). Both forms exhibit similar magnitude and phase responses. However, using (19), the magnitude frequency deviation at low frequency is better than that of (20), while the phase response of (20) at low frequency is better than that of (19). But, the significant improvement of (19) over the one in

(20) is evident by the order reduction. For example, if one wishes to discretize a system with Laplace operators of two different orders, one would need an 18^{th}-order model when using (2), while a 4^{th}-order one will be sufficient when using (19), which is faster and requires less hardware design.

Notice that since both poles and zeros of (8) or (13) are symmetric and lie in the stable region. Interchanging the poles and zeros with each other approximates the fractional-order discrete-time integrators, $(1/s^{\alpha})$, or equivalently, by using the reciprocal of (8) or (13) one can also discretize DTFoIs.

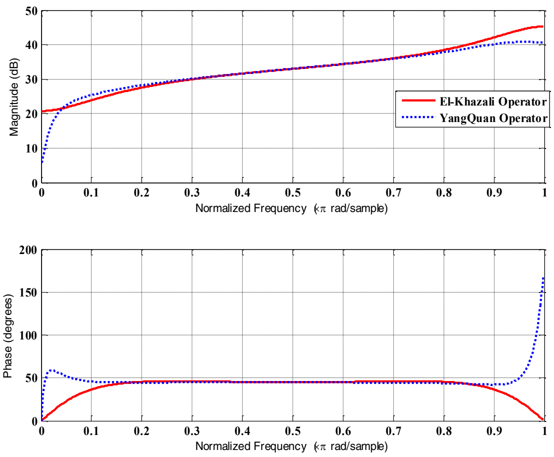

Figure 11: Frequency response of (19) and (20) that discretize $s^{0.5}$

VI. CONCLUSIONS

Two closed-forms of first and second-order DTOs were introduced to discretize the fractional-order Laplacian operator, $s^{\pm\alpha}$. The operators depends solely on the fractional-order, α, which can accommodate systems of "variable" orders. The two operators exhibit flat phase and constant gain frequency response over a wide range of frequency with a perfect symmetry in the phase response. The proposed solution is straightforward and yields a stable operator of low order. It is an IIR-type z-transfer function that discretize $s^{\pm\alpha}$. It is worth noting that the first-order operator can be used for reasonable high fractional orders; (say $0.8 \le \alpha \le 1$). The proposed closed-form operators exhibit a flat phase frequency response with less magnitude deviation, at both ends of the spectrum, than other forms obtained when using the CFE technique. The significance of the proposed operators becomes evident in system analysis and controller design, which is left for future consideration.

VII. REFERENCES

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